EXTENDED SUPERSYMMETRIES FOR THE SCHRÖDINGER-PAULI EQUATION

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It is argued that extended, reducible and generalized supersymmetry (SUSY) are common in many systems of standard non-relativistic quantum mechanics. For example, it is proved, that well-studied quantum mechanical system of a spin 1/2 particle interacting with constant and homogeneous magnetic field admits the $N = 4$ SUSY and has the internal symmetry $so(3,3)$. Then an approach to search for systems whose spectra have a specific SUSY degeneration is presented and developed. It is applied to a wide class of systems admitting $N = 3, N = 4$ and $N = 5$ SUSY. Some of these symmetries have a very peculiar property - the related supercharges are realized without usual fermionic variables. It is shown that for them the usual extension $N = 3$ to $N = 4$ SUSY is no more guaranteed.

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I. INTRODUCTION

A beautiful and rich concept of supersymmetry (SUSY) has been introduced by several authors in various contexts (see, e.g.,ref.[1] but also refs[2] where the idea of SUSY was formulated in somewhat rudimentary form). Since that time it plays more and more important role in physics and mathematics in general and in modern particle physics and quantum mechanics[3] in particular. This is due to the fact that SUSY presents a powerful tool for transforming bosons to fermions and vice versa, for formulating theories with non-trivial unification of space-time and internal symmetries, for formulating string theories and their most powerful dualities (refer, e.g., to Refs.[4−6]), for understanding the relations between spectra of different Hamiltonians as well as for explaining degeneracy of their spectra, for constructing exactly or quasi-exactly solvable systems, for justifying formulations of initial and boundary problems, etc., etc.; see, e.g., surveys[4,7,8].

In this work we shall concentrate on quantum mechanical systems since they provide a ground for testing the principal question: whether SUSY is realized in Nature or not, free of the complexities of field theories. Examples of such systems (like interaction of spin 1/2 particle with the Coulomb or constant and homogeneous magnetic field) which admit exact $N = 2$ SUSY are well known [9,10] (see also Refs. [7,8] and references therein). Here we search for problems with *extended* ($N > 2$) SUSY.

In this connection, let us remind that the quantum mechanical models which include $N > 2$ supercharges were investigated, e.g., in Refs.[11] , and examples of quantum mechanical systems with extended SUSY where discussed in Refs.[12−16]. In papers[17] the so called "generalized SUSY" was proposed; it includes extended SUSY as a particular case.

It was pointed out in Refs.[12−14] that some quantum mechanical models are invariant w.r.t. reducible representations of SUSY algebra; we will refer to the related symmetry as "reducible SUSY".

Of course it is interesting to search for physical systems which admit exact (especially extended, reducible, or generalized) SUSY. First, they brink additional indications that SUSY is indeed the symmetry of Nature, and secondly, for such systems we have standard methods for their analysis at our disposal.

In fact it will be shown in the present paper that the extended, reduced and generalized SUSYs are common in many problems of standard nonrelativistic quantum mechanics. For example, we prove that the well studied system of spin 1/2 particle interacting with a constant and homogeneous magnetic field, which can be described by the Schrödinger-Pauli equation, admits $N = 4$ SUSY and $N = 2$ reducible SUSY as well.

In Section II we show that the extended, reducible, and generalized SUSY appear naturally in a wide class of problems of standard one-dimensional SUSY quantum mechanics. In Section III we consider the quantum mechanical system of a spin 1/2 particle interacting with a constant and homogeneous magnetic field (III.1) and prove that it has $N = 4$ extended SUSY (III.2). The reducible SUSY and $so(3, 3)$ symmetry of this model is discussed in Section III.3.

In Section IV we search for extended and reducible SUSY of the Schrödinger-Pauli equation for a particle interacting with a static inhomogeneous magnetic field. We find a wide class of systems admitting these supersymmetries and discuss briefly their physical consequences.

II. ADDITIONAL EXTENDED AND REDUCIBLE SUSY OF SUPERSYMMETRIC QUANTUM MECHANICS

Supersymmetric quantum mechanical systems are described by the Schrödinger equation with a matrix potential[3]

$$\hat{H}\psi == \frac{1}{2}\left(p^2 + W^2 + \sigma_3 W'\right)\psi = E\psi, \qquad (2.1)$$

where $p = -i\frac{\partial}{\partial x}$, $W = W(x)$ is a superpotential, $W' = \frac{\partial W}{\partial x}$, and σ_3 is the Pauli matrix of the form $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

It is well known that equation (2.1) admits specific symmetries (supercharges)

$$Q_1 = \frac{1}{\sqrt{2}}\left(\sigma_1 p + \sigma_2 p\right), \quad Q_2 = \frac{1}{\sqrt{2}}\left(\sigma_2 p - \sigma_1 W\right) \qquad (2.2)$$

which satisfy the following superalgebra

$$\{Q_a, Q_b\} = 2\delta_{ab}\hat{H}, \quad [Q_a, \hat{H}] = 0, \qquad (2.3)$$

where $a, b = 1, 2$; [.,.] and {.,.} denote a commutator and anticommutator respectively.

Let us demonstrate that in addition to the transparent $N = 2$ SUSY, equation (2.1) admits $N = 3$ extended SUSY provided the corresponding superpotential $W(x)$ is an even function of x (cf. ref. [12]).

Proposition 1. Let $W(-x)=W(x)$, then there exist the third supercharge

$$Q_3=i\sigma_1 RQ_1 \tag{2.4}$$

satisfying relations (2.3) for $a=1,2,3$ together with operators (2.2). Here R is defined by

$$R\psi(x)=\psi(-x). \tag{2.5}$$

Proof. The proof of this proposition can be done by a simple direct calculation taking into account the relations

$$[\sigma_1 R, Q_2]=\{\sigma_1 R, Q_1\}=0, \quad (\sigma_1 R)^2=1.$$

Thus even the simplest SUSY model (2.1) can admit the extended SUSY generated by three supercharges.

Another interesting possibility is connected with the fact that the representation of superalgebra (2.2), (2.3) can be reducible. This happens for the systems described by equation (2.1) with odd superpotentials (cf. refs. [12,13]).

Proposition 2 . Let $W(-x)=-W(x)$, then the superalgebra (2.2), (2.3) is reducible.

Proof. For $W(x)$ odd there exists the invariant operator, namely

$$I=\sigma_3 R, \tag{2.6}$$

which commutes with any element of algebra (2.3). Using the mapping $I\rightarrow I'=UIU^\dagger$, where

$$U=R_+ - i\sigma_2 R, \quad R_\pm=\frac{1}{2}(1\pm R), \tag{2.7}$$

the operator (2.6) is transformed to the diagonal matrix

$$I'_3=\sigma_3. \tag{2.8}$$

The corresponding transformed supercharges $Q'_q=UQ_aU^\dagger$ and the Hamiltonian $\hat{H}'=U\hat{H}U^\dagger$ commute with σ_3 and thus are diagonal too:

$$Q'_a=\begin{pmatrix} q_+^\alpha & 0 \\ 0 & q_-^\alpha \end{pmatrix}, \quad \hat{H}'=\begin{pmatrix} \hat{H}_+ & 0 \\ 0 & \hat{H}_- \end{pmatrix}, \quad \alpha=1,2. \tag{2.9}$$

Here

$$q_\pm^1=iRP\pm W, \quad q_\pm^2=\pm p-iRW, \quad \hat{H}_\pm=\frac{1}{2}\left(p^2+W^2\pm W'R\right). \tag{2.10}$$

Thus supercharges Q'_α are expressed as a direct sum of q_+^α and q_-^α . Q.e.d.

Propositions 1 and 2 indicate how to find extended and reducible SUSY of realistic three-dimensional systems by first determining and then applying the appropriate involutive *discrete symmetries* (e.g., parities) of the system.

It is easy to see that the above obtained extended and reducible supersymmetries are equivalent to generalized ones[17], i.e., supersymmetries satisfying the relations

$$Q^2 = \hat{H}, \quad \{I_a, Q\} = 0, \quad I_a^2 = 1, \quad I_a I_b = \pm I_b I_a, \quad a = 1, 2, ... \tag{2.11}$$

where involutions I_a either all commute or all anticommute.

Indeed, for even superpotentials there exist anticommuting involutions $I_1 = \sigma_3$ and $I_2 = \sigma_1 R$ which together with $Q = Q_1$ satisfy relations (2.11). In the case of odd superpotentials, relations (2.11) are satisfied by supercharge Q equal to Q_1 and commuting involutions $I_1 = \sigma_3$ and $I_2 = R$ (compare with Refs. [17]).

In the systems analyzed later on we shall find their extended and reducible SUSY too. However, in contradistinction to the systems studied in the present section, the existence of these SUSYs will not imply the existence of the corresponding generalized SUSY.

III. SPIN 1/2 PARTICLE IN CONSTANT, HOMOGENEOUS MAGNETIC FIELD

III.1 Degeneracy of the spectrum of energy

Consider a quantum mechanical system consisting of a spin 1/2 charged particle interacting with a constant and homogeneous magnetic field. In a non-relativistic approximation this system is described by the Schrödinger-Pauli equation

$$2mE\psi = \hat{H}\psi, \quad \hat{H} = \boldsymbol{\pi}^2 - \frac{1}{2}eg\boldsymbol{\sigma}\cdot\mathbf{H}, \tag{3.1}$$

where

$$\pi_a = -i\frac{\partial}{\partial x_a} - eA_a, \quad a = 1, 2, 3, \quad \boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3), \quad g = 2, \tag{3.2a}$$

and

$$A_1 = eHx_2, \quad A_2 = A_3 = 0, \quad e\mathbf{H} = i\boldsymbol{\pi}\times\boldsymbol{\pi} = e(0, 0, H). \tag{3.2b}$$

Here σ_a are the Pauli matrices, and H is a constant characterizing the strength of the magnetic field $\mathbf{H}$.

The problem (3.1) is exactly solvable[18] . The corresponding eigenvalues E (Landau levels) are given by

$$2mE = 2neH + p_3^2, \qquad n = 0, 1, 2, \cdots . \tag{3.3}$$

For any $n \neq 0$ there exist two independent eigenfunctions (see, e.g., ref. [19])

$$\begin{aligned} \psi_{1,p_1,p_3} &= \exp(ip_1x_1 + ip_3x_3)\exp(-y^2/2)\begin{pmatrix} H_n(y) \\ H_{n-1}(y) \end{pmatrix}, \\ \psi_{2,p_1,p_3} &= \exp(ip_1x_1 + ip_3x_3)\exp(-y^2/2)\begin{pmatrix} H_n(y) \\ -H_{n-1}(y) \end{pmatrix} \end{aligned} \tag{3.4}$$

with H_n being Hermite polynomials, $H_{-1} = 0$, and

$$y = \sqrt{eH}x_2 - \frac{p_1}{\sqrt{eH}}. \tag{3.5}$$

For $n = 0$ the eigenfunctions ψ_{1,p_1,p_2} and ψ_{2,p_1,p_2} coincide. For $n > 0$ any energy level is two-fold degenerate due to $N = 2$ SUSY of equation (3.1). Moreover, there exists the infinite degeneration of any energy level due to independence of E on p_1[18] .

In spite of the fact that symmetries and supersymmetries of equation (3.1) have been studied quite intensively (see, e.g., Refs. [6–8,20,21]), we shall find a new additional (extended) SUSY for this equation.

Starting with (3.4) and taking into account the quadratic dependence of energy E on p_3 and independence of E on p_1, we can write, for instance, six additional solutions corresponding to the same energy (3.3), namely

$$\begin{aligned} &\psi_{3,p_1,p_3} = \psi_{1,-p_1,p_3}, \quad \psi_{4,p_1,p_3} = \psi_{1,-p_1,-p_3}, \\ &\psi_{5,p_1,p_3} = \psi_{1,p_1,-p_3}, \quad \psi_{6,p_1,p_3} = \psi_{2,-p_1,p_3}, \\ &\psi_{7,p_1,p_3} = \psi_{2,-p_1,-p_3}, \quad \psi_{8,p_1,p_3} = \psi_{2,p_1,-p_3}. \end{aligned} \tag{3.6}$$

A bit surprisingly, the corresponding eight-fold degeneration of energy levels can be interpreted as caused by $N = 4$ extended SUSY of system (3.1).

III.2. Extended SUSY

It is well known that whenever the gyromagnetic ratio g of the particle is equal to 2, equation (3.1) admits $N = 2$ SUSY[10] . Here we demonstrate that this SUSY is reducible and that there exists a more extended, namely, $N = 4$ SUSY for (3.1) in addition.

The standard supercharge for equation (3.1) has the form[7]

$$Q_1 = \boldsymbol{\sigma} \cdot \boldsymbol{\pi} \ , \qquad Q_1^2 = \hat{H}. \tag{3.7}$$

The remaining three additional supercharges can be constructed using the fact that (3.1) is invariant w.r.t. the following three discrete transformations:

$$\psi \to R_3\psi, \qquad \psi \to CR_1\psi, \qquad \psi \to CR_2\psi, \tag{3.8}$$

where $R_a (a = 1, 2, 3)$ are the space reflection transformations

$$R_a\psi(\mathbf{x}) = \sigma_a\theta_a\psi(\mathbf{x}), \qquad \theta_a\psi(\mathbf{x}) = \psi(r_a\mathbf{x}). \tag{3.9a}$$

Here

$$r_1\mathbf{x} = (-x_1, x_2, x_3), \quad r_2\mathbf{x} = (x_1, -x_2, x_3), \quad r_3\mathbf{x} = (x_1, x_2, -x_3), \tag{3.9b}$$

and $C = i\sigma_2 c$, where c is the operator of complex conjugation

$$c\psi(\mathbf{x}) = \psi^*(\mathbf{x}). \tag{3.10}$$

Note that operators defined in (3.8)-(3.10) satisfy the following relations

$$\begin{array}{c} \{R_a, \boldsymbol{\sigma} \cdot \boldsymbol{\pi}\} = \{R_a, C\} = \{CR_1, \boldsymbol{\sigma} \cdot \boldsymbol{\pi}\} = \{CR_2, \boldsymbol{\sigma} \cdot \boldsymbol{\pi}\} = 0, \\ R_a^2 = -C^2 = 1, \qquad a = 1, 2, 3. \end{array} \tag{3.11}$$

Thus, using (3.7), (3.11) we can see that the operators

$$Q_1 = \boldsymbol{\sigma} \cdot \boldsymbol{\pi}, \quad Q_2 = iR_3\boldsymbol{\sigma} \cdot \boldsymbol{\pi}, \quad Q_3 = CR_1\boldsymbol{\sigma} \cdot \boldsymbol{\pi}, \quad Q_4 = CR_2\boldsymbol{\sigma} \cdot \boldsymbol{\pi} \tag{3.12}$$

fulfill the following relations

$$\{Q_k, Q_l\} = 2g_{kl}\hat{H}, \qquad \left[Q_k, \hat{H}\right] = 0, \tag{3.13}$$

where $k, l = 1, 2, 3, 4, g_{11} = g_{22} = -g_{33} = -g_{44} = 1;\ g_{kl} = 0, k \neq l$. In other words, operators (3.12) are supercharges generating the $N = 4$ extended SUSY of equation (3.1).

We notice that choosing the basis

$$\begin{array}{ll} \hat{Q}_1 = \frac{1}{\sqrt{2}}\left(Q_1 + Q_3\right), & \hat{Q}_2 = \frac{1}{\sqrt{2}}\left(Q_2 + Q_4\right), \\ \hat{Q}_1^+ = \frac{1}{\sqrt{2}}\left(Q_1 - Q_3\right), & \hat{Q}_2^+ = \frac{1}{\sqrt{2}}\left(Q_2 - Q_4\right), \end{array} \qquad (3.14)$$

it is possible to represent the commutation and anticommutation relations (3.13) in a more familiar form

$$\left\{\hat{Q}_\alpha, \hat{Q}_\beta\right\} = 0, \qquad \left\{\hat{Q}_\alpha, \hat{Q}_\beta^+\right\} = 2\delta_{\alpha\beta}\hat{H},$$
$$\left[\hat{Q}_\alpha, \hat{H}\right] = 0, \qquad \alpha, \beta = 1, 2.$$

Thus we have proved that the well-known $N = 2$ SUSY of equation (3.1) can be extended to $N = 4$ SUSY taking into account involutive symmetries (3.8). Acting by supercharges (3.12) on standard solutions (3.4) we obtain the set of eight linearly independent solutions (3.4), (3.6). The interpretation of the corresponding eight-fold degeneracy is given in the next section.

III.3. Internal symmetries and reducible SUSY

A direct consequence of the $N = 4$ SUSY is a specific four-fold degeneration of the corresponding non-ground states[11]. However, we have found that system (3.1) has eight-fold degeneracy. Let us demonstrate that this is due to the existence a special internal symmetry algebra. This algebra appears as follows.

First, for any non-zero eigenvalue E of Hamiltonian (3.1) we can choose the following set of symmetry operators:

$$S_{6k} = \frac{1}{2\sqrt{E}}Q_k, \quad S_{65} = \frac{1}{2}R_3, \quad S_{mn} = \left[S_{6m}, S_{6n}\right]. \qquad (3.15)$$

Here Q_k and R_3 are operators defined in (3.9), (3.12), $k = 1, 2, 3, 4$ and $m, n = 1, 2, 3, 4, 5$.

However, there exists an additional symmetry operator, namely, the operator

$$I_1 = i\left(\sigma_1\pi_2 - \sigma_2\pi_1\right)p_3R_3, \qquad (3.16)$$

which commutes with any of the operators (3.15).

Thus, taking into account that operators S_{6n} form the Clifford algebra

$$\{2S_{6n}, 2S_{6m}\} = 2g_{mn} \qquad (3.17)$$

with nonzero components of g_{mn} being $g_{11} = g_{22} = -g_{33} = -g_{44} = g_{55} = 1$, we can easily find that symmetry operators (3.15) and (3.16) satisfy the following commutation relations

$$[S_{kl}, S_{mn}] = g_{kl}S_{ln} + g_{ln}S_{kl} - g_{kn}S_{lm} - g_{lm}S_{kn}, \qquad (3.18a)$$

$$[S_{kl}, I_1] = 0 \qquad (3.18b)$$

(with $k, l, m, n = 1, 2, 3, 4, 5, 6$ and $g_{66} = -1$), i.e., form the central extension of Lie algebra so(3,3) by I_1[22]. Its invariant operators are given by

$$\begin{array}{c} C_1 = \frac{1}{2}S_{kl}S^{kl} == \frac{15}{4}, \qquad C_2 = \frac{1}{2}S_{kl}S^{ln}S_{nf}S^{fk} == \frac{315}{16}, \\ C_3 = \frac{1}{6!}\varepsilon_{mnrslk}S^{mn}S^{rs}S^{lk} = \frac{1}{8}R_1R_2, \qquad C_4 = I_1. \end{array} \qquad (3.19)$$

Using (3.11), (3.12) it is easy to show that eigenvalues of operators C_3 and C_4 are $\pm\frac{1}{8}$ and $\pm p_3\sqrt{2neH}$ respectively. Four possible combinations of these eigenvalues specify four orthogonal subspaces of solutions of equation (3.1) for p_3 and n different from zero. Thus operators (3.15) realize the direct sum of finite- dimensional irreducible representations of $so(3,3)$, namely, the direct sum of irreducible representations $2D\left(\frac{1}{2}\frac{1}{2}\frac{1}{2}\right) \oplus 2D\left(\frac{1}{2}\frac{1}{2} - \frac{1}{2}\right)$ of the algebra $so(3,3)$ [23]. The corresponding representation space is 16-dimensional over $\mathbb{R}$, so effectively we have the eight-fold degeneracy over the field of complex numbers.

Restricting ourselves to linear symmetries (i.e., to those including no the antiunitary operator of complex conjugation) $N = 4$ SUSY is reduced to $N = 2$ SUSY which is generated by supercharges Q_1 and Q_2 specified in (3.12). However, this SUSY is reducible since there exist two linear symmetries for (3.1), namely C_3 and C_4 (3.19), which are involutive up to constant factors and commute with supercharges Q_1 and Q_2:

$$[C_3, Q_a] = [I_1, Q_a] = 0, \qquad [I_1, C_3] = 0, \qquad a = 1, 2. \qquad (3.20)$$

Analogously as in the Proof of Proposition 2 we can diagonalize C_3 and C_4 and reduce any of supercharges Q_1, Q_2 to a direct sum of four supercharges. This yields four invariant subspaces $\Phi^{(\alpha)}(\alpha = 1, 2, 3, 4)$ of supercharges Q_1 and Q_2 with basis elements $\Phi_1^{(\alpha)}, \Phi_2^{(\alpha)}$, where

$$\begin{array}{l} \Phi_1^{(1)} = \psi_{1,p_1,p_3} + \psi_{1,-p_1,p_3} + i\psi_{1,p_1,-p_3} + i\psi_{1,-p_1,-p_3}, \\ \Phi_2^{(1)} = \psi_{2,p_1,p_3} + \psi_{2,-p_1,p_3} - i\psi_{2,p_1,-p_3} - i\psi_{2,-p_1,-p_3}; \end{array} \qquad (3.21)$$

$$\begin{aligned}\Phi_1^{(2)} &= \psi_{1,p_1,-p_3} + \psi_{1,-p_1,-p_3} + i\psi_{1,p_1,p_3} + i\psi_{1,-p_1,p_3},\\ \Phi_2^{(2)} &= \psi_{2,p_1,-p_3} + \psi_{2,-p_1,-p_3} - i\psi_{2,p_1,p_3} - i\psi_{2,-p_1,p_3};\end{aligned} \tag{3.22}$$

$$\begin{aligned}\Phi_1^{(3)} &= -\psi_{1,p_1,p_3} + \psi_{1,-p_1,p_3} - i\psi_{1,p_1,-p_3} + i\psi_{1,-p_1,-p_3},\\ \Phi_2^{(3)} &= -\psi_{2,p_1,p_3} + \psi_{2,-p_1,p_3} + i\psi_{2,p_1,-p_3} - i\psi_{2,-p_1,-p_3};\end{aligned} \tag{3.23}$$

$$\begin{aligned}\Phi_1^{(4)} &= -\psi_{1,p_1,-p_3} + \psi_{1,-p_1,-p_3} - i\psi_{1,p_1,p_3} - i\psi_{1,-p_1,p_3},\\ \Phi_1^{(4)} &= -\psi_{2,p_1,-p_3} + \psi_{2,-p_1,-p_3} + i\psi_{2,p_1,p_3} - i\psi_{1,-p_1,p_3},\end{aligned} \tag{3.24}$$

where ψ_{1,p_1,p_2} and ψ_{2,p_1,p_2} are functions defined in (3.4).

IV. SPIN 1/2 PARTICLE IN AN ARBITRARY EXTERNAL MAGNETIC FIELD

IV.1. Extended SUSY

In this section we shall show that the system of spin 1/2 particle interacting with various magnetic fields has extended SUSY too, provided the external field has definite parity properties. Starting with reflections (3.9b) we find that the corresponding parity properties of vector-function $\mathbf{A}(\mathbf{x})$ (3.2b) are of the form

$$\mathbf{A}(r_1\mathbf{x}) = -r_1\mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r_2\mathbf{x}) = -r_2\mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r_3\mathbf{x}) = r_3\mathbf{A}(\mathbf{x}). \qquad (4.1)$$

Relations (4.1) are satisfied by a large class of potentials which includes (3.2b) as a particular case. For all such potentials the corresponding equation (3.1) is invariant w.r.t. involutions (3.8) and so admits the extended SUSY generated by supercharges (3.12). Moreover, equation (3.1) for $g = 2$ and an arbitrary uniform magnetic field , i.e., the field

$$A_1 = A_1(x_1, x_2), \quad A_2 = A_2(x_1, x_2), \quad A_3 = 0, \qquad (4.2)$$

admits all internal symmetries described in Section III.2 provided $\mathbf{A}(\mathbf{x})$ satisfies relations (4.1).

Other systems with extended SUSY can be found by extending reflections (3.9b) to the eight-dimensional group of involutions, i.e., by adding the transformations

$$\begin{array}{c} r_{12}\mathbf{x} = (-x_1, -x_2, x_3), \quad r_{31}\mathbf{x} = (-x_1, x_2, -x_3), \\ r_{23}\mathbf{x} = (x_1, -x_2, -x_3), \quad r_{123}\mathbf{x} = (-x_1, -x_2, -x_3), \quad I\mathbf{x} = \mathbf{x} \end{array} \qquad (4.3)$$

to reflections (3.9b).

We notice that $r_a (a = 1, 2, 3)$ and r_{123} are reflections while $r_{ab}(a, b = 1, 2, 3)$ are rotations.

Let us suppose that the vector potential $\mathbf{A}(\mathbf{x})$ has definite parities w.r.t. a subset of transformations (3.9b), (4.3). All possible transformations for the vector-potential with definite parities w.r.t. (9.3b) and (6.3), which are compatible with the Lorentz gauge $\mathbf{p} \cdot \mathbf{A} = 0$ are given by the formulae

$$\mathbf{A}(r_{ab}\mathbf{x}) = r_{ab}\mathbf{A}(\mathbf{x}), \qquad a, b = 1, 2, 3, \qquad (4.4a)$$

$$\mathbf{A}(\mathbf{x}) = r_a\mathbf{A}(\mathbf{x}), \qquad (4.4b)$$

$$\mathbf{A}(r_{123}\mathbf{x}) = -\mathbf{A}(\mathbf{x}) \tag{4.4c}$$

and

$$\mathbf{A}(r_{ab}\mathbf{x}) = -r_{ab}\mathbf{A}(\mathbf{x}), \tag{4.5a}$$

$$\mathbf{A}(r_a\mathbf{x}) = -r_a\mathbf{A}(\mathbf{x}), \tag{4.5b}$$

$$\mathbf{A}(r_{123}\mathbf{x}) = \mathbf{A}(\mathbf{x}). \tag{4.5c}$$

It is easy to see that whenever $\mathbf{A}(\mathbf{x})$ transforms according one of relations (4.4a)-(4.4c) or (4.5a)-(4.5c) (for fixed values of indices a, b) the equation (3.1) remains invariant w.r.t. this transformation provided $\psi(\mathbf{x})$ co-transforms accordingly, i.e., via relations

$$\psi(\mathbf{x}) \to iR_aR_b\psi(\mathbf{x}) \equiv R_{ab}\psi(\mathbf{x}), \tag{4.6a}$$

$$\psi(\mathbf{x}) \to R_a\psi(\mathbf{x}), \tag{4.6b}$$

$$\psi(\mathbf{x}) \to R_1R_2R_3\psi(\mathbf{x}) \equiv R\psi(\mathbf{x}) \tag{4.6c}$$

or

$$\psi(\mathbf{x}) \to i\sigma_2cR_a\psi(\mathbf{x}) \equiv CR_{ab}\psi(\mathbf{x}), \tag{4.7a}$$

$$\psi(\mathbf{x}) \to i\sigma_2cR_a\psi(\mathbf{x}) \equiv CR_a\psi(\mathbf{x}), \tag{4.7b}$$

$$\psi(\mathbf{x}) \to i\sigma_2cR\psi(\mathbf{x}) \equiv CR\psi(\mathbf{x}) \tag{4.7c}$$

respectively. Here R_a and c are operators introduced in (3.9), (3.10).

Transformations (4.6b)-(4.7c) are involutions anticommuting with $\boldsymbol{\sigma}\cdot\boldsymbol{\pi}$, so yielding $N = 2$ SUSY with supercharges given by

$$Q_1 = \boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \qquad Q_2 = i\hat{R}\boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \tag{4.8}$$

where $\hat{R}$ denotes the relevant operators (4.6b)-(4.7c) (i.e., for the symmetry (4.4b) the operator $\hat{R} = R_a$, for (4.4c) the operator $\hat{R} = R$, etc.).

More complicated cases, in which the vector $\mathbf{A}(\mathbf{x})$ has definite transformation properties w.r.t. combined parities, can be discussed analogously. First, using the group properties of involutions (3.9b), (4.3) it is easy to show that whenever $\mathbf{A}(\mathbf{x})$ has definite parities w.r.t. two of these involutions, it has the definite parity w.r.t. their product. Requiring definite parities w.r.t. various triplets of involutions enumerated in (3.9b), (4.3), we receive the cases which are either equivalent to those with definite the transformation properties w.r.t. doublets of parities or w.r.t. all eight involutions (3.9b), (4.3).

If $\mathbf{A}(\mathbf{x})$ satisfies two compatible relations from (4.4), (4.5) simultaneously then equation (3.1) with $g = 2$ admits $N = 2$ or $N = 3$ SUSY. All these nonequivalent possibilities are enumerated in the Appendix. Here we consider only one example, namely when the vector-potential has the property

$$\mathbf{A}(r_1\mathbf{x}) = r_1\mathbf{A}(\mathbf{x}), \qquad \mathbf{A}(r_2\mathbf{x}) = r_2\mathbf{A}(\mathbf{x}), \tag{4.9}$$

but has no definite parity w.r.t. reflection r_3. The related supercharges are of the form

$$Q_1 = \boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \quad Q_2 = iR_1\boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \quad Q_3 = iR_2\boldsymbol{\sigma}\cdot\boldsymbol{\pi}. \tag{4.10}$$

They satisfy relations (2.3) (where $\hat{H}$ is Hamiltonian (3.1), $a, b = 1, 2, 3$) and thus generate the symmetry algebra equal to $N = 3$ SUSY for the system. This SUSY causes a four-fold degeneration of the corresponding (non-ground) energy levels, since for any nonzero E it yields the four-dimensional representation $D\left(\frac{1}{2}\frac{1}{2}\right) \oplus D\left(\frac{1}{2} - \frac{1}{2}\right)$ of Lie algebra $so(4)$ generated by

$$S_{4a} = \frac{1}{2\sqrt{E}}Q_a, \qquad S_{ab} = -i\left[S_{4a}, S_{4b}\right]. \tag{4.11}$$

The most extended, $N = 4$ and $N = 5$ SUSY appears in the cases for which the vector-potential has definite parities w.r.t. all involutions (3.9b), (4.3). In addition to (4.1) there are three more possible transformation properties of $\mathbf{A}(\mathbf{x})$:

$$\mathbf{A}(r_a\mathbf{x}) = \mathbf{A}(\mathbf{x}), \quad a = 1, 2, 3, \tag{4.12}$$

$$\begin{gathered}\mathbf{A}(r_a\mathbf{x}) = \mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r_b\mathbf{x}) = \mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r_c\mathbf{x}) = -\mathbf{A}(\mathbf{x}),\\ a \neq b, \quad b \neq c, \quad c \neq a, \quad c \quad is \quad fixed,\end{gathered} \tag{4.13}$$

and

$$\mathbf{A}(r_a\mathbf{x}) = -\mathbf{A}(\mathbf{x}), \quad a = 1, 2, 3. \tag{4.14}$$

They allow to construct the corresponding supercharges, namely

$$Q_1 = iR_1\boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \quad Q_2 = iR_2\boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \quad Q_3 = iR_3\boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \quad Q_4 = \boldsymbol{\sigma}\cdot\boldsymbol{\pi}; \tag{4.15}$$

$$Q_0 = CR_c\boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \quad Q_1 = \boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \quad Q_2 = iR_a\boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \quad Q_3 = iR_b\boldsymbol{\sigma}\cdot\boldsymbol{\pi}; \tag{4.16}$$

and

$$\begin{gathered}Q_1 = CR_1\boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \quad Q_2 = CR_2R_1\boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \quad Q_3 = CR_3\boldsymbol{\sigma}\cdot\boldsymbol{\pi},\\ Q_4 = CR\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}, \quad Q_5 = \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\end{gathered} \tag{4.17}$$

for the cases (4.12), (4.13) and (4.14) respectively.

Operators (4.15) and Hamiltonian (3.1) satisfy relations (2.3) for $a, b = 1, 2, 3, 4$ and thus generate $N = 4$ extended SUSY. The corresponding internal symmetries reduce to the algebra $so(5)$ whose basis elements (constructed analogously to (4.11)) generate the four-dimensional irreducible representation $D(\frac{1}{2}\frac{1}{2}\frac{1}{2})$. Thus for the system (3.1), (3.2a) we can expect a four-fold degeneracy of non-ground energy levels whenever the vector-potential of an external field satisfies the relations (4.12).

Operators (4.16) and (4.17) satisfy relations (3.13) for $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$ and $-g_{11} = -g_{22} = -g_{33} = g_{44} = g_{55} = 1$ respectively and thus generate $N = 4$ and $N = 5$ SUSY .

IV.2. Reducible SUSY

In this subsection the involutions (4.6), (4.7) are used to find out reducible SUSY for the systems described by the equation (3.1) with $g = 2$ and vector-potential $\mathbf{A}(\mathbf{x})$.

First let us assume that the parity properties of the vector-potential are specified by relations (4.12), (4.13). Then the corresponding equation (3.1) admits $N = 4$ SUSY. Moreover, there exist the involutions

$$I = R_{23} \tag{4.18}$$

and

$$I^{(c)} = CR_{ab}, \qquad a, b \neq c. \tag{4.19}$$

commuting with pairs of supercharges Q_1, Q_2 from (4.15a) and Q_0, Q_1 from (4.16) respectively, so that the corresponding $N = 2$ SUSY is reducible.

If parities of the vector-potential are specified by relations (4.14) then there exists the involution

$$I = R_{23} \tag{4.20}$$

which commutes with a triplet of supercharges, namely with supercharges Q_1, Q_2 and Q_3 of (4.16). Consequently the related equation (3.1) admits $N = 3$ reducible SUSY.

If the vector-potential satisfies all relations (4.1) then there exists the involution

$$I = R_{12} \tag{4.21}$$

commuting with all four supercharges (3.12) and so the corresponding system has $N = 4$ reducible SUSY.

Indeed, diagonalizing involutions (4.18)-(4.21), the corresponding supercharges are transformed to block diagonal forms. For instance, for involutions (4.18) and supercharges (4.15) we obtain

$$I \to UIU^\dagger = \sigma_3, \quad Q_a \to UQ_aU^\dagger = \tfrac{1}{2}\left(1+\sigma_3\right)Q_a^+ + \tfrac{1}{2}\left(1-\sigma_3\right)Q_a^-, \quad a = 1,2 \tag{4.22}$$

where

$$U = \frac{1}{\sqrt{2}}\left(1+\sigma_3 I\right), \quad U^\dagger = \frac{1}{\sqrt{2}}\left(1-\sigma_3 I\right), \tag{4.23}$$

and

$$Q_1^+ = \left(\pi_1 - i\pi_2\right)\theta_{23} + \pi_3, \quad Q_2^+ = \left(i\pi_1 + \pi_2\right)\theta_1 + i\pi_3\theta_{123}, \tag{4.24}$$

$$Q_1^- = \left(-\pi_1 - i\pi_2\right)\theta_{23} - \pi_3, \quad Q_2^+ = \left(i\pi_1 - \pi_2\right)\theta_1 - i\pi_3\theta_{123} \tag{4.25}$$

with $\theta_{ab} = \theta_a\theta_b$, $\theta_{123} = \theta_1\theta_2\theta_3$, and operators θ_a defined in (3.9a).

The operators (4.23) together with

$$\hat{H} = \hat{H}^+ = \boldsymbol{\pi}^2 - 2e\left[H_3 + \left(iH_2 - H_1\right)\theta_{23}\right], \tag{4.26}$$

form superalgebra (2.3), while operators (4.24) satisfy (2.3) with the Hamiltonian of the form

$$\hat{H} = \hat{H}^- = \boldsymbol{\pi}^2 + 2e\left[H_3 - \left(H_1 + iH_2\right)\theta_{23}\right]. \tag{4.27}$$

Here H_1, H_2 and H_3 denote the components of the magnetic field strength.

The supercharges generating reducible SUSY for other systems described by (3.1) can be diagonalized in a similar way. The explicit form of the corresponding transformation operators is given in the Appendix..

Let us note that supercharges (4.23), (4.24) depend on three variables x_1, x_2, x_3 and have a very peculiar property: they include neither fermionic variables nor matrices.

V. DISCUSSION

In this article we have described an approach for systematical study of quantum mechanical systems whose symmetry group includes extended SUSY and whose degeneracy of energy spectra is of SUSY nature.

In Section 4, requiring definite parity properties of the vector-potential, we find a number of quantum mechanical systems with $N = 3, N = 4$ and $N = 5$ SUSY.

It is necessary to stress, that there exist a lot of realistic physical systems whose parities satisfy required relations (4.1), (4.12)-(4.14). In addition to the vector-potential of the constant magnetic field, given by relations (3.2b), we present here as examples the potential of an infinite straight conductor with the constant current I directed along the third co-ordinate axis

$$A_1 = A_2 = 0, \quad A_3 = -\frac{I}{4\pi}\ln\left(x_1^2 + x_2^2\right), \tag{5.1}$$

superpositions of potential (5.1) which are generated by two or four infinite straight conductors shifted by distance $2b$ (two neibour currents have opposite directions),

$$A_1 = A_2 = 0, \quad A_3 = -\frac{I}{4\pi}\ln\left[\frac{(x_1 - b)^2 + x_2^2}{(x_1 + b)^2 + x_2^2}\right], \tag{5.2}$$

or

$$\begin{gathered} A_1 = A_2 = 0, \\ A_3 = -\frac{I}{4\pi}\ln\left\{\frac{\left[(x_1+b)^2+(x_2+b)^2\right]\left[(x_1-b)^2+(x_2-b)^2\right]}{[(x+b)^2+(x_2-b)^2][(x_1-b)^2+(x_2+b)^2]}\right\}, \end{gathered} \tag{5.3}$$

and the magnetic octopole potential[16]

$$\begin{gathered} A_1 = \frac{a^2 m}{4\pi}\frac{x_1(x^2-x_2^2)}{x^7}, \quad A_2 = -\frac{a^2 m}{4\pi}\frac{x_2(x^2-x_1^2)}{x^7}, \quad A_3 = 0, \\ x^2 = x_1^2 + x_2^2 + x_3^2. \end{gathered} \tag{5.4}$$

Parities of potentials (5.1), (5.2), (5.3) and (5.4) are given by relations (4.1), (4.13), (4.14) and (4.12) respectively.

Moreover, analyzing various superpositions of magnetic dipole and straight conductor potentials, it is possible to generate models of physical systems with any parity properties enumerated in (A.1)-(A.3), (A.8)-(A.12).

The other examples of potentials having well defined parity properties and yielding extended SUSY are those of Aharonov-Bohm and magnetic dipole potentials.

The very existence of such systems presents a strong indication that the extended SUSY is indeed realized in Nature. Moreover, knowledge of extended SUSY for systems described by the Schrödinger-Pauli equation enables us to predict the specific degeneracy of the corresponding energy levels.

This degeneracy can be removed by adding a small symmetry-breaking term corresponding, e.g., to interaction with a week external electric field and thus experimentally verified.

We did not discuss the question whether the found extended SUSY is exact or broken. To this end it is necessary to analyze degeneracy of the ground state of the considered systems. For two-dimensional quantum mechanical systems such analysis was made in Ref. [25].

Our approach to extended SUSY can be compared with that using generalized SUSY[17] whenever all supercharges of the considered systems do not include complex conjugation and can be constructed starting with involutions satisfying (2.11). Since for some of our systems the corresponding supercharges include the antiunitary operator of complex conjugation, the above mentioned correspondence does not exist (in this case (2.11) does not necessary hold). Consequently, our approach covers more general situations then the approach proposed in Ref. [17].

It is well known that $N = 3$ SUSY can always be extended to that of $N = 4$[26] for the systems in which SUSY is realized by Grassmanian variables. Our paper shows that such an extension of odd N SUSY to even one is not guaranteed in general.

Analogously to the above mentioned cases with time-independent magnetic fields, it is possible to search for systems with extended SUSY described by the Schrödinger-Pauli equation with a time- dependent magnetic field. The case with $N = 2$ SUSY was discussed in paper [27].

We notice that our approach can be extended to the relativistic Dirac equation with a similar result (for particular examples see Refs[13,14]). However, Dirac's equation admits an extended SUSY also for the cases with external electric fields and scalar potentials[15].

Another intriguing problem is to generalize the above results for particles with spin $s > 1/2$. This can be done, e.g., in the framework of the week SUSY approach[28] .

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APPENDIX

COMBINED PARITIES, SUPERCHARGES AND EXACT REDUCTIONS

Here explicit forms of supercharges are presented for the cases when $\mathbf{A}(\mathbf{x})$ satisfies all possible combinations of relations (4.6), (4.7).

First we consider systems with $N = 2$ SUSY. They correspond to the following parity properties of the electromagnetic field:

$$\mathbf{A}(r_a\mathbf{x}) = r_a\mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r_{bc}\mathbf{x}) = r_{bc}\mathbf{A}(\mathbf{x}), \tag{A.1}$$

$$\mathbf{A}(r_a\mathbf{x}) = -r_a\mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r_{bc}\mathbf{x}) = r_{bc}\mathbf{A}(\mathbf{x}), \tag{A.2}$$

and

$$\mathbf{A}(r_a\mathbf{x}) = -r_a\mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r_{bc}\mathbf{x}) = -r_{bc}\mathbf{A}(\mathbf{x}), \tag{A.3}$$

where $b, c \neq a$ and a is fixed.

The related supercharges have the form

$$Q_1 = \boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \quad Q_2 = iR_a\boldsymbol{\sigma}\cdot\boldsymbol{\pi} \tag{A.4}$$

for parities (A.1), and

$$Q_1 = \boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \quad Q_2 = CR_a\boldsymbol{\sigma}\cdot\boldsymbol{\pi} \tag{A.5}$$

for the cases when $\mathbf{A}(\mathbf{x})$ satisfies (A.2) or (A.3).

In all these cases the corresponding $N = 2$ SUSY is reducible. The involution $I_1 = R_{bc}$, commutes with supercharges (A.4) and (A.5) provided relations (A.1) or (A.2) are satisfied, while for the parities (2.3) an involution commuting with supercharges (A.5) have the form $I_2 = CR_{bc}$. Particular cases of these involutions are expressed in the formulae (4.18)-(4.21).

The operators diagonalizing I_1 are

$$\begin{array}{c} U = \frac{1}{2}\left(1 - i\sigma_2\right)\left(1 + i\sigma_2\theta_{12}\right), \quad for\ a = 3; \\ U = \frac{1}{\sqrt{2}}\left(1 + \sigma_3 I_1\right) for\ a \neq 3, \end{array} \tag{A.6}$$

whereas the expressions for the operators diagonalizing I_2 are given by

$$\begin{array}{c} U = U_1 = \frac{1}{2}\left(1 - i\sigma_2\right)\left(1 + i\sigma_2\theta_{23}\right) \quad for\ a = 1; \\ U = U_2 = \frac{1}{2}\left(1 + C\right)\left(1 - i\sigma_1\theta_{31}\right) \quad for\ a = 2, \\ U = U_3 = \frac{1}{\sqrt{2}}\left(1 + \sigma_3 I_3\right) \quad for\ a = 3 \end{array} \tag{A.7}$$

Now we shall present systems with $N = 3$ SUSY. In addition to (4.11) we have the following nonequivalent combinations of parity properties

$$\begin{array}{c}\mathbf{A}(r_{12}\mathbf{x}) = r_{12}\mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r_{23}\mathbf{x}) = r_{23}\mathbf{A}(\mathbf{x}),\\ Q_1 = R_{23}\boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \quad Q_2 = R_{31}\boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \quad Q_3 = R_{12}\boldsymbol{\sigma}\cdot\boldsymbol{\pi};\end{array} \qquad (A.8)$$

$$\begin{array}{c}\mathbf{A}(r_a\mathbf{x}) = r_a\mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r_{bc}\mathbf{x}) = -r_{bc}\mathbf{A}(\mathbf{x}),\\ Q_1 = i\boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \quad Q_2 = iR_a\boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \quad Q_3 = CR_{bc}\boldsymbol{\sigma}\cdot\boldsymbol{\pi};\end{array} \qquad (A.9)$$

$$\begin{array}{c}\mathbf{A}(r_a\mathbf{x}) = -r_a\mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r_b\mathbf{x}) = -r_b\mathbf{A}(\mathbf{x}),\\ Q_0 = \boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \quad Q_1 = CR_a\boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \quad Q_2 = CR_b\boldsymbol{\sigma}\cdot\boldsymbol{\pi};\end{array} \qquad (A.10)$$

$$\begin{array}{c}\mathbf{A}(r_a\mathbf{x}) = r_a\mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r_b\mathbf{x}) = -r_b\mathbf{A}(\mathbf{x}),\\ Q_0 = CR_b\boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \quad Q_1 = \boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \quad Q_2 = iR_a\boldsymbol{\sigma}\cdot\boldsymbol{\pi};\end{array} \qquad (A.11)$$

$$\begin{array}{c}\mathbf{A}(r_{ab}\mathbf{x}) = r_{ab}\mathbf{A}(\mathbf{x}), \quad \mathbf{A}(r_{bc}\mathbf{x}) = -r_{bc}\mathbf{A}(\mathbf{x}),\\ Q_0 = iR_{ab}\boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \quad Q_1 = CR_{bc}\boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \quad Q_2 = CR_{ac}\boldsymbol{\sigma}\cdot\boldsymbol{\pi}.\end{array} \qquad (A.12)$$

The supercharges in (A.8) satisfy relations (2.3) for $a, b = 1, 2, 3$; supercharges (A.9), (A.10) and (A.11), (A.12) satisfy relations (3.13) for $g_{11} = -g_{22} = -g_{33} = 1$ and $-g_{11} = g_{22} = g_{33} = 1$ respectively.

The supercharges in (A.10) commute with the involution R_{ab} and thus generate $N = 3$ reducible SUSY. The supercharges Q_1 and Q_2 in (A.12) also commute with this involution and generate the $N = 2$ reducible SUSY. The remaining supercharges, i.e., those in (A.8), (A.9) and (A.11) are irreducible.

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