TENSOR-BISPINOR EQUATIONS FOR DOUBLETS

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We consider tensor-bispinor equation which describes doublets of particles with arbitrary half-integer spins and nonzero masses interacting with external electromagnetic field. We use this equation to describe charged particles interacting with constant magnetic and electric and magnetic fields.

1 Introduction

Theory of higher spin particles is an important subject of modern theoretical physics. Theoretically, it is an essential part of modern theories of unification of fundamental interactions [1]. Experimentally, a number of baryonic resonances with spin $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$,..., $\frac{13}{2}$ were indicated, and so particles with higher spins are real physical objects which need a theoretical description.

The problem of deduction of relativistic wave equation for particles with arbitrary spins started with the Dirac paper [3]. It attracted attention of great many of investigators including such outstanding physicists and mathematician as Pauli, Bargman, Wightman, Wigner and others [4]-[9].

However, till now we do not have a satisfactory theory of a single relativistic particle of spin $s > 1$, in as much as all known relativistic wave equations for particles of higher spin lead to serious difficulties by description of interaction of charged particle with external fields. Here we mention the causality violation discovered by Velo and Zwanziger [10] for the Rarita-Schwinger equation, incompatibility of the experimental value of gyromagnetic ratio $g = 2$ with theoretical $g = \frac{1}{s}$ and others [11]-[14].

In papers [15]-[16] the tensor-bispinor equations for particles of arbitrary half-integer spin were proposed, which do not lead to the above mentioned difficulties. In absence of interaction these equations are equivalent to ones described in [17]-[20]. To avoid the difficulty with incorrect value of the gyromagnetic ratio a special interaction with the external electromagnetic field was introduced, which, in contrast with the usual minimal interaction, leads to hermitian quasi-relativistic Hamiltonian for a particle of spin $s > \frac{1}{2}$.

In the present paper we consider tensor-bispinor equation with quadratic anomalous interaction. In Section 2 we formulate a free tensor-bispinor equation for a particle with arbitrary half-integer spin. Minimal and anomalous interaction is considered in Section 3. In Section 4 we introduce the generalized anomalous interaction quadratic in strengths of the electromagnetic field. The problem of interaction of a charged particle of spin $s = \frac{3}{2}$

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with a constant and homogeneous magnetic field is considered in Section 5. In Section 6 we describe motion of a particle with spin $\frac{3}{2}$ in electric and magnetic fields.

2 Free equation for doublets

In this section we present a model of a particle with an arbitrary half-integer spin described in terms of irreducible antisymmetric tensor-bispinor $\Psi_{\gamma}^{[\mu_1,\nu_1][\mu_2,\nu_2]\dots[\mu_n,\nu_n]}$ of rank $2n$ ($n = s - \frac{1}{2}$). Here the indexes μ, ν take the values $0,1,2,3$ and $\gamma = 0,1,2,3$ is a bispinor index. More precisely we find a casual equation of motion for two particles with arbitrary half-integer spin and opposite parity [15]-[16]. The tensor-bispinor is symmetric w.r.t permutations of pairs of indices $[\mu_k,\nu_k] \longleftrightarrow [\mu_l,\nu_l]$, antisymmetric w.r.t. permutations of indices $\mu_k \longleftrightarrow \nu_k$ and has zero convolutions with metric tensor $g_{\mu\nu}$ and absolutely antisymmetric tensor $\varepsilon_{\mu_k\nu_k\mu_l\nu_l}$. In addition, tensor-bispinor $\Psi^{[\mu_1\nu_1][\mu_2\nu_2]\dots[\mu_n\nu_n]}$ satisfies the condition

$$\gamma_\mu\gamma_\nu\Psi^{[\mu\nu][\mu_1\nu_1]\dots[\mu_{n-1}\nu_{n-1}]} = 0 \tag{2.1}$$

and solves the Dirac equation

$$(\gamma_\lambda p^\lambda - m)\Psi^{[\mu_1,\nu_1]\dots[\mu_n,\nu_n]} = 0. \tag{2.2}$$

Here $p_\mu = i\frac{\partial}{\partial x^\mu}$ and γ_μ are the Dirac matrices acting on spinorial index γ of $\Psi_{\gamma}^{[\mu_1,\nu_1][\mu_2,\nu_2]\dots[\mu_n,\nu_n]}$ which we omit . A consequence of (2.1) and (2.2) is the following constraint

$$p_\mu\gamma_\nu\Psi^{[\mu\nu][\mu_1\nu_1]\dots[\mu_{n-1}\nu_{n-1}]} = 0. \tag{2.3}$$

Below we consider a particular case of (2.1)-(2.3) for spin $s = \frac{3}{2}$ (for the general case see [15]-[16]). For $s = \frac{3}{2}$ equations (2.1)-(2.3) can be written as a single equation

$$\begin{aligned}
&(\gamma_\lambda p^\lambda - m)\Psi^{[\mu\nu]} + \frac{1}{12}(p^\mu\gamma^\nu - p^\nu\gamma^\mu)(\gamma_\rho\gamma_\sigma - \gamma_\sigma\gamma_\rho)\Psi^{[\rho\sigma]} - \\
&-\frac{1}{12}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(p_\rho\gamma_\sigma - p_\sigma\gamma_\rho)\Psi^{[\rho\sigma]} + \\
&+\frac{1}{24}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\gamma_\lambda p^\lambda(\gamma_\rho\gamma_\sigma - \gamma_\sigma\gamma_\rho)\Psi^{[\rho\sigma]} = 0.
\end{aligned} \tag{2.4}$$

The Lagrangian corresponding to equation (2.4) has the form

$$\Im = \bar{\Psi}_{[\mu\nu]}L^{[\mu\nu][\rho\sigma]}\Psi_{[\rho\sigma]} \tag{2.5}$$

where

$$\begin{aligned}
&L^{[\mu\nu][\rho\sigma]} = (\gamma_\lambda p^\lambda - m)(g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma}) + \frac{1}{12}(p^\mu\gamma^\nu - p^\nu\gamma^\mu)(\gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho) - \\
&-\frac{1}{12}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(p^\rho\gamma^\sigma - p^\sigma\gamma^\rho) + \frac{1}{24}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\gamma_\lambda p^\lambda(\gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho)
\end{aligned} \tag{2.6}$$

and $\bar{\Psi}^{[\mu\nu]} = (\Psi^{[\mu\nu]})^\dagger \gamma_0$.

The related propagator reads

$$\begin{aligned}
&G^{[\mu\nu][\rho\sigma]}(p) = \frac{(\gamma_\lambda p^\lambda + m)}{p^2 - m^2}[\frac{1}{2}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) - \\
&-\frac{1}{12m}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(p^\rho\gamma^\sigma - p^\sigma\gamma^\rho) + \\
&+\frac{1}{12m}(p^\mu\gamma^\nu - p^\nu\gamma^\mu)(\gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho) + \\
&+\frac{1}{24m}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\gamma_\lambda p^\lambda(\gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho)].
\end{aligned} \tag{2.7}$$

3 Anomalous Interaction Linear in Electromagnetic Field

The minimal interaction can be introduced into equation (2.4) via the substitution

$$p_\mu \longrightarrow \pi_\mu = p_\mu - eA_\mu \tag{3.1}$$

where A_μ is the vector-potential of electromagnetic field. As a result equation (2.4) takes the following form

$$\begin{aligned}
&(\gamma_\lambda\pi^\lambda - m)\Psi^{[\mu\nu]} + \frac{1}{12}(\pi^\mu\gamma^\nu - \pi^\nu\gamma^\mu)(\gamma_\rho\gamma_\sigma - \gamma_\sigma\gamma_\rho)\Psi^{[\sigma\rho]} - \\
&-\frac{1}{12}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(\pi_\rho\gamma_\sigma - \pi_\sigma\gamma_\rho)\Psi^{[\rho\sigma]} + \\
&+\frac{1}{24}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\gamma_\lambda\pi^\lambda(\gamma_\rho\gamma_\sigma - \gamma_\sigma\gamma_\rho)\Psi^{[\rho\sigma]} = 0.
\end{aligned} \tag{3.2}$$

Following Pauli [4] we can generalize (3.2) introducing anomalous interaction

$$\begin{aligned}
&(\gamma_\lambda\pi^\lambda - m)\Psi^{[\mu\nu]} + \frac{1}{12}(\pi^\mu\gamma^\nu - \pi^\nu\gamma^\mu)[\gamma_\rho, \gamma_\sigma]\Psi^{[\rho\sigma]} - \\
&-\frac{1}{12}[\gamma_\mu, \gamma_\nu](\pi_\rho\gamma_\sigma - \pi_\sigma\gamma_\rho)\Psi^{[\rho\sigma]} + \\
&+\frac{1}{24}[\gamma_\mu, \gamma_\nu]\gamma_\lambda\pi^\lambda[\gamma_\rho, \gamma_\sigma]\Psi^{[\rho\sigma]} + T^{\mu\nu}_{\rho\sigma}\Psi^{[\rho\sigma]} = 0
\end{aligned} \tag{3.3}$$

where $T^{[\mu\nu]}_{[\rho\sigma]}$ depends on strengths of electromagnetic field.

Contracting (3.3) with $\frac{i}{4}[\gamma_\mu, \gamma_\nu]$ and $(\pi_\mu\gamma_\nu - \pi_\nu\gamma_\mu)$ we come to the following constraints

$$\gamma_\mu\gamma_\nu\Psi^{[\mu\nu]} = \frac{1}{m}\gamma_\mu\gamma_\nu T^{\mu\nu}_{\rho\sigma}\Psi^{[\rho\sigma]}, \tag{3.4a}$$

$$\begin{aligned}
&\pi_\mu\gamma_\nu\Psi^{[\mu\nu]} = \frac{ie}{m}(F_{\mu\nu} - \gamma^\lambda\gamma_\nu F^{\mu\lambda})\Psi^{[\mu\nu]} + \\
&+\frac{1}{48m}[\gamma_\rho, \gamma_\sigma]F^{\rho\sigma}[\gamma_\mu, \gamma_\nu]T^{\mu\nu}_{\kappa\lambda}\Psi^{[\kappa\lambda]} - \\
&-\frac{1}{4m^2}\pi_\lambda\pi^\lambda[\gamma_\mu, \gamma_\nu]T^{\mu\nu}_{\rho\sigma}\Psi^{[\rho\sigma]} + \frac{1}{m^2}\pi_\mu\gamma_\nu T^{\mu\nu}_{\rho\sigma}\Psi^{[\rho\sigma]}
\end{aligned} \tag{3.4b}$$

where $F^{\mu\nu} = i(p^\mu A^\nu - p^\nu A^\mu)$

Substituting (3.4a) and (3.4b) into (3.3) we come to the equations which in general include second order derivative terms. It is not difficult to show [15]-[16] that this equation is causal and remains first order equation provided $T^{\mu\nu}_{\rho\sigma}$ satisfies the conditions

$$\gamma_\mu \gamma_\nu T^{\mu\nu}_{\rho\sigma} = 0, \tag{3.5a}$$

$$\pi_\mu \gamma_\nu T^{\mu\nu}_{\rho\sigma} = 0. \tag{3.5b}$$

Both conditions (3.5a) and (3.5b) are satisfied if we impose the following restriction

$$\gamma_\mu T^{\mu\nu}_{\rho\sigma} = 0. \tag{3.6}$$

Starting with tensor $F^{\mu\nu}$, $\epsilon^{\mu\nu\rho\sigma}$, $g_{\mu\nu}$ and γ_μ one can construct the basis of antisymmetric tensor-bispinors linear in $F^{\mu\nu}$:

$$\begin{aligned}
&T^{\mu\nu}_{1\ \rho\sigma} = F^\nu_\rho \gamma^\mu \gamma_\sigma - F^\mu_\rho \gamma^\nu \gamma_\sigma - F^\nu_\sigma \gamma^\mu \gamma_\rho + F^\mu_\sigma \gamma^\nu \gamma_\rho,\\
&T^{\mu\nu}_{2\ \rho\sigma} = F^\mu_\rho \delta^\nu_\sigma - F^\nu_\rho \delta^\mu_\sigma - F^\mu_\sigma \delta^\nu_\rho + F^\nu_\sigma \delta^\mu_\rho,\\
&T^{\mu\nu}_{3\ \rho\sigma} = \gamma^\nu \gamma^\lambda F_{\rho\lambda} \delta^\mu_\sigma - \gamma^\mu \gamma^\lambda F_{\rho\lambda} \delta^\nu_\sigma - \gamma^\nu \gamma^\lambda F_{\sigma\lambda} \delta^\mu_\rho + \gamma^\mu \gamma^\lambda F_{\sigma\lambda} \delta^\nu_\rho + \gamma_\rho \gamma^\lambda F^\nu_\lambda \delta^\mu_\sigma -\\
&-\gamma_\rho \gamma^\lambda F^\mu_\lambda \delta^\nu_\sigma + \gamma_\sigma \gamma^\lambda F^\mu_\lambda \delta^\nu_\rho - \gamma_\sigma \gamma^\lambda F^\nu_\lambda \delta^\mu_\rho,\\
&T^{\mu\nu}_{4\ \rho\sigma} = (\delta^\mu_\sigma \delta^\nu_\rho - \delta^\nu_\sigma \delta^\mu_\rho) \gamma_\alpha \gamma_\beta F^{\alpha\beta},\\
&T^{\mu\nu}_{5\ \rho\sigma} = \gamma_4 (\tilde{F}^\mu_\rho \delta^\nu_\sigma - \tilde{F}^\nu_\rho \delta^\mu_\sigma - \tilde{F}^\mu_\sigma \delta^\nu_\rho + \tilde{F}^\nu_\sigma \delta^\mu_\rho),\\
&T^{\mu\nu}_{6\ \rho\sigma} = \gamma_4 (F^\mu_\alpha \epsilon^{\alpha\nu}_{\ \ \rho\sigma} - F^\nu_\alpha \epsilon^{\alpha\mu}_{\ \ \rho\sigma} + F_{\alpha\rho} \epsilon^{\alpha\mu\nu}_{\ \ \ \ \sigma} - F_{\alpha\sigma} \epsilon^{\alpha\mu\nu}_{\ \ \ \ \rho}),\\
&T^{\mu\nu}_{7\ \rho\sigma} = (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) F_{\rho\sigma} + F^{\mu\nu} (\gamma_\rho \gamma_\sigma - \gamma_\sigma \gamma_\rho),\\
&T^{\mu\nu}_{8\ \rho\sigma} = \gamma^\nu \gamma^\lambda F_{\rho\lambda} \delta^\mu_\sigma - \gamma^\mu \gamma^\lambda F_{\rho\lambda} \delta^\nu_\sigma - \gamma^\nu \gamma^\lambda F_{\sigma\lambda} \delta^\mu_\rho + \gamma^\mu \gamma^\lambda F_{\sigma\lambda} \delta^\nu_\rho - \gamma_\rho \gamma^\lambda F^\nu_\lambda \delta^\mu_\sigma +\\
&+\gamma_\rho \gamma^\lambda F^\mu_\lambda \delta^\nu_\sigma - \gamma_\sigma \gamma^\lambda F^\mu_\lambda \delta^\nu_\rho + \gamma_\sigma \gamma^\lambda F^\nu_\lambda \delta^\mu_\rho,\\
&T^{\mu\nu}_{9\ \rho\sigma} = \gamma_4 (F^\mu_\alpha \epsilon^{\alpha\nu}_{\ \ \rho\sigma} - F^\nu_\alpha \epsilon^{\alpha\mu}_{\ \ \rho\sigma} - F_{\alpha\rho} \epsilon^{\alpha\mu\nu}_{\ \ \ \ \sigma} + F_{\alpha\sigma} \epsilon^{\alpha\mu\nu}_{\ \ \ \ \rho}),\\
&T^{\mu\nu}_{10\ \rho\sigma} = (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) F_{\rho\sigma} - F^{\mu\nu} (\gamma_\rho \gamma_\sigma - \gamma_\sigma \gamma_\rho),
\end{aligned} \tag{3.7}$$

here $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu}_{\rho\sigma} F^{\rho\sigma}$. Then the general form of $T^{\mu\nu}_{\rho\sigma}$ is

$$T^{\mu\nu}_{\rho\sigma} = \sum_{i=1}^{10} \alpha_i T^{\mu\nu}_{i\ \rho\sigma} \tag{3.8}$$

where α_i are arbitrary constants.

Using (3.6) and taking into account that the Lagrangian corresponding to equation (3.3) must be real and so $T^{\mu\nu}_{\rho\sigma} = T^{\dagger\mu\nu}_{\rho\sigma}$, we come to the conditions

$$\begin{cases} \alpha_1 = \alpha_6 = -\alpha_9 = \frac{\lambda}{2} \\ \alpha_2 = 2\lambda \\ \alpha_3 = \alpha_8 = \frac{\lambda}{4} \\ \alpha_4 = \alpha_5 = \alpha_7 = \alpha_{10} = 0 \end{cases} \tag{3.9}$$

Substituting (3.6)-(3.9) into (3.3) we write equation (3.3) as

$$\begin{aligned} &(\gamma_\lambda \pi^\lambda - m)\Psi^{[\mu\nu]} + \frac{ie}{m}((1+\lambda)(F^\nu_\rho \gamma^\mu \gamma_\sigma - F^\mu_\rho \gamma^\nu \gamma_\sigma)\Psi^{[\rho\sigma]} + \\ &+(1+\lambda)(\gamma^\nu \gamma^\lambda F_{\rho\lambda}\Psi^{[\rho\mu]} - \gamma^\mu \gamma^\lambda F_{\rho\lambda}\Psi^{[\rho\nu]}) + (1+4\lambda)(F^\mu_\rho \Psi^{[\rho\nu]} - F^\nu_\rho \Psi^{[\rho\mu]}) + \\ &+(1+2\lambda)\gamma_4 \epsilon^{\mu\nu\lambda}_\sigma F_{\rho\lambda}\Psi^{[\rho\sigma]}) = 0. \end{aligned} \tag{3.10}$$

Equation (3.10) can be reduced to the Dirac form [15]-[16], [21]. It can be shown using the substitution

$$\begin{aligned} \Psi^{ab} &= \frac{1}{2}\epsilon_{abc}(\Phi^{(1)}_c + \Phi^{(2)}_c) \\ \Psi^{0c} &= \frac{i}{2}(\Phi^{(2)}_c - \Phi^{(1)}_c), \quad a,b,c = 1,2,3, \end{aligned} \tag{3.11}$$

where $\Phi^{(1)}_c$ and $\Phi^{(2)}_c$ are bispinors. Using (3.11) equation (3.10) can be written in the form:

$$(\Gamma_\mu \pi^\mu - m + \frac{e}{4m}(1 - i\Gamma_4)(\frac{i}{4}(g-2)[\Gamma_\mu, \Gamma_\nu] + g\tau_{\mu\nu})F^{\mu\nu})\Psi^{(1)} = 0, \tag{3.12a}$$

$$(\Gamma_\mu \pi^\mu - m + \frac{e}{4m}(1 + i\Gamma_4)(\frac{i}{4}(g-2)[\Gamma_\mu, \Gamma_\nu] + g\tau_{\mu\nu})F^{\mu\nu})\Psi^{(2)} = 0 \tag{3.12b}$$

where $g = \frac{2}{3}(1 - \frac{\lambda}{2})$, $\Psi^{(1)} = (\Phi^{(1)}_1, \Phi^{(1)}_2, \Phi^{(1)}_3)^T$, $\Psi^{(2)} = (\Phi^{(2)}_1, \Phi^{(2)}_2, \Phi^{(2)}_3)^T$, $S_{\mu\nu} = \frac{i}{4}[\Gamma_\mu, \Gamma_\nu] + \tau_{\mu\nu}$ and $\tau_{\mu\nu}$ satisfy the relations $\tau_{ab} = \epsilon_{abc}\tau_c$; $\tau_{0a} = i\tau_a$, $\tau_a\tau_a = \tau(\tau+1)$; $[\tau_a, \tau_b] = i\epsilon_{abc}\tau_c$, $a,b,c = 1,2,3$. Matrices Γ_μ and τ_a can be represented as

$$\Gamma_\mu = \gamma_\mu \otimes I_3, \quad \tau_a = I_4 \otimes \hat{\tau}_a, \tag{3.13}$$

symbol $\otimes$ denotes the direct product of matrices, $\hat{\tau}_a$ are 3×3 matrices, realizing the representation $D(3)$ of the algebra $AO(3)$, I_3 and I_4 are the unit 3×3 and 4×4 matrices correspondingly.

Constraints (3.4) in the representation (3.11) are reduced to the forms

$$[(\Gamma_\mu \pi^\mu + m)(1 + i\Gamma_4)(S_{\mu\nu}S^{\mu\nu} - 3)]\Psi^{(1)} = 24m\Psi^{(1)}, \tag{3.14a}$$

$$[(\Gamma_\mu \pi^\mu + m)(1 - i\Gamma_4)(S_{\mu\nu}S^{\mu\nu} - 3)]\Psi^{(2)} = 24m\Psi^{(2)}. \qquad (3.14b)$$

Thus introducing the anomalous interaction we obtain an additional freedom in definition of the value of constant g. For example, we can choose such value of g which corresponds to the simplest from of equations (3.12). Thus value is nothing but $g = 2$! We see that it is the generally accepted value of g which corresponds to the most compact version of the motion equations (3.12).

Finally we notice that Hamiltonian corresponding to equation (3.12a) (or (3.12b)) can be diagonalized using Fouldy-Wouthuysen transformation [21] and reduced to the following form in the non-relativistic approximation [3] makes the small parameter $\frac{1}{c}$ "invisible"):

$$\begin{aligned} H = m + \frac{\pi^2}{2m} - \frac{\pi^4}{8m^3} - eg\frac{\vec{S}\vec{H}}{2m} + eA_0 + \frac{e}{48m^2}Q^{ab}\frac{\partial E_a}{\partial x_b} - \\ - \frac{eg^2}{16m^2}\vec{S}(\vec{E}\times\vec{p} - \vec{p}\times\vec{E}) - \frac{5eg^2}{16m^2}\mathrm{div}\vec{E} - \frac{(g-2)e}{12m^2}(Q_{ab}\frac{\partial H_a}{\partial x_b} - \\ - \frac{1}{3}\vec{S}(\vec{H}\times\vec{p} - \vec{p}\times\vec{H})). \end{aligned} \qquad (3.15)$$

Here $Q_{ab} = 3[S_a,\ S_b]_+ - 15\delta_{ab}$, S_a are 4×4 matrices realizing irreducible representation $D(\frac{3}{2})$ of algebra $AO(3)$, $\vec{E}$ and $\vec{H}$ are vectors of the electric and magnetic field strengths.

The terms $m + \frac{\pi^2}{2m} + eA_0$ represent the Schrodinger Hamiltonian (with the rest energy m), $\frac{\pi^4}{8m^3}$ is the relativistic correction to the kinetic energy, $eg\frac{\vec{S}\vec{H}}{2m}$ corresponds to the dipole coupling with arbitrary constant of interaction g, term $\frac{eg^2}{16m^2}\vec{S}(\vec{E}\times\vec{p} - \vec{p}\times\vec{E})$ represents spin-orbit coupling, $\frac{e}{48m^2}Q^{ab}\frac{\partial E_a}{\partial x_b}$ describe quadruple coupling and $\frac{5eg^2}{16m^2}\mathrm{div}\vec{E}$ is the Darwin coupling. The last term, which is proportional to $(g-2)$ is non-Hermitian, but it vanishes when $g = 2$.

4 Anomalous interaction quadratic in electromagnetic field

In Section 3 we considered the anomalous interaction $T^{\mu\nu}_{\rho\sigma} = T^{\mu\nu}_{\rho\sigma}(F)$ linear in $F^{\mu\nu}$. Here we introduce interaction quadratic in $F^{\mu\nu}$.

The simplest way to introduce such anomalous interaction consists in generalization of equation (3.3) to the form

$$\begin{aligned} (\gamma_\lambda p^\lambda - m)\Psi^{[\mu\nu]} + \frac{1}{12}(\pi^\mu\gamma^\nu - \pi^\nu\gamma^\mu)[\gamma_\rho,\gamma_\sigma]\Psi^{[\rho\sigma]} - \\ - \frac{1}{12}[\gamma_\mu,\gamma_\nu](\pi_\rho\gamma_\sigma - \pi_\sigma\gamma_\rho)\Psi^{[\rho\sigma]} + \frac{1}{24}[\gamma_\mu,\gamma_\nu]\gamma_\lambda\pi^\lambda[\gamma_\rho,\gamma_\sigma]\Psi^{[\rho\sigma]} + \\ + T^{\mu\nu}_{\rho\sigma}\Psi^{\rho\sigma} + T^{\mu\nu}_{\rho\sigma}T^{\rho\sigma}_{\delta\epsilon}\Psi^{[\delta\epsilon]} = 0 \end{aligned} \qquad (4.1)$$

[3] More precisely we consider non-relativistic approximation up to the terms of order $\frac{1}{c^2}$, but using the units with $c = \hbar = 1$

where $T^{\mu\nu}_{\rho\sigma}(F)$ was found in Section 3 (see (3.7)-(3.9)). Using (3.11) we come to the following equations for $\Psi^{(1)}$ and $\Psi^{(2)}$ instead of (3.10)

$$(\Gamma_\mu \pi^\mu - m + \frac{e}{4m}(1 - i\Gamma_4)(gS_{\mu\nu}F^{\mu\nu} + g_1(S_{\mu\nu}F^{\mu\nu})^2 - i\Gamma_\mu\Gamma_\nu F^{\mu\nu}))\Psi^{(1)} = 0, \qquad (4.2a)$$

$$(\Gamma_\mu \pi^\mu - m + \frac{e}{4m}(1 + i\Gamma_4)(gS_{\mu\nu}F^{\mu\nu} + g_1(S_{\mu\nu}F^{\mu\nu})^2 - i\Gamma_\mu\Gamma_\nu F^{\mu\nu}))\Psi^{(2)} = 0, \qquad (4.2b)$$

Equations (4.1) and (4.2) assume the Lagrangian formulation and include two coupling constants g and g_1. The presence of the additional (in comparison with (3.12a), (3.12b)) constant g_1 makes it possible to overcome difficulties with complex energy levels for the particle in constant magnetic field when $g \neq \frac{1}{s}$. .

5 Particle with spin $s = \frac{3}{2}$ in constant magnetic field

In this section we consider the problem of interaction of a charged, spin $\frac{3}{2}$ particle with a constant and homogeneous magnetic field. We demonstrate that for $g \neq \frac{1}{s}$ this problem leads to the known difficulties with complex energies and demonstrate that it is possible to get over this difficulty using the generalized model (4.2) with a bilinear in $F_{\mu\nu}$ anomalous interaction.

We start with equation (4.2a), which can be expressed in the form

$$(\pi_\mu\pi^\mu - m^2 + \frac{eg}{2}S_{\mu\nu}F^{\mu\nu} + \frac{eg_1}{2}(S_{\mu\nu}F^{\mu\nu})^2)\Psi^{(1)}_+ = 0, \qquad (5.1a)$$

$$(S_{\mu\nu}S^{\mu\nu} - 15)\Psi^{(1)}_+ = 0, \qquad (5.1b)$$

$$\Psi^{(1)}_- = \frac{1}{m}\Gamma_\mu\pi^\mu\Psi^{(1)}_+ \qquad (5.1c)$$

(a similar equation can be obtained starting with (4.2b)).

For the case of the constant and homogeneous magnetic field the vector-potential A_μ and the field tensor $F_{\mu\nu}$ are

$$\begin{aligned} &A_0 = A_2 = A_3 = 0,\ A_1 = Hx_2 \\ &F_{0a} = F_{23} = F_{31} = 0, \quad a = 1,2,3 \\ &F_{12} = H_3 = H, H \geq 0, \end{aligned} \qquad (5.2)$$

H is the magnetic field strength.

The solution of equation (5.1b) can be represented as [21]

$$\Psi^{(1)}_+ = \begin{pmatrix} \Phi^{(1)}_{\frac{3}{2}} \\ \hat{0} \\ \frac{1}{m}(\varepsilon + \frac{2}{3}S_a\pi_a)\Phi^{(1)}_{\frac{3}{2}} \\ -\frac{2}{3m}K^{\frac{3}{2}}_a\pi_a\Phi^{(1)}_{\frac{3}{2}} \end{pmatrix}. \qquad (5.3)$$

Here $(K_3^{\frac{3}{2}})_{mm'} = \delta_{mm'}\sqrt{\frac{9}{4}-m^2}$; $m, m' = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$, $\hat{0} = (0,0)^T$, $(K_1^{\frac{3}{2}})_{mm'} \pm i(K_2^{\frac{3}{2}})_{mm'} = \pm\delta_{m\pm m'}\sqrt{\frac{3}{2} \mp m(m\mp 1) \pm 3m}$, $\Phi^{(1)}_{\frac{3}{2}}$ is a four-component spinor which satisfy the equation

$$[p^2 + e^2H^2x_2^2 - eH(gS_3 + 2g_1S_3^2H + 2x_2p_1]\Phi^{(1)}_{\frac{3}{2}} = (\varepsilon^2 - m^2)\Phi^{(1)}_{\frac{3}{2}}. \tag{5.4}$$

So the problem of describing the motion of particle with spin $\frac{3}{2}$ in the constant magnetic field reduces to solving equation (5.4).

Using eigenvectors $\Omega_m^{\frac{3}{2}}$ of matrix S_3 we can represent $\Phi_{\frac{3}{2}}$ in the form

$$\Phi^{(1)}_{\frac{3}{2}} = \exp(i(p_1x_1 + p_3x_3)) \sum_{m=-\frac{3}{2}}^{\frac{3}{2}} f_\nu^{\frac{3}{2}}(x_2)\Omega_m^{\frac{3}{2}}, \tag{5.5}$$

where $f_m^{\frac{3}{2}}(x_2)$ are unknown functions. The functions $f_\nu^{\frac{3}{2}}$ satisfy the equation

$$\left(\frac{d^2}{dy^2} + y^2\right) f_\nu^{\frac{3}{2}}(y) = \eta f^{\frac{3}{2}}{}_\nu(y)$$

where $\eta = \frac{\varepsilon^2 - m^2 - p_3^2}{eH} + \nu(g + 2g_1\nu H)$, $x_2 = \frac{1}{eH}(p_1 + \sqrt{eH}y)$.

Requiring $f^{\frac{3}{2}}{}_\nu(y) \longrightarrow 0$ when $y \longrightarrow \pm\infty$ we have

$$\eta = 2n+1, \quad n = 0, 1, 2, 3, ...$$

Then the energy levels are

$$\varepsilon^2 = m^2 + p_3^2 + eH(2n + 1 - \nu(g + 2g_1\nu H)) \tag{5.6}$$

and eigenfunctions $f_\nu^{\frac{3}{2}}(y)$ take the form

$$f_\nu^{\frac{3}{2}}(x_2) = \exp\left(-\frac{eHx_2 - p_2}{2eH}\right) h_n\left(\frac{eHx_2 - p_2}{\sqrt{eH}}\right),$$

where $h_n(y) = \dfrac{H_n(y)}{||H_n(y)||}$, $H_n(y)$ are Hermitian polynomials.

The eigenfunctions $\Phi_\nu^{\frac{3}{2}}$ were obtained in [21].

If the gyromagnetic ratio takes the physical value $g = 2$ and $g_1 = 0$ the energy eigenvalues (5.6) can take complex values provided $\frac{eH}{2m^2} > 1$. However, if the coupling constant g_1 is non-trivial, it is possible to get over this difficulty. Namely, in order that $\varepsilon^2 \geq 0$ for all H, g and g_1 must satisfy the following condition

$$g_1 \leq -\frac{(3g-2)^2 e}{72m^2}. \tag{5.7}$$

Finally let us discuss the physical meaning of (5.6). Expanding it in power series of $\frac{1}{m}$ we have

$$
\begin{aligned}
&\mid \varepsilon \mid = \sqrt{m^2 + p_3^2 + eH(2n + 1 - \nu(g + 2g_1\nu H))} \simeq \\
&\simeq m + \frac{p_3^2}{2m} + \Omega(n + \frac{1}{2} - \frac{\nu g}{2} - \nu g_1 H).
\end{aligned}
\tag{5.8}
$$

Here $\Omega = \frac{eH}{m}$ and $\nu = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$.

We see that $\mid \varepsilon \mid$ contains the kinetical energy of motion of the particle with spin $\frac{3}{2}$ in magnetic field and quantum part which includes two parameters, i.e., n and ν. If $g = 2$ and $g_1 = 0$, the energy levels (5.8) have a specific degeneration typical for theories admitting parasupersymmetry with paraquantization parameter $p = 3$ [23]. And it is possible to show that the related equation (5.4) indeed admits this parasupersymmetry.

Supposing arbitrary parameter g_1 satisfy (5.7) we overcome the inconsistency with complex energy levels which appears by description of particles with spin $\frac{3}{2}$ and $g = 2$ in the constant magnetic field.

6 Particle with spin $s = \frac{3}{2}$ in crossed electric and magnetic field.

Extending the results present in the previous section let us use equation (5.1) with nonlinear anomalous interaction to describe a particle with spin $\frac{3}{2}$ interacting with crossed constant electric and magnetic fields.

We notice that such problem was considered in papers [24]-[26] for particles with spin 0, $\frac{1}{2}$ and in paper [27] for particles with an arbitrary spin and minimal interaction.

We can restrict ourselves to the parallel and orthogonal configurations of electric and magnetic fields (all others configurations can be obtained starting with the mentioned ones using Lorentz transformations [24]-[26]).

a) $\vec{E}||\vec{H}$. For constant and uniform electric field $\vec{E}$ and magnetic field $\vec{H}$ directed along z axis we may choose $\vec{E} = (0, 0, E)$, $\vec{H} = (0, 0, H)$, $A_\mu = (x_3 E, x_2 H, 0, 0)$ and $E \neq H$. After substituting (5.3) into the related equation (5.1) we obtain

$$
\begin{aligned}
&[(p_0 - ex_3E)^2 - (p_1 - ex_2H)^2 - p_2^2 - p_3^2 - m^2 \\
&+ egS_3(H - iE) + 2eg_1S_3^2(H - iE)^2)\Phi^{(1)}_{\frac{3}{2}} = 0
\end{aligned}
\tag{6.1}
$$

Presenting $\Phi^{(1)}_{\frac{3}{2}}$ in the form

$$
\Phi^{(1)}_{\frac{3}{2}} = \exp(ip_1x_1 - \varepsilon x_0) f(x_2) \sum_{\nu=-\frac{3}{2}}^{\frac{3}{2}} g_\nu(x_3)\Omega_\nu^{\frac{3}{2}}
\tag{6.2}
$$

where $f(x_2)$ and $g_\nu(x_3)$ are unknown functions, we can decouple (6.1) to two separate equations for $f(x_2)$ and $g_\nu(x_3)$. Solving these equations with using the results [24]-[26]

we find $\Phi^{(1)}_{\frac{3}{2}}$ in the form

$$
\begin{aligned}
&\Phi^{(1)}_{\frac{3}{2}} = \exp(ip_1x_2 - \varepsilon x_0)\exp\left(-\frac{(p_1+ex_2Hj)^2}{2eH}\right)h_n(x_2)\\
&\times\exp\left(\frac{iz^2}{2}\right)\sum_{-\frac{3}{2}}^{\frac{3}{2}} G^j_\nu(-i\delta_\nu,\ -iz^2)\Omega^{\frac{3}{2}}_\nu, \quad j=1,2,
\end{aligned}
\tag{6.3}
$$

where p_1 and ε are constants, $h_n(x_2) = \frac{H_n(x_2)}{|H_n(x_2)|}$, $H_n(x_2)$ are Hermitian polynomials, $z = \frac{1}{\sqrt{|eH|}}(\varepsilon - ex_2E)$, $\delta_\nu = \frac{m^2 - e\nu g(H-iE) - 2e\nu^2 g_1(H-iE)^2}{|eH|-(2n+1)}$, $n = 0,1,2,3,...$, $G^1_\nu(-i\delta_\nu,\ -iz^2) = F(\frac{1}{4}(1-i\delta_\nu),\ \frac{1}{2}, -iz^2)$ and $G^2_\nu(-i\delta_\nu,\ -iz^2) = F(\frac{1}{4}(1-i\delta_\nu)+\frac{1}{2},\ \frac{3}{2}, iz^2)\sqrt{iz^2}$, F is the confluent hypergeometric function. So, in this case, the energy levels are not quantized.

b) $\vec{E} \perp \vec{H}$. Setting $\vec{E} = (0,E,0)$, $\vec{H} = (0,0,H)$, $A_\mu = (x_2E,\ x_2H, 0, 0)$ we obtain the following equation for $\Phi^1_{\frac{3}{2}}$

$$
\begin{aligned}
&[(p_0 - ex_2E)^2 - (p_1 - ex_2H)^2 - p_2^2 - p_3^2 - m^2\\
&+eg(S_3H - iS_2E) + 2eg_1(S_3H - iS_2E)^2]\Phi^1_{\frac{3}{2}} = 0.
\end{aligned}
\tag{6.4}
$$

Representing $\Phi^1_{\frac{3}{2}}$ in the form

$$
\Phi^1_{\frac{3}{2}} = \exp(ip_1x_1 + ip_3x_3 - i\varepsilon x_0)\sum_{\nu=-\frac{3}{2}}^{\frac{3}{2}} P_\nu(x_2)\Omega^{\frac{3}{2}}_\nu, \tag{6.5}
$$

substituting (6.5) into (6.4) and using transformation $\hat{P}_\nu = U_{\nu,\nu'}P_{\nu'}(x_2)$ we come to the equation

$$
\begin{aligned}
&[(\varepsilon - ex_2E)^2 - (p_1 - ex_2H)^2 + \frac{d^2}{dx_2{}^2} - p_3^2 - m^2]\hat{P}_\nu(x_2) =\\
&eU_{\nu\nu'}\Lambda_{\nu'\nu''}U^{-1}_{\nu''\nu'''}\hat{P}_{\nu'''}(x_2).
\end{aligned}
\tag{6.6}
$$

where

$$
\Lambda_{\nu,\nu'} = eg(S_3H - iS_2E)_{\nu.\nu'} + 2eg_1(S_3H - iS_2E)^2_{\nu,\nu'}
$$

and $U_{\nu,\nu'}\Lambda_{\nu',\nu''}U^{-1}_{\nu'',\nu'''} = \lambda_\nu\delta_{\nu,\nu'}$.

Solution of equation (6.6) have the following form

$$
E = H: \quad \hat{P}_\nu = \Phi(\alpha - 2eH\gamma(p_1-\varepsilon)x_2), \tag{6.7}
$$

where Φ is Airy function, $\alpha = (p_3^2 + p_1^2 + m^2 - \varepsilon^2)\gamma$ and $\gamma = (4e^2h^2(p_1-\varepsilon)^2)^{-\frac{1}{3}}$. So, the energy levels are not quantized;

$$
E \neq H: \ \hat{P}_\nu(x_2) = \exp\left(\frac{iz^2}{2}\right)G^j_\nu(-i\alpha_\nu, -iz^2), \quad j=1,2. \tag{6.8}
$$

Here $\alpha_\nu = -\left(\frac{p_3^2+p_1^2+m^2-\varepsilon^2+e\lambda_\nu}{e\eta} + \frac{(p_1H-\varepsilon E)^2}{e\eta^3}\right)$, $\eta = \sqrt{E^2-H^2}$, $\lambda_\nu = ig\nu\eta - 3g_1\nu^2\eta^2$, $z = \sqrt{e\eta}(x_2 + \frac{p_1H-\varepsilon E)}{e\eta^2})$.

If $E > H$, $-iz^2$ becomes purely imaginary and energy levels, like in the case $E \parallel H$, are not quantized. When $E < H$, $-iz^2$ is purely real and energy levels are quantized:

$$\left(\varepsilon - \frac{p_1E}{H}\right)^2 = \left(\frac{\eta'}{H}\right)^2 ((2n+1)|e|\eta' + |e|\lambda_\nu + p_3^2 + m^2) \tag{6.9}$$

where $\eta' = -i\eta$. If we put $E \longrightarrow 0$ we come to formula (5.6). For $g_1 \longrightarrow 0$ relation (6.9) reduces to the form obtained in [27].

Finally we notice that the elements $U_{\nu\nu'}$ of matrix U can be found from the system of algebraic equations

$$\begin{aligned} &\left(\left(\frac{3}{2}-\nu\right)\left(\frac{3}{2}+\nu+1\right)\right)^{\frac{1}{2}} U_{\nu\nu'+1} + 2\left(\nu-\lambda_\nu\right)U_{\nu\nu'} + \\ &+\left(\left(\frac{3}{2}+\nu\right)\left(\frac{3}{2}-\nu+1\right)\right)^{\frac{1}{2}} U_{\nu\nu'-1} = 0, \end{aligned} \tag{6.10}$$

here $U_{\nu\frac{5}{2}} = U_{\nu-\frac{3}{2}} = 0$.

7. Discussion

We present equations for doublets of particles with arbitrary half-integer spin s, which describe tensor-spinor wave function and does not heave acausal solutions. Moreover the equation for particles with spin-$\frac{3}{2}$ was considered in detail.

We also show that equations with anomalous interaction and $g = 2$ generate the hermitian Hamiltonian at least in the quasi-relativistic approximation and describe spin-orbit, quadruple and Darwin couplings. These equations are easy-to-use for solving standard quantum mechanical problems. Moreover, as it was shown in [21] the equation for doublets in representation (4.2) does not become much more complicated with the growth of spin value.

Finally, for $g = 2$ and $g_1 = 0$ the inconsistency with complex energy levels arises for the problem of motion of particle with spin in the constant magnetic field. Using the nonlinear anomalous interaction with a non-trivial constant g_1 we found the condition for the parameters g and g_1 when energy level for particle with spin $s = \frac{3}{2}$ be real for any magnetic field strength.

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