“Extremal” Laurent polynomials in 2 dimensions

March 2012

Mathematical background

For a Laurent polynomial

$$P(x_1, \ldots, x_n) \in \mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$$

consider the sequence of the coefficients of degree 0 in the powers of P, i.e.

$$a_m \;=\; \mathrm{coeff}_{\mathbf{1}} P(x_1, \ldots, x_n)^m\,, \quad m = 0, 1, 2, \ldots \tag{1}$$

This sequence is known to satisfy a recursive relation of the form

$$\sum_{k=0}^{r-1} R_k(n-k) a_{n-k} \;=\; 0\,, \tag{2}$$

where all R_k are certain polynomials ([1, 2]). One can give estimates on the length r of this recursion and also on the degrees of the polynomials R_k based on the *Newton polytope* $\mathrm{Newt}(P)$. The Newton polytope $\mathrm{Newt}(P)$ is the convex hull of the set which consists of all the exponent vectors appearing in a collection of monomials of P. In particular, it is a lattice polytope, i.e. all its vertices are vectors with integer coordinates.

Let us fix a lattice polytope Δ and consider the family of all Laurent polynomials P with $\mathrm{Newt}(P) = \Delta$. Based on example claculations, it is expected that there is a number $r = r(\Delta)$ such that for a generic polynomial P in this family the length of the recursion (2) equals r, and for some polynomials it gets smaller. We call these latter polynomials *extremal*.

A lattice polytope is called *reflexive* if its only internal integer point is the origin $(0, \ldots, 0)$. If we consider reflexive polytopes modulo unimodular transformations, there is exactly one reflexive polytope in 1 dimension, 16 of them in 2 dimensions, 4319 reflexive polytopes in 3 dimensions and more than 473 million in 4 dimensions ([3, 4]). It is expected that modulo homotheties

$$P(x_1, \ldots, x_n) \mapsto P(\alpha_1 x_1, \ldots, \alpha_n x_n)$$

(homothetic polynomials produce the same sequence a_m) and scalings

$$P \mapsto \lambda P$$

(the sequence for λP will be $\lambda^m a_m$) there are finitely many extremal Laurent polynomials for a given reflexive polytope.

Project statement

Our ultimate goals are

(i) find all reflexive polygons in 2 dimensions

(ii) for each reflexive polygon write the formula for generic recursion and find extremal polynomials; check if there are indeed finitely many of them modulo scalings and homotheties

(iii) create tables of results

One sample calculation

Let $n = 2$ and Δ be a triangle with vertices $(1, 0)$, $(0, 1)$ and $(-1, -1)$. The generic polynomial with this Newton polygon is

$$P(x, y) \;=\; \alpha_1 x + \alpha_2 y + \frac{\alpha_3}{xy} + \beta\,, \quad \alpha_i \neq 0\,. \tag{3}$$

This polynomial is homothetic to

$$P(x, y) \;=\; x + y + \frac{\alpha}{xy} + \beta\,, \quad \alpha \;=\; \alpha_1\alpha_2\alpha_3 \neq 0\,.$$

Then the sequence (1) is

$$a_m \;=\; \sum_{s=0}^{[\frac{m}{3}]} \frac{(m-s)!}{(m-3s)!(s!)^2}\alpha^s\beta^{m-3s} \;=\; \sum_{s=0}^{[\frac{m}{3}]} \binom{m-s}{m-3s, s, s}\alpha^s\beta^{m-3s}$$

and for the generating series we then have

$$\sum_{m=0}^{\infty} a_m t^m \;=\; \sum_{k,s=0}^{\infty} \binom{k+2s}{k, s, s}\alpha^s\beta^k t^{k+3s}\,.$$

With the help of computer we find the recursion

$$\begin{aligned} n^3 a_n - \beta(1 + 3(n-1) + 3(n-1)^2)a_{n-1} + 3\beta^2(n-1)^2 a_{n-2} \\ - (27\alpha + \beta^3)(2 + 3(n-3) + (n-3)^2)a_{n-3} \;=\; 0\,. \end{aligned}$$

(This formula can be proved using properties of trinomial coefficients.) Therefore extremal Laurent polynomials are those with

$$27\alpha + \beta^3 \;=\; 0\,.$$

We can take a homothetic polynomial with $\alpha_1 = \alpha_2 = \alpha_3 = -\frac{\beta}{3}$

$$P(x, y) \;=\; -\frac{\beta}{3}\Big(x + y + \frac{1}{xy} - 3\Big)\,.$$

Finally, $r(\Delta) = 4$ and we see that up to scalings and homotheties there is a unique extremal Laurent polynomial for the polygon Δ

$$P(x, y) \;=\; x + y + \frac{1}{xy} - 3$$

with the corresponding sequence of 0 degree coefficients being

$$1\,,\ -3\,,\ 9\,,\ -21\,,\ 9\,,\ 297\,,\ -2421\ldots$$

Comments

The idea behind this little research program is closely related to the project "Fano Varieties and Extremal Laurent Polynomials" done by Tom Coates, Alessio Corti, Sergey Galkin, Vasily Golyshev and Al Kasprzyk. The details can be found in their research blog [5]. The actual definition of an extremal Laurent polynomial due to Golyshev ([6]) is that the local system on $\mathbb{P}^1 \setminus S$ (where $S \subset \mathbb{P}^1$ is a finite set) corresponding to the above recursion is extremal, that is nontrivial, irreducible, and of smallest possible ramification. The simplified definition we take here comes from a personal communication with Vasily Golyshev and is closely related to the actual one. In particular, results of this project have to be compared with the list of extremal Laurent polynomials in dimension 2 given by Sergey Galkin in [7].

References

[1] J. Stienstra, F. Beukers, On the Picard-Fuchs equation and the formal Brauer group of certain elliptic K3- surfaces, Mathematische Annalen 271 (1985) pp.269–304.

[2] I. M. Gelfand, M. M. Kapranov, A. V. Zelevinsky, Discriminants, Resultants and Multidimensional Determinants, Birkhauser, Boston (1994)

[3] M. Kreuzer, H. Skarke, Classification of reflexive polyhedra in three dimensions, Adv. Theor. Math. Phys. 2 (1998), pp. 853–871.

[4] M. Kreuzer, H. Skarke, Complete classification of reflexive polyhedra in four dimensions, Adv. Theor. Math. Phys. 4 (2000), pp. 1209–1230.

[5] T. Coates, A. Corti, S. Galkin, V. Golyshev and A. Kasprzyk, Fano Varieties and Extremal Laurent Polynomials (http://coates.ma.ic.ac.uk/fanosearch/)

[6] V. Golyshev, Spectra and Strain, arXiv:0801.0432 (hep-th)

[7] S. Galkin, Del Pezzo surfaces and low ramified pencils of elliptic curves, preprint MIAN (2006) (in Russian)