Extremal Laurent Polynomials in Two Dimensions

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Abstract

This is a report on the summer internship project we did under the guidance of Masha Vlasenko. In two dimensions there are 16 distinct reflexive lattice polygons modulo unimodular tansformations. We calculate corresponding extremal Laurent polynomilas for each one of these polygons and compare sequences of zero degree terms of their powers to those found in the paper "Integral Solutions of Apéry-Like Recurrence Equations" by Don Zagier.

1 preamble

For a Laurent polynomial

$$P(x_1, \ldots, x_n) \in \mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$$

consider the sequence of the coefficients of degree 0 in the powers of P, i.e.

$$a_m \;=\; \mathrm{coeff}_{\mathbf{1}} P(x_1, \ldots, x_n)^m\,, \quad m = 0, 1, 2, \ldots$$

This sequence is known to satisfy a recursive relation of the form

$$\sum_{k=0}^{r-1} R_k(n-k) a_{n-k} \;=\; 0\,,$$

where all R_k are certain polynomials. One can give estimates on the length r of this recursion and also of the degrees of the polynomials R_k based on the *Newton polytope* Newt(P). The Newton polytope Newt(P) is the convex hull of the set which consists of all the exponent vectors appearing in a collection of monomials of P. In particular, it is a lattice polytope, i.e. all its vertices are vectors with integer coordinates.

Let us fix a lattice polytope Δ and consider the family of all Laurent polynomials P with Newt$(P) = \Delta$. Based on example claculations, it is expected that there is a number $r = r(\Delta)$ such that for a generic polynomial P in this family the length of the recursion equals r, and for some polynomials it's less than r. We call these latter polynomials *extremal*.

A lattice polytope is called *reflexive* if its only internal integer point is the origin $(0, \ldots, 0)$. If we consider reflexive polytopes modulo unimodular transformations, there is exactly one reflexive polytope in 1 dimension, 16 of them in 2 dimensions, 4319 reflexive polytopes in 3 dimensions and more than 473 million in 4 dimensions . It is expected that modulo homotheties

$$P(x_1, \ldots, x_n) \mapsto P(\alpha_1 x_1, \ldots, \alpha_n x_n)$$

(homothetic polynomials produce the same sequence a_m) and scalings

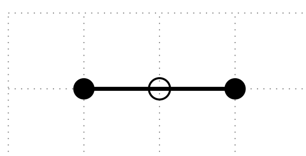
$$P \mapsto \lambda P$$

(the sequence for λP will be $\lambda^m a_m$) there are finitely many extremal Laurent polynomials for a given reflexive polytope.

1.1 overview

The purpose of this project was to uncover extremal polynomials for the 16 newton polytopes in dimension 2, and to make a list readily avaiable. Also, to investigate whether or not these extremals are unique, and justify them. In section 2 one can find a diagram of the 16 different newton polytopes in dimension 2. As can be seen, the number of lattice points increases from 4 to 10. An increase in variables upped the challenge of finding extremals and as a result different approaches were required. In the end we could subdivide the 16 into 3 groups, whereby each group had a different approach taken to locate the polynomial.

1.2 illustration



Although working with dimension 2, we will use the 1 dimensional case for means of an example, partly due to its simplicity. There is only one trivial newton polytope in dimension 1, that of the “line” , represented by the Laurent Polynomial

$$\alpha_1 x + \alpha_2 + \frac{\alpha_3}{x}$$

This can be rewritten as

$$x + \alpha + \frac{1}{x}$$

using the substitution

$$x = x\sqrt{\frac{\alpha_3}{\alpha_1}}$$

and then dividing across to give $\alpha = \alpha_2\sqrt{\alpha_3\alpha_1}$
While working with dimension 2, one deals with mutlivariates and can reduce three coefficients to 1 by subbing in suitably for x and y, and in general working with polynomials of n variables, one can reduce n coefficients to $n-3$.

We create a function $f(n)$ which gives the zero coefficient term, of the polynomial, **P**, when raised the the n^{th} power. To calculate the recursive formula for this sequence, one must figure out the degree of n in it's largest polynomiall coefficient and also its length(see conjectures). Assume for now that this is worked out in tour de force fashion. Create a matrix, of length $(deg * len)$ by $(deg * len)$ whereby the intial row would be of the form

$$f(0), 1 * f(0)....1 * f(0), 1 * f(1), 2 * f(1), .., 2^{deg-1} * f(1), f(2)...$$

where the last term in the first row will be

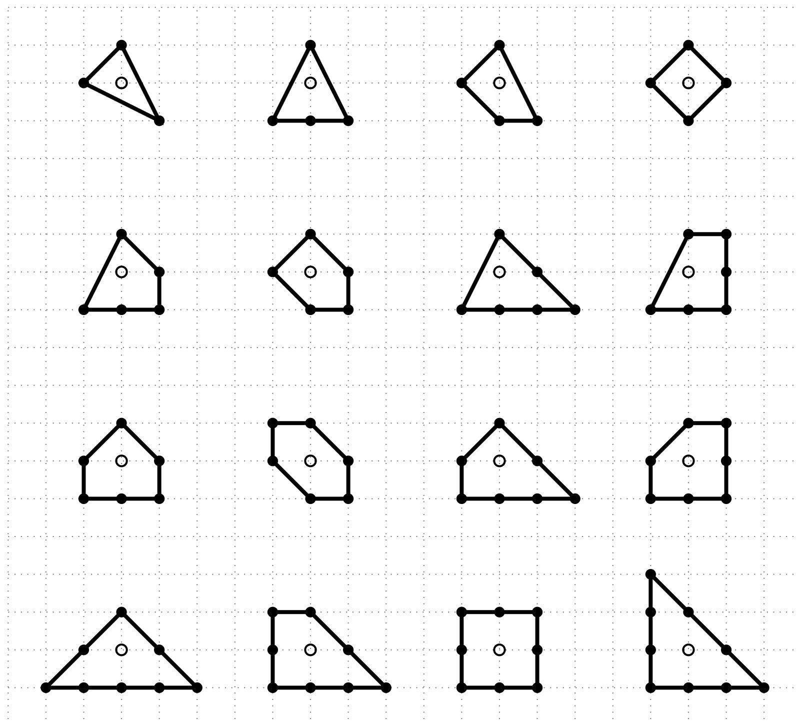
$$len^{deg-1} * f(len-1)$$

and the first term in the second row will be f(2).
Once this matrix is filled it, its determinant can be checked quickly by Pari. If this is zero, then a kernel exists, which pari will also calculate quickly. Starting at length 1 and degree 1 and moving upwards means a 1 dimensional kernel will eventually be found. the entries of the kernel are the coefficients of the polynomial coefficients of the recurrence equation.
For this example the recurrence is

$$na_n - a(2n-1)a_{n-1} + (n-1)(a^2-4)a_{n-2}$$

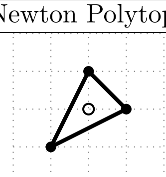
It's length 3, and by letting a $=\pm 2$ the end term is zeroed which shortens the length to two, it's extremal form.

2 dimension 2



2.1 1 variable

For the cases that can be reduced to having 1 variable coefficients we can can compile code that will use the general idea in the illustration. Computing the kernel gives the coefficients of the terms in the recurrence. By examination, or using an equation solver in Sage, these can be zeroed. The values for the zeros are the extremal values.

Number	Newton Polytope	Extremal Laurent Polynomials
1		$x+y+\frac{1}{xy}-3$

The generic polynomial with this Newton polygon is

$$P(x,y) \ = \ \alpha_1 x+\alpha_2 y+\frac{\alpha_3}{xy}+\beta\,, \quad \alpha_i\neq 0\,.$$

This polynomial is homothetic to

$$P(x,y) \;=\; x+y+\frac{\alpha}{xy}+\beta\,, \quad \alpha \;=\; \alpha_1\alpha_2\alpha_3 \neq 0\,.$$

Then the sequence of zero degree terms is

$$a_m \;=\; \sum_{s=0}^{[\frac{m}{3}]} \frac{(m-s)!}{(m-3s)!(s!)^2}\alpha^s\beta^{m-3s} \;=\; \sum_{s=0}^{[\frac{m}{3}]} \binom{m-s}{m-3s,s,s}\alpha^s\beta^{m-3s}\,.$$

With the help of computer we find the recursion

$$n^3a_n - \beta(1+3(n-1)+3(n-1)^2)a_{n-1} + 3\beta^2(n-1)^2a_{n-2} \\ -(27\alpha+\beta^3)(2+3(n-3)+(n-3)^2)a_{n-3} \;=\; 0\,.$$

(This formula can be proved using properties of trinomial coefficients.) Therefore extremal Laurent polynomials are those with

$$27\alpha+\beta^3 \;=\; 0\,.$$

We can take a homothetic polynomial with $\alpha_1 = \alpha_2 = \alpha_3 = -\frac{\beta}{3}$

$$P(x,y) \;=\; -\frac{\beta}{3}\Big(x+y+\frac{1}{xy}-3\Big)\,.$$

Finally, $r(\Delta) = 4$ and we see that up to scalings and homotheties there is a unique extremal Laurent polynomial for the polygon Δ

$$P(x,y) \;=\; x+y+\frac{1}{xy}-3$$

with the corresponding sequence of 0 degree coefficients being

$$1\,,\ -3\,,\ 9\,,\ -21\,,\ 9\,,\ 297\,,\ -2421\ldots$$

2.2 2-3 variables

It is found that in the cases reduced to 2 variable coefficients, that the extremal polynomials have length 3 and degree 3. It turns out that every one of the 16 laurent polynomials has a zero degree coefficient recursion with an extremal length and degree both of three(See conjectures).

For the cases which can be reduced to having either 2 or 3 coefficients, we had not the computational power to just find kernels, of 20 by 20 matrices, where each entry had polynomials of degree in excess of ten, with thousand-digit-long coefficients. To find the recursive sequences of these polynomials, one has to sub in values for say β and γ to generate a kernel in terms of α for the three variable case. When selecting values for β and γ one has to ensure they aren't extremal values(check that the kernel doesn't become two dimensional).

Doing likewise for the other two coefficients, one obtains three kernels with the entries of the polynomial coefficients for two variables subbed in. Now for some linear algebra.
In some instances the kernels had rational polynomials. These couldn't be worked with in the next step of the approach. So the lcm of the denominators of the three kernels was calculated. Then the values of α, β and γ were subbed in, to calculate the scaler factor in the three kernels. this scalar factor was multiplied into the respective polynomial lcm for that variable. These three polynomials of one variable could then be made into their one true polynomial of three variable using much the same linear algebra as was used in the next part of the approach, described below. Suppose β has the highest degree to be found among the three variables. Place it on the outside of the following system, so that the polynomials in the brackets are minimal.

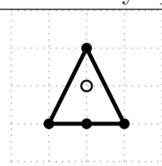
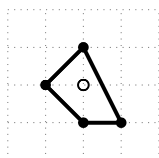
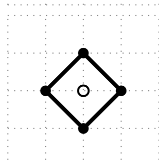
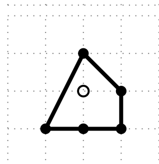
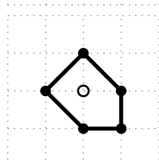
$$\begin{gathered}[(\lambda_{m,n}\alpha^n + \lambda_{m,n-1}\alpha^{n-1} + ...\lambda_{m,0}\alpha^0)\gamma^m \\ +(\lambda_{m-1,n}\alpha^n + \lambda_{m-1,n-1}\alpha^{n-1} + ...\lambda_{m-1,0}\alpha^0)\gamma^{m-1} \\ +...(\lambda_{0,n}\alpha^n + \lambda_{0,n-1}\alpha^{n-1} + ...\lambda_{0,0}\alpha^0)]\beta^k + \\ ... \\ +[(\lambda_{m,n}\alpha^n + \lambda_{m,n-1}\alpha^{n-1} + ...\lambda_{m,0}\alpha^0)\gamma^m \\ +(\lambda_{m-1,n}\alpha^n + \lambda_{m-1,n-1}\alpha^{n-1} + ...\lambda_{m-1,0}\alpha^0)\gamma^{m-1} \\ +...(\lambda_{0,n}\alpha^n + \lambda_{0,n-1}\alpha^{n-1} + ...\lambda_{0,0}\alpha^0)]\beta^{k-1}\end{gathered}$$

Now to solve for the coefficients, we set up the following matrix equations.

$$\begin{bmatrix} \alpha_1^n\gamma_1^m & \alpha_1^{n-1}\gamma_1^m & \dots & \alpha_1^0, \gamma_1^0 \\ & \ddots & & \\ \alpha_{(m+1)(n+1)}^n\gamma_{(m+1)(n+1)}^m & \alpha_{(m+1)(n+1)}^{n-1}\gamma_{(m+1)(n+1)}^m & \dots & \alpha_{(m+1)(n+1)}^0, \gamma_{(m+1)(n+1)}^0 \end{bmatrix}$$

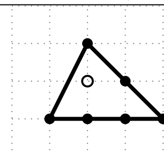
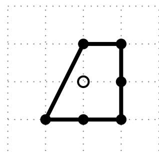
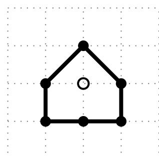
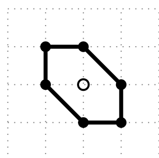
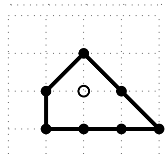
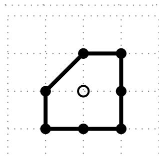
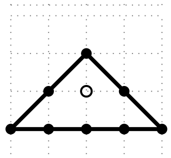
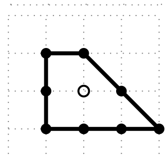
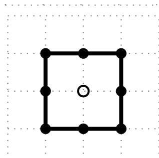
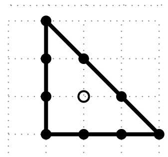
is multiplied by $\begin{bmatrix} \lambda_{m,n} \\ \dots \\ \lambda_{0,0} \end{bmatrix}$ to give the matrix $\begin{bmatrix} ker_{1,1} \\ \dots \\ ker_{m+1,n+1} \end{bmatrix}$

where $ker_{i,j}$ is the value of the coefficient of b^r to be solved for in the kernel when the i and j values have been subbed in for α and γ. Using the matsolve function in pari, one can determine the values of the lambdas, which gives the coefficients for a term in a polynomial of the recursion. Repeat the same process for the other terms in the polynomial. And then repeat this whole process in a for loop to obtain the other polynomials. Having now solved the recurrence equation, using sage software to make terms vanish, we could find the extremal polynomials, which all turn out to be of length and degree 3. The recurrences quickly became too big to work out the extremals by examination without a computer and needed equation solvers which are readily available, such as sage. for example the second three variable case has a recurrence equation of over 32,000 characters. Yet it extremal reduces to something a lot more pleasant. By now it had become apparent that recurrence relations would be out of the question for the larger variable cases. It would have taken an entire library rather than a paper to print all 16 recurrence relations. With our conjecture of length and degree both being 3 for the extremals in all cases, we decided to focus on obtaining all 16 forms of this.

Number	Newton Polytope	Extremal Laurent Polynomials
2		$y + 2/y + x/y + 1/xy$
3		$-3y + 1/y + 16/3x + x/y + 6$
4		$y + 1/y + x + 1/x$
5		$4x - y + 4/xy + 8/y + 4x/y - 4$
6		$x - 4y - 4/x + 1/y + x/y - 4$

2.3 4 -10 variables

Once we got to the 4 variable cases a new difficulty arose. We did not have the computational power to compute the determinant. Backtracking, we ensured every term in the laurent polynomial had a coefficient. This would ensure that the kernel would have integer entries rather than rational ones. Next we generated for loops to run through values of α_1 up to α_k from 1 to 4, computing a new determinant for each, of length and degree both of three. Coefficients can go to 0 if that would not disfigure the newton polytope. So for example the constant term can always go to zero. In this way we were able to compute the remaining extremal polynomials.

Number	Newton Polytope	Extremal Laurent Polynomials
7		$x^2/y + 2x + 3x/y + y + 3/y + 1/xy + 2$
8		$2x + x/y + y + 2/y + 1/xy + xy + 2$
9		$x + 1/x + y + 2/y + 1/xy + x/y + 2$
10		$x + 1/x + y + 1/y + y/x + x/y + 2$
11		$x^2/y + 2x + y + 1/x + 3/y + 3x/y + 1/xy + 3$
12		$2x + y + 1/x + 2/y + x/y + 1/xy + 3 + xy$
13		$y + 2/y + x^2/y + 1/x^2y$
14		$x^2/y + x + 3x/y + y + 2 + 3/y + y/x + 1/x + 1/xy$
15		$xy + x/y + 1/xy + y/x$
16		$1/xy + x^2/y + y^2/x - 3$

comments (by Masha Vlasenko)

The idea behind this little research program is closely related to the project "Fano Varieties and Extremal Laurent Polynomials" done by Tom Coates, Alessio Corti, Sergey Galkin, Vasily Golyshev and Al Kasprzyk. The details can be found in their research blog [5]. The actual definition of an extremal Laurent polynomial due to Golyshev ([6]) is that the local system on $\mathbb{P}^1 \setminus S$ (where $S \subset \mathbb{P}^1$ is a finite set) corresponding to the above recursion is extremal, that is nontrivial, irreducible, and of smallest possible ramification. The simplified definition we take here comes from a personal communication with Vasily Golyshev and is closely related to the actual one. In particular, results of this project have to be compared with the list of extremal Laurent polynomials in dimension 2 given by Sergey Galkin in [7].

The extremal polynomials found in this project correspond to families of elliptic curves with small number of singular points, which is seen in particular from the fact that some of our sequences appear in Zagier's tables [8] and correspond to Beauvilles' semi-stable families.

References

[1] J. Stienstra, F. Beukers, On the Picard-Fuchs equation and the formal Brauer group of certain elliptic K3- surfaces, Mathematische Annalen 271 (1985) pp.269–304.

[2] I. M. Gelfand, M. M. Kapranov, A. V. Zelevinsky, Discriminants, Resultants and Multidimensional Determinants, Birkhauser, Boston (1994)

[3] M. Kreuzer, H. Skarke, Classification of reflexive polyhedra in three dimensions, Adv. Theor. Math. Phys. 2 (1998), pp. 853–871.

[4] M. Kreuzer, H. Skarke, Complete classification of reflexive polyhedra in four dimensions, Adv. Theor. Math. Phys. 4 (2000), pp. 1209–1230.

[5] T. Coates, A. Corti, S. Galkin, V. Golyshev and A. Kasprzyk, Fano Varieties and Extremal Laurent Polynomials (http://coates.ma.ic.ac.uk/fanosearch/)

[6] V. Golyshev, Spectra and Strain, arXiv:0801.0432 (hep-th)

[7] S. Galkin, Del Pezzo surfaces and low ramified pencils of elliptic curves, preprint MIAN (2006) (in Russian)

[8] D. Zagier, Integral solutions of Apéry-like recurrence equations, CRM Proceedings and Lecture Notes 47, 349-366 (2009).

Newton Polytope	Sequence of Free Terms	Recursion	Position in Zagier's Table
1, 16	$1, -3, 9, -21, 9, 297, -2421, \ldots$	$(n+1)^2 a_{n+1} + (9n^2+9n+3)a_n + 27n^2 a_{n-1}$	#7
2, 4, 13, 15	$1, 0, 4, 0, 36, 0, 400, \ldots$	$(n+1)^2 a_{n+1} - 16n^2 a_{n-1}$	#2
7, 8, 9, 10	$1, 2, 10, 56, 346, 2252, 15184, \ldots$	$(n+1)^2 a_{n+1} + (7n^2+7n-2)a_n - 8n^2 a_{n-1}$	#5
11, 12	$1, 3, 19, 147, 1251, 11253, 104959, \ldots$	$(n+1)^2 a_{n+1} + (11n^2+11n-3)a_n - n^2 a_{n-1}$	#9
14	$1, 2, 18, 140, 1330, 12852, 130284, \ldots$	$(n+1)^2 a_{n+1} - (8n^2+8n+2)a_n - (48n^2-12)a_{n-1}$	n/a
3	$1, 6, 30, 12, -2250, -35244, -329364, \ldots$	$(n+1)^2 a_{n+1} - (56/3n^2+56/3n+6)a_n + (144n^2-4)a_{n-1}$	n/a
5, 6	$1, -4, 0, 224, -2240, 5376, 139776, \ldots$	$(n+1)^2 a_{n+1} + (13n^2+13n+4)a_{n-1} + (128n^2-8)a_{n-1}$	n/a