What is a motivic gamma function?

- Joint project with Spencer Bloch

- Started on ideas from Vasily Golyshev

$L$ is a differential operator on $P^1 \setminus S = U$

finite set of singular points

$\int_0^\gamma x^s \psi(x) \frac{dx}{x}$ defined up to

$\pm i\pi \text{ms mod } \mathbb{Z}$

$\gamma$ is an oriented closed path in $U$

avoiding $0$ and $\infty$

and contractible in $P^1 \setminus \{0, \infty\}$

$\psi$ is a solution to $L \psi = 0$

defined in a neighbourhood of $\gamma$

and having trivial monodromy around $\gamma$

Remarks:

1) $\gamma \sim \sum n_i [0_i]$, $\psi_i \sim \text{const} \cdot \psi_i$

gamma functions form a module over $\mathbb{C}[e^{2\pi i}]$

2) $\int_0^\gamma x^s \psi(x)$ is called motivic when

$L$ is of Picard-Fuchs type

and $\psi(x)$ is a period function

$\mathcal{X}$

$\psi(x) = \int \omega(x)$

$\Rightarrow$ de Rham form

$\mathcal{X} / \mathbb{Q}$

$\Rightarrow$ Betti cycle

Module of finite rank over $\mathbb{Q}[e^{2\pi i}]$
Example \[ \psi(x) = (1-x)^{-\frac{1}{2}} \}

\[ \int_{\bar{C_i}} \psi(s) = \int_{\bar{C_i}} \frac{x^s}{\sqrt{1-x}} \frac{dx}{x} = \]

\[ = \int_0^1 -e^{-2\pi i s} \int_0^1 e^{-2\pi i s} \int_1^0 \frac{x^s}{1-x} \frac{dx}{x} = 2(1-e^{-2\pi i s}) \frac{\Gamma(s) \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+s)} \]

entire, motivic

Applications: - interpolation of recurrences
- Apéry constants (this talk)
- ...

Motivation:

\[ \gamma \rightarrow s_2 \]

\[ s_1 \text{ path between two singular pts} \]

\[ \{ \psi^{(k)}_i \} \]

basis in the space of solutions to \( \psi \) near \( s_k \)

\[ [x] \psi^{(k)}_i = \sum_j \psi^{(k)}_j C_{ij} \]

connection matrix

Special choice: Frobenius basis

\[ \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \]

Jordan block of \( [\delta_0] \)

local monodromy

\[ \lambda = e^{2\pi i \rho} \]

regular singularity

unique under the condition \( \psi^{(n)}_0(0) = 1 \)

\( \psi^{(n)}_i(0) = 0 \quad i > 0 \)
Frobenius basis spans de Rham structure of the limiting MHS in mirror symmetry.

Quantum diff. e.q. of a Fano manifold \( \Rightarrow \) Picard-Fuchs diff. e.q. \( L \) connection matrices contain info about the original Fano.

**Special case**

\( \sigma_2 \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) SS point + pseudo-reflection

\( \sigma_1 \sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) MUM point

\( \sigma_0, \ldots, \sigma_{n-1} \) Frobenius basis near \( s_1 \)

Apéry constants are \( x_0 = 1, x_1, \ldots, x_{n-1} \in \mathbb{C} \)

s.t. \( [\mathcal{A}] (\varphi_0 - x_0 \varphi_0) \) is \( [\sigma_2] \)-invariant.

Ambiguity: \( x_0 = 0 \) \( \varphi_q (x) = \varphi_0 (x) \log (x) + \varphi_i^{ \alpha_i (x)} \)

\( x_i \sim x_i + 2\pi i \frac{\alpha_i}{\alpha_i - 1} \) \( x_i \in \mathbb{C} / 2\pi i \mathbb{Z} \)

\( \sigma_0^m \varphi_0, \ldots, \sigma_1^m \varphi_{n-1} \) is also a Frobenius basis.

\( [\sigma_1] \sum \varphi_i (x) e_i (e) = e \sum \varphi_i (e) e_i \)

\( \Rightarrow x(e) = \sum x_i e_i \) is defined up to \( \times e \)

(Note similarity with the ambiguity of \( \prod (x) \! \) )

(In fact, one can define "higher" \( x_n, x_{n+1}, \ldots \) and "complete" \( \mathcal{A}(x) \) to (almost) a gamma function! I am coming to this point after a short example.)
\[ L = D^3 - x (2D + 1) (17D^2 + 17D + 5) + x^2 (D + 1)^3 \]
\[ = x^3 (-34x + x^2) \frac{d^3}{dx^3} + \ldots \]

\[ S = \{ 0, \infty, 17 \pm \sqrt{17^2 - 1} \} \]

\( x = 0 \) is MUM

\[ g_0(x) = 1 + 5x + 73x^2 + \ldots \]

\[ g_1(x) = g_0(x) \log x + \left( \frac{12x + 210x^2 + \ldots}{g_1(x)} \right) \]

\[ g_2(x) = g_0(x) \frac{\log^2 x}{2!} + g_1(x) \log x + \left( \frac{72x^2 + 2160x^3 + \ldots}{g_2(x)} \right) \]

\[ C = 17 - \sqrt{17^2 - 1} = 0.0294 \ldots \]

Reflection point \( (0, 0, 0) \)

\( x_0 = 1 \quad x_1 = 0 \quad x_2 = -\frac{\pi^2}{3} \)

Lemma \( L = \sum_{i=0}^{r} x^i P_i(D) \) polynomial diff operator of order \( N \)

has MUM at \( x = 0 \) iff \( P_0(D) = D^N \)

\( \Rightarrow \) all \( D^k L \) have MUM at \( x = 0 \)

can construct higher Frobenius functions (Golyshen-Zagier)

\[ \Phi_0(x), \Phi_1(x), \ldots \]

\[ \Phi(x, \varepsilon) = \sum_{i=0}^{\infty} \Phi_i(x) \varepsilon^i \]

\[ L \Phi = \varepsilon^N x^N \]

More interestingly; \( C = 0.0294 \ldots \) seems to staff a reflection point for all \( D^k L \)

G-Z compute

\[ x_3 = 17 \log (3), \quad x_4, \ldots \quad x_{10}, \quad x_{11}, \quad \frac{x_{11}}{x_{10}} \]

ratio of combinations of ZVs of weight 41, \ldots, 110

\( S(3, 5, 3) \)!
Problem: understanding higher \( \xi_k \)'s as periods?

Geometric origin of \( D^k \)?

\( k = 1 \) Mahler measure, normal functions

\( k = 2 \) ... more \( K \)-theory

**Theorem**

\( c = 0.0294... \) remains

semisimple for all \( D^k \), \( k = 0, 1, 2, ... \)

(\( \Rightarrow \) \( [\xi] \) is a reflection).

Therefore all higher Apéry constants exist and in fact

\[\mathcal{X}(\xi) = \sum_{i=0}^{\infty} x_i \xi^i = \left( \frac{2\pi i \xi}{e^{2\pi i \xi} - 1} \right)^3 \prod_{\sigma} f_0(\xi) \]

\[6^3 = 6_0^{-3} 6_1 \left( 6_0^3 6_1^3 \right)\]

\[\psi = \text{unique } \left[ 6^3 \text{-invariant} \right]

solution to \( \mathcal{L} \) normalized so that \( \prod_{\sigma} f_0(0) = 1 \)

\[= -\frac{1}{3} \cdot \frac{1}{2\pi i} f_0 - \frac{1}{2} \cdot \frac{1}{(2\pi i)^2} f_1 + \frac{1}{3} \cdot \frac{1}{(2\pi i)^3} f_2\]

Remark: the presentation (RHS here) is non-canonical we can have

\[P(e^{2\pi i \xi}) \; \mathcal{X}(\xi) = \xi^N \prod_{\sigma} f_0(\xi)\]

a poly with root 1 of multiplicity \( 2N \)

the canonical presentation is

\[\mathcal{X}(\xi) = \xi^N \int_0^\xi x^2 \delta(x) \frac{dx}{x} \]

\[\delta = \text{(uniquely normalized)} \]

\([\sigma_c] \delta = -\delta \]

*reflection!*
What is our benefit in understanding $\mathcal{H}$ as periods

Theorem $\Rightarrow \mathcal{H} = \text{linear combinations of iterated integrals}

(\text{As Francis Brown and Richard Heintz just explained to us, iterated integrals are periods of a relative completion!})

... work in progress

$$\int_{\sigma_1}^{(0)} (x) = \int_{\sigma}^{0} \log^N(x) \psi(x) \frac{dx}{x}$$

**Lemma** $[\sigma]$ $\mathcal{D}^{-\mathcal{K}} \psi(x) = \mathcal{D}^\mathcal{K} \psi(x) + \sum_{j=0}^{k-1} \frac{\mathcal{D}_0^{(j)}(0) \log^{k-1-j}}{j!} (x)$

in definite iterated integral

**Example** polylogs $\psi = \frac{1}{1-x}$ $L_2 = (1-x) \frac{d}{dx} - 1$

$$\int_{\sigma_1}^{(0)} (s) = \int_{\sigma_1}^{0} \frac{x^5}{1-x} \frac{dx}{x} = -2\pi i$$

What does this tell us?

$[\sigma_1] \operatorname{Li}_k(x) = \operatorname{Li}_k(x) - 2\pi i \frac{\log^{k-1}(x)}{(k-1)!}$