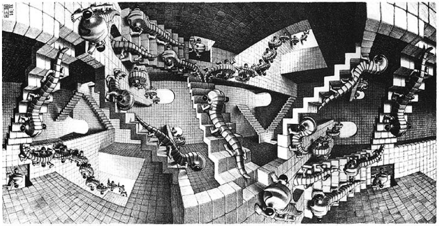
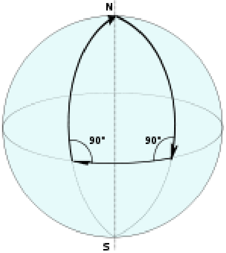
Internal geometry of surfaces



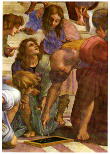
(M.C.Escher, *Stairs*)

Masha Vlasenko, November 10

A fellow took a morning stroll. He first walked 2 km South, then 2 km West, and then 2 km North. It so happened that he found himself back at his house door. How can this be?



The fellow lives on the North pole!



Euclid of
Alexandria
$\approx$ 300 BC

Oxyrhynchus papyrus
showing fragment
of Euclid's Elements

Euclid's plane geometry *axioms*:

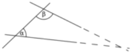
I. Between any two points a straight line may be drawn.

II. Every straight line can be continued infinitely in both directions.

III. A circle can be drawn with any centre and radius.

IV. All right angles are equal to one another.

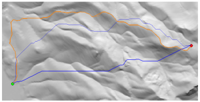
V. If the sum of the interior angles α and β is less than 180°, the two straight lines, produced infinitely, meet on that side.

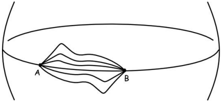


Do the axioms hold on a surface?

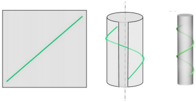
What are straight lines?

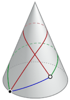
The *distance* between two points A and B is the length of the shortest path joining A and B.

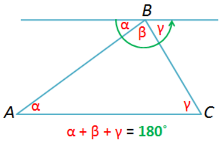


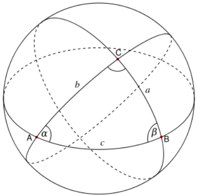


Geodesic curve is a curve such that for any two close enough points of this curve the shortest path between them is the one along the curve.

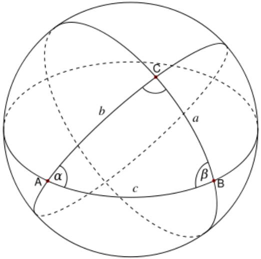








plane	sphere
geodesics are lines	geodesics are great circles
lines have infinite length	finite length
two lines either intersect at 1 point or do not intersect at all (parallel lines)	two "lines" always intersect at 2 points
in a triangle $\alpha + \beta + \gamma = 180^\circ$	$\alpha + \beta + \gamma > 180^\circ$



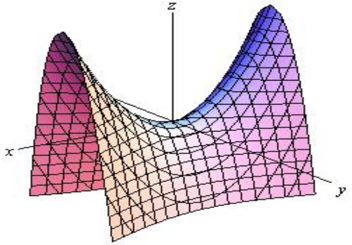
$S =$ the area of the entire sphere

$S_{ABC} =$ the area of the triangle ABC

$$S \cdot \frac{2\alpha}{360^\circ} + S \cdot \frac{2\beta}{360^\circ} + S \cdot \frac{2\gamma}{360^\circ} = S + 4 \cdot S_{ABC}$$

$$S\Big(\frac{\alpha+\beta+\gamma}{180^\circ} - 1\Big) = 4 \cdot S_{ABC}$$

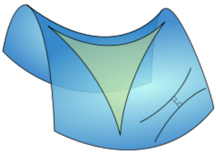
$$S_{ABC} = \frac{S}{4 \cdot 180^\circ} \cdot \Big(\alpha+\beta+\gamma-180^\circ\Big) \ > \ 0$$



$z = x^2 - y^2$

hyperbolic paraboloid

geodesics are hyperbolas and parabolas



$$S_{ABC} = K \cdot \left(180^\circ - \alpha - \beta - \gamma\right)$$

$$\alpha + \beta + \gamma < 180^\circ$$

surfaces of constant curvature

plane	sphere	hyperbolic paraboloid
in a triangle $\alpha + \beta + \gamma = 180^\circ$	$> 180^\circ$	$< 180^\circ$
through an external point one can draw one line parallel to a given one	no parallel lines	infinitely many parallel lines

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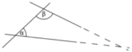

For two thousand years, many attempts were made to prove the fifth postulate using Euclid's first four postulates.

Attempts to prove the fifth axiom using the first four axioms:
Proclus (410–485)
Ibn Alhazen (965–1039)
Omar Khayyam (1050–1123)
Nasir al-Din al-Tusi (1201–1274)
Giordano Vitale (1633–1711)
Johann Lambert (1728–1777)

Nikolai Lobachevsky in 1829 and Janos Bolyai in 1831 published accounts of acute (hyperbolic) geometry, which were later developed by Lobachevsky, Riemann and Poincare. The independence of the parallel postulate from Euclid's other axioms was finally demonstrated by Eugenio Beltrami in 1868.

Thank you!