Geometricity Problem for Differential Equations

Masha Vlasenko

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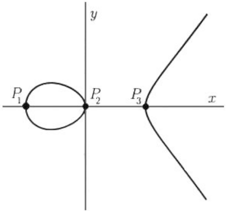
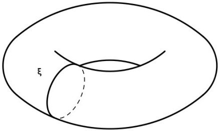
Periods of an algebraic manifold

X algebraic variety
periods of X = numbers which one can obtain integrating algebraic differential forms along topological cycles

$$X \ : \ y^2 = x(x-1)(x+1) \qquad \omega \ = \ \frac{dx}{y}$$





$$\Omega \ = \ \int_{\xi} \omega \ = \ 2\int_{-1}^{0} \frac{dx}{\sqrt{x(x-1)(x+1)}}$$

Periods in families of manifolds

$$X_t\,,\ \omega_t\,,\ \xi_t \qquad \Omega(t) \ = \ \int_{\xi_t} \omega_t$$

$$X_t \ : \ y^2 = x(x-1)(x-t)$$

$$\Omega(t) \ = \ \int_0^t \frac{dx}{\sqrt{x(x-1)(x-t)}}$$

$$t(t-1)\Omega''(t) \ + \ (2t-1)\Omega'(t) + \frac{1}{4}\Omega(t) \ = \ 0$$

is the Picard-Fuchs differential equation for this family of curves

Geometricity problem

Given a differential equation to determine whether it is "geometric", i.e. whether it is a Picard-Fuchs equation for some 1-parametric family of algebraic varieties.

Properties of Picard-Fuchs differential operators:

- global monodromy satisfies certain restrictions (variation of Hodge structures)
- they are globally nilpotent, i.e. p-curvature operators Ψ_p are nilpotent for almost all primes p

Conjecture (B. Dwork, C. Siegel, 70s): a differential equation satisfying these two conditions is "geometric".

N. Katz: A rigid differential equation with quasi-unipotent local monodromies is "geometric".

$$t(t-1)\frac{d^2}{dt^2} + ((a+b+1)t - c)\frac{d}{dt} + ab$$

Taylor coefficients of solutions of Picard-Fuchs equations become integral after simple rescaling:

$$\frac{1}{\pi}\Omega(t) = \frac{1}{\pi}\int_0^t \frac{dx}{\sqrt{x(x-1)(x-t)}} = 1 + \frac{1}{4}t + \frac{9}{32}t^2 + \dots$$
$$= \sum_{n=0}^{\infty} \frac{n!}{16^n}\binom{2n}{n}^2 t^n \in \mathbb{Z}\left[\left[\frac{t}{16}\right]\right]$$

D.V. Chudnovsky and G.V. Chudnovsky, 1985: this property is more or less equivalent to the two properties listed earlier.

Dwork-Siegel conjecture ⇔ a differential equation is "geometric" if and only if solutions have "almost integral" Taylor expansions at a given point

Determinantal differential equations

For a matrix $A = (a_{ij})_{i,j=0}^N$ satisfying

$$\begin{aligned} a_{ij} &= 0\,, \quad i-j>1 \\ a_{ij} &= 1\,, \quad i-j=1 \\ a_{ij} &= a_{N-j,N-i} \quad i-j<1 \end{aligned}$$

the determinantal differential operator of order N (a DN-operator) is

$$\mathcal{L}_A(z) = \det{}_{right} \left(\delta_{ij} z \frac{d}{dz} - a_{ij} \left(\frac{d}{dz}\right)^{j-i+1}\right) \left(\frac{d}{dz}\right)^{-1}.$$

V. Golyshev, J.Stienstra, "Fuchsian equations of type DN", 2007

V. Golyshev, "Classification problems and mirror duality", 2005

E.g. $N = 2$

$$\mathcal{L}_A(z) \;=\; \det{}_{right} \begin{pmatrix} (z - a_{00})\frac{d}{dz} & -a_{01}\left(\frac{d}{dz}\right)^2 & -a_{02}\left(\frac{d}{dz}\right)^3 \\ -1 & (z - a_{11})\frac{d}{dz} & -a_{01}\left(\frac{d}{dz}\right)^2 \\ 0 & -1 & (z - a_{00})\frac{d}{dz} \end{pmatrix} \left(\frac{d}{dz}\right)^{-1}$$

$$= -a_{02}\left(\frac{d}{dz}\right)^2 - a_{01}\left(\frac{d}{dz}\right)^2 (z - a_{00}) + (z - a_{00})\frac{d}{dz}\left((z - a_{11})\frac{d}{dz}(z - a_{00}) - a_{01}\frac{d}{dz}\right)$$

$$= F(z)\left(\frac{d}{dz}\right)^2 + F'(z)\,\frac{d}{dz} + (z - a_{00})$$

where

$$F(z) \;=\; \det\bigl(z - A\bigr) \;=\; z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0$$

$$\alpha_2 \;=\; -a_{11} - 2a_{00}$$

$$\alpha_1 \;=\; 2a_{00}a_{11} + a_{00}^2 - 2a_{01}$$

$$\alpha_0 \;=\; 2a_{00}a_{01} - a_{00}^2 a_{11} - a_{02}$$

Quantum cohomology construction

X	$\rightsquigarrow$	$QH^*(X)$	$\rightsquigarrow$	$\mathcal{L}$
Fano		small quantum		differential operator
variety		cohomology ring		(Dubrovin's connection)

After a certain change of variables, $\mathcal{L} = \mathcal{L}_A$ where

$$a_{ij} = \text{two-pointed genus 0 Gromov-Witten invariants of } X$$

Homological mirror symmetry conjecture

M.Kontsevich, V.Batyrev, A.Givental, K.Hori, C.Vafa:

X	$\rightsquigarrow$	$\mathcal{L}$	$\rightarrow$	Y_t
Fano variety		defferential operator		family of Calabi-Yau varieties

Quantum differential equations of Fano varieties are "geometric".

"Geometricity" of D3 equations: a solution

Fano 3-folds X of Picard rank 1 $\rightsquigarrow$ D3 equations

A generic D3 equation is

$$F(z)\left(\frac{d}{dz}\right)^3 + \frac{3}{2}F'(z)\left(\frac{d}{dz}\right)^2 + \left(\frac{1}{2}F''(z) + G(z)\right)\frac{d}{dz} + \frac{1}{2}G'(z)$$

where

$$\begin{aligned} F(z) &= \det\left(z - A\right) = z^4 + \alpha_3 z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0 \\ G(z) &= z^2 + \beta_1 z + \beta_0 \end{aligned}$$

Construction: Frobenius basis in the space of solutions near $z = \infty$

$$\phi_0(z) = \frac{1}{z} + \frac{\beta_1}{z^2} + \frac{-\frac{3}{8}\alpha_3\beta_1 + \frac{3}{8}\beta_1^2 - \frac{1}{8}\alpha_2 - \frac{1}{8}\beta_0}{z^2} + \ldots$$

$$\phi_1(z) = -\log z\,\phi_0(z) + \frac{-\frac{1}{2}\alpha_3 - \beta_1}{z^2} + \ldots$$

$$\phi_2(z) = (\log z)^2\,\phi_0(z) + \ldots$$

$$Q(z) = \exp\Big(\frac{\phi_1(z)}{\phi_0(z)}\Big) = \frac{1}{z}\exp\Big(\frac{-\frac{1}{2}\alpha_3 - \beta_1}{z} + \ldots\Big) = \frac{1}{z} + \ldots$$

$$\frac{1}{z} = Q + \big(\frac{1}{2}\alpha_3 + \beta_1\big)\,Q^2 + \ldots$$

$$\phi_0(Q) = Q + \big(\frac{1}{2}\alpha_3 + 2\beta_1\big)\,Q^2 + \ldots = \sum_{n=1}^{\infty} C_n Q^n$$

The *spectral curve* of a D3 equation

$$F(z)\Big(\frac{d}{dz}\Big)^3 + \frac{3}{2}F'(z)\Big(\frac{d}{dz}\Big)^2 + \Big(\frac{1}{2}F''(z) + G(z)\Big)\frac{d}{dz} + \frac{1}{2}G'(z)$$

$$\begin{aligned} F(z) &= z^4 + \alpha_3 z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0 \\ G(z) &= z^2 + \beta_1 z + \beta_0 \end{aligned}$$

is the elliptic curve birational to

$$y^2 = z^4 + \alpha_3 z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0 \, .$$

Theorem

Suppose the D3 equation under consideration is "geometric" and the sequence $\{C_n; n \geq 1\}$ is defined by

$$\phi_0(Q) = \sum_{n=1}^{\infty} C_n Q^n, \qquad Q = \exp\left(\frac{\phi_1(z)}{\phi_0(z)}\right).$$

Then

$$\sum_{n=1}^{\infty} \frac{C_n}{n^s}$$

is the Hasse-Weil L-unction of the spectral elliptic curve

$$y^2 = z^4 + \alpha_3 z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0 .$$

Corollary: one has $C_{mn} = C_n \cdot C_m$ whenever $(m, n) = 1$.

Since $C_n = C_n(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1)$ are simple rational functions we can find all "geometric" D3 equations by solving multiplicativity equations

$$\begin{aligned} C_6 &= C_2 \cdot C_3 \\ C_{10} &= C_2 \cdot C_5 \\ C_{15} &= C_3 \cdot C_5 \\ &\dots \end{aligned}$$

α_3	α_2	α_1	α_0	β_1	β_0
4	0	0	0	0	0
2	1	0	0	0	-1
-2	-3	0	0	0	6
-4	-88	-150	-304	0	-8
0	0	-54	0	0	0
-2	-43	-78	-216	0	-5
-6	-135	-270	-648	0	-9
-2	-59	-68	-80	0	-5
4	-80	96	0	0	-16
2	9	-108	432	0	-9

...

(30 cases found)

Classification of Fano varieties

dim $= 1$
$\mathbb{P}^1$

dim $= 2$
$\mathbb{P}^1 \times \mathbb{P}^1$ or the blow up of $\mathbb{P}^2$ in ≤ 8 general points

dim $= 3$
105 deformation families of nonsingular Fano 3-folds: 17 families with $\beta_2 = 1$ (Fano,Iskovskih) and 88 families with $\beta_2 \geq 2$ (Mori-Mukai)