FUNCTIONAL EQUATION FOR ZETA FUNCTIONS

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ABSTRACT. These are notes of my talk on the seminar on J.Tate's thesis held at MPI in Bonn in Spring 2006.

1. Haar measure on Ideles

Let k be a number field, A be it's ring of adeles. Then $I = A^\times$ is a group of ideles with multiplication restricted from A. An element $(x_p) \in I$ satisfies $x_p \in o_p^\times = u_p$ for all but finite number of places p. We define a topology on I as on restricted product of $k_p^\times$ with respect to subgroups u_p. Note that this topology differs from the one restricted from A: the set $I^\infty \times \prod u_p$ is a neighbourhood of 1 in I but it cannot be lifted to a neighbourhood of 1 in A since such a neighbourhood should contain o_p at infinitely many places and $o_p \cap k_p^\times \neq u_p$.

$k_p^\times$ is an abelian locally compact group, also we have the modular function $|x|_p : k_p^\times \longrightarrow \mathbb{R}_+$. Recall that for each Haar measure dx_p on the additive group k_p we have a corresponding Haar measure on $k_p^\times$ denoted by $\frac{dx_p}{|x|_p}$. This measure is defined by the Haar integral (for $k_p^\times$ it means "multiplication invariant" !)

$$f \mapsto \int_{k_p - \{0\}} \frac{f(x)}{|x|_p} dx_p$$

for $f \in C_0(k_p^\times)$ — continuous function with compact support. This integral is well defined since $f \mapsto \frac{f(x)}{|x|_p}$ is an isomorphism between $C_0(k_p^\times)$ and $C_0(k_p - \{0\})$.

We could define the Haar measure on I simply by $\prod_p \frac{dx_p}{|x|_p}$, but we prefer the following modification by constant:

$$dx_I = \prod_{p\ infinite} \frac{dx_p}{|x|_p} \times \prod_{p\ finite} \frac{Np}{Np-1} \frac{dx_p}{|x|_p}.$$

This modification will be important in Theorems **??** and **??**.

2. Embedding of $k^\times$ into Ideles

Multiplication by an idele is a continuous automorphism of A, so we have a modular function $|\cdot| : I \mapsto \mathbb{R}_+$. Obviosly $|(x_p)| = \prod |x_p|_p$ since $dx = \prod dx_p$ on A.

Proposition 1. $|x| = 1$ *for* $x \in k^\times \subset I$.

Proof. Let $D \subset A$ be a fundamental domain for k in A. For $x \in k^\times$ we have $xk = k$, so xD is a fundamental domain for k again. Thus D and xD can be devided into countable number of pairwise congruent pieces $D \cap (y + xD)$ and $xD \cap (-y + D)$ ($y \in k$). Then measure of xD equals measure of D, and by definition of modular function we have $|x| = 1$. □

Let $J = Ker(|\cdot| : I \longrightarrow \mathbb{R}_+)$ be a subset of ideles of norm 1. It is closed, and we have $k^\times \longrightarrow J$ due to the proposition. All ideles now can be considered as a product of two subgroups

$$I = J \times \mathbb{R}_+$$

(layers consist of elements with fixed value of $|\cdot|$). We fix an embedding of $\mathbb{R}_+$ into I as follows. Let p_∞ be an arbitrary chosen infinite place, and we embed t as $x(t) = (t, 1, 1, 1, \cdot)$ with t at p_∞ if p_∞ is real and $x(t) = (\sqrt{(t)}, 1, 1, 1, \cdot)$ if p_∞ is complex. Then $|x(t)| = t$ and any element y is represented as $\frac{y}{x(|y|)} \times x(|y|)$ with $\frac{y}{x(|y|)} \in J$. We fix a Haar measure on $\mathbb{R}_+$ as $\frac{dt}{t}$. Then a Haar measure dx_J on J should exist such that $dx_I = dx_J \times \frac{dt}{t}$.

Theorem 1. *An embedding $k^\times \subset J$ is discrete and the quotient $J/k^\times$ is compact.*

Proof. We prove this by constructing explicitly a fundamental doman E such that

$$J = \bigcup_{x \in k^\times} xE.$$

Consider an infinite part of ideles $I^\infty = \mathbb{R}^\times \times \ldots \mathbb{R}^\times \times \mathbb{C}^\times \ldots \mathbb{C}^\times$ with r_1 factors $\mathbb{R}^\times$ and r_2 factors $\mathbb{C}^\times$. It is mapped onto $\mathbb{R}^{r_1+r_2}$ by the logarithm

$$\mathrm{Log}(x_1, \ldots, x_{r_1}, y_1, \ldots, y_{r_2}) = (log|x_1|, \ldots, 2log|y_{r_2}|),$$

and we have the $Tr : \mathbb{R}^{r_1+r_2} \to \mathbb{R}$ which is simply the sum of coordinates. So, we have maps

$$k^\times \subset I^\infty \xrightarrow{\mathrm{Log}} \mathbb{R}^{r_1+r_2} \xrightarrow{\mathrm{Tr}} \mathbb{R}.$$

Let $r := r_1 + r_2 - 1$. Let $u = o_k^\times$ be global units. Obviosly $u \subset \mathrm{Ker}(\mathrm{Tr} \circ \mathrm{Log})$, and it is known (Dirihlet unit theorem) that $u \cap \mathrm{Ker}(\mathrm{Log}) = \mu_k$ (roots of unity in k) and $\mathrm{Log}(u) \cong \mathbb{Z}^r$ is a lattice of maximal rank in $\mathbb{R}^r \cong \mathrm{Ker}(\mathrm{Tr})$. We pick $\varepsilon_1, \ldots, \varepsilon_r \in u$ such that $\mathrm{Log}(\varepsilon_i)$ generate $\mathrm{Log}(u)$, and let $P \subset \mathrm{Ker}(\mathrm{Tr})$ be a paralelotope spanned by $\mathrm{Log}(\varepsilon_i)$. μ_k is a cyclic group, let $w = \#(\mu_k)$ be its order.

We again need an arbitrary infinite place p_∞. Put

$$E_0 = \{x \in \mathrm{Log}^{-1}(P) | 0 \le arg(x_{p_\infty}) < \frac{2\pi}{w}\}.$$

Obviously E_0 is bounded (thus relatively compact), and has an interior in sense of usual topology on the subspase of elements of I^∞ with norm 1. Then $E_1 = E_0 \times \prod_{p\ finite} u_p$ is also relatively compact and has interior as a subset of J. Now we show that a finite number of translates of E_1 is a fundamental domain we are looking for.

Let $I^0 = \prod_{p\ finite} k_p^\times$ be finite part of ideles. Recall the map from I^0 to fractional ideals of k. Then $\prod u_p$ is its kernel

$$I^0 \Big/ \prod u_p \cong Ideals(k)$$

and

$$I^0 \Big/ k^\times \prod u_p \cong Cl(k)$$

Let $h = \#Cl(k)$ be the class number and $x_1, \ldots, x_h \in I^0$ be ideles which represent all different classes. Obviously we can lift them all to J. Then

$$E = x_1 E_1 \cup x_2 E_1 \cup \ldots x_h E_1$$

is a fundamental domain for $k^\times$ in J. Indeed, let $x = x^\infty \times x^0 \in J$. Then for exactly one i we have $(x x_i^{-1})^0 \in k^\times \prod u_p$. So, for some $y \in k^\times$ we have $(x x_i^{-1} y)^0 \in \prod u_p$ and this y is defined up to a unit from $u = o_k^\times$. Now a unit $z \in u$ can be chosen so that $Log((x x_i^{-1} y z)^\infty) \in P$, and this unit is defined up to a root of unity in μ_k. Now we finally choose a root of unity v so that $0 \le arg((x x_i^{-1} y z v)^\infty_{p_\infty}) < \frac{2\pi}{w}$. So we get for our $x \in I$ that $x \times (yzv) \in x_i E_1$, where $yzv \in k^\times$ is unique by construction. □

Although a fundamental domain E we have just constructed depends on a number of choises, its measure is a fixed number which doesn't depend on the choises. In fact

Theorem 2. *Let E be a fundamental domain for $k^\times$ in J. Then*

$$\int_E dx_J = \frac{2^{r_1}(2\pi)^{r_2} hR}{w\sqrt{|d|}}$$

where $R = |\det(\log|\varepsilon_i|_{p_j})|$ is the regulator of the field k (here p_j runs over all but one $r_1 + r_2$ infinite places, and the value of R doesn't depend on the choise of the place excluded).

Proof. We use the fundamental domain E constructed in the proof of the theorem above.

Let $\mu_\infty = \prod_{p\ infinite} \frac{dx_p}{|x|_p}$ be the measure in multiplicative Minkovsky space I^∞, $\mu_p = \frac{Np}{Np-1}\frac{dx_p}{|x|_p}$ be the measure in $k_p^\times$ for finite places. Then

$$\mu_p(u_p) = \frac{Np}{Np-1}\int_{u_p}\frac{dx_p}{|x|_p} = \frac{Np}{Np-1}\int_{u_p} dx_p = \int_{o_p} dx_p = [o_p : \delta_p]^{-\frac{1}{2}},$$

so

$$\frac{1}{\sqrt{|d|}} = (N\delta)^{-\frac{1}{2}} = \prod_p [o_p : \delta_p]^{-\frac{1}{2}} = \prod_p \mu_p(u_p).$$

Obviously,

$$\int_E dx_J = h\int_{E_1} dx_J = h\frac{\int_{[1,t]\times E_1} dx_I}{\log t} = h\frac{\mu_\infty([1,t]\times E_0)}{\log t}\prod_{p\ finite}\mu_p(u_p)$$

$$= \frac{h}{\sqrt{|d|}}\frac{\mu_\infty([1,t]\times E_0)}{\log t} = \frac{h}{w\sqrt{|d|}}\frac{\mu_\infty\left([1,t]\times Log^{-1}(P)\right)}{\log t}$$

because $Log^{-1}(P)$ is a disjoint union of translates of E_0 by roots of unity in k.

It is easy to check (separately for each infinite place) that for Lebesque measure λ in $\mathbb{R}^{r_1+r_2}$ we have $\mu_\infty(Log^{-1}X) = 2^{r_1}(2\pi)^{r_2}\lambda(X)$ for measurable $X \subset \mathbb{R}^{r_1+r_2}$. Then, P is a subset of the hyperplane $\mathrm{Tr} = 0$, image of $[1,t]$ is the interval $[0, \log t]$ along some axis in $\mathbb{R}^{r_1+r_2}$. Since all axes are under the same angle α to $\mathrm{Tr} = 0$, the "volume" $\lambda([0,\log t]\times P)$ doesn't depend on the choises, and obviously equals $\log t \sin(\alpha)$ times the "area" of P. Since $\sin(\alpha) = \cos(\frac{\pi}{2} - \alpha)$, this volume is $\log t$ times the area of the projection of P onto the hyperplane orthogonal to our chosen axis. This area is obviously $R = |\det(\log|\varepsilon_i|_{p_j})|$. □

3. Multiplicative characters

The quasi-character c on I is a continuous homomorphism to $\mathbb{C}^\times$, so it is of the form

$$c(x) = \prod c_p(x_p)$$

where c_p are quasi-characters on k_p and all but finite number of them are trivial on u_p.

We consider only those quasi-characters c, which are trivial on $k^\times$. For them:

1) c restricted to J is a character. Indeed, since $J/k^\times$ is compact $|c(x)| = 1$ for $x \in J$.

2) If c is trivial on J then $c(y) = |y|^s$ for some $s \in \mathbb{C}$ uniquely defined by c.

3) For given c there exist a number $\sigma \in \mathbb{R}$ such that $|c(y)| = |y|^\sigma$. Indeed, $|c(\cdot)|$ is a quasi-character trivial on J, so in is $|\cdot|^\sigma$ for some $\sigma \in \mathbb{C}$. And $\sigma \in \mathbb{R}$ because this quasi-character takes values in $\mathbb{R}_+$.

The number σ is called an exponent of c. Quasi-character is a character if and only if its exponent is 0.

4. Zeta functions

The function $f : A \longrightarrow \mathbb{C}$ is "good" if
(i) f is continuous and in $L_1(A)$
(ii) $f(x)\Big|_I |x|^\sigma \in L_1(I)$ for $\sigma > 1$.
(iii) $\sum_{\xi\in k} f(x(y+\xi))$ is convergent for each idele x and each adele y, uniformly in (x, y) ranging over D times any fixed compact subset of I

Definition 2. *Suppose f and it's Fourier transform $\hat{f}$ are both "good". Then the following function of quasi-characters of exponent greater then 1*

$$\zeta(f, c) = \int_I f(x)c(x)dx_I$$

is called zeta function of the field k.

Let us call two quasi-characters equvalent if they are equal on J. Then equivalence class is $c_0(\cdot)|\cdot|^s$, $s \in \mathbb{C}$ where c_0 is any representative of the class. So, each equivalence class is a complex plane.

For a quasi-character c we define $\hat{c}(\cdot) = \frac{|\cdot|}{c(\cdot)}$. If σ is an exponent of c then $1 - \sigma$ is an exponent of $\hat{c}$.

Theorem 3. *We can extend $\zeta(f, \cdot)$ to the domain of all quasi-characters so that an extension is analytic on each equivalence class except the trivial one, where it has poles at $c = 1$ and $c = |\cdot|$ with residues $-\kappa f(0)$ and $+\kappa\hat{f}(0)$ correspondingly with $\kappa = \int_E dx_J$. Moreover,*

$$\zeta(f, c) = \zeta(\hat{f}, \hat{c}).$$

Proof.

$$\int_I f(x)c(x)dx_I = \int_0^\infty \left(\int_J f(tx)c(tx)dx_J\right)\frac{dt}{t},$$

so we consider $\zeta_t(f, c) = \int_J f(tx)c(tx)dx_J$. Then due to (iii)

$$\zeta_t(f, c) + f(0)c(t)\int_E c(x)dx_J = \sum_{\xi\in k^\times}\int_{\xi E} f(tx)c(tx)dx_J + f(0)c(t)\int_E c(x)dx_J$$

$$= \sum_{\xi\in k}\int_E f(\xi tx)c(tx)dx_J = \int_E\left(\sum_{\xi\in k} f(\xi tx)\right)c(tx)dx_J,$$

and applying Poisson summation formula to expression in brackets we get

$$= \int_E\left(\frac{1}{|tx|}\sum_{\xi\in k}\hat{f}(\frac{\xi}{tx})\right)c(tx)dx_J = \int_E\left(\sum_{\xi\in k}\hat{f}(\frac{\xi}{tx})\right)\hat{c}(\frac{1}{tx})dx_J$$

$$= \int_E\left(\sum_{\xi\in k}\hat{f}(\frac{\xi}{t}x)\right)\hat{c}(\frac{1}{t}x)dx_J$$

since modular function of $x \mapsto \frac{1}{x}$ is 1, and expression which we integrate is periodic under $x \mapsto \xi x$ for $\xi \in k^\times$. Analogously we get

$$\zeta_t(f, c) + f(0)c(t)\int_E c(x)dx_J = \zeta_{\frac{1}{t}}(\hat{f}, \hat{c}) + \hat{f}(0)\hat{c}(\frac{1}{t})\int_E \hat{c}(x)dx_J$$

If c is nontrivial on J we have $\int_E c(x)dx_J = 0$, otherwise it equals κ and $c(x) = |x|^s$. So we write

$$\zeta_t(f,c) = \zeta_{\frac{1}{t}}(\hat{f},\hat{c}) + \{\{\hat{f}(0)\kappa t^{s-1} - f(0)\kappa t^s\}\}$$

where expression in brackets is present only for characters of trivial class. So

$$\zeta(f,c) = \int_1^\infty \zeta_t(f,c)\frac{dt}{t} + \int_0^1 \zeta_t(f,c)\frac{dt}{t}$$

where expression under the first integral is convergent for characters of any exponent. Indeed, $\frac{|c_1(x)|}{|c_2(x)|} = |x|^{\sigma_1-\sigma_2} \geq 1$ when $|x| \geq 1$ and $\sigma_1 \geq \sigma_2$. Then

$$\zeta_t(f,c) = \int_1^\infty \zeta_t(f,c)\frac{dt}{t} + \int_1^\infty \zeta_t(\hat{f},\hat{c})\frac{dt}{t} + \{\{\hat{f}(0)\kappa\int_0^1 t^{s-2}dt - f(0)\kappa\int_0^1 t^{s-1}dt\}\}$$

$$\int_1^\infty \zeta_t(f,c)\frac{dt}{t} + \int_1^\infty \zeta_t(\hat{f},\hat{c})\frac{dt}{t} + \{\{\frac{\hat{f}(0)\kappa}{s-1} - \frac{f(0)\kappa}{s}\}\}.$$

□