Apéry's differential equations and elliptic curves

Masha Vlasenko

(join work with Vasily Golyshev)

Dublin, 13/03/2013

R.Apéry, 1978: $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} \notin \mathbb{Q}$

$$(n+1)^3 u_{n+1} \ = \ (34n^3 + 51n^2 + 27n + 5)u_n - n^3 u_{n-1}$$

$$u_0 = 1, \quad u_1 = 5 \quad \rightsquigarrow \ u_n \in \mathbb{Z}$$

$$u_0 = 0, \quad u_1 = 6 \quad \rightsquigarrow \ \ell_n^3 u_n \in \mathbb{Z}, \ell_n \ = \ \text{l.c.m.}(1, 2, \ldots, n)$$

We denote the first sequence by $\{a_n; n = 0, 1, \ldots\}$ and the second one by $\{b_n; n = 0, 1, \ldots\}$:

$$a_n : 1, 5, 73, 1445, 33001, 819005, \ldots$$

$$b_n : 0, 6, 351/4, 62531/36, 11424695/288, \ldots$$

$$0 < b_n - a_n\zeta(3) < (\sqrt{2} - 1)^{4n}$$

$$\ell_n^3(b_n - a_n\zeta(3)) \to 0 \Rightarrow \zeta(3) \notin \mathbb{Q}$$

How to prove that $a_n \in \mathbb{Z}$?

$$(n+1)^3 a_{n+1} \;=\; (34n^3 + 51n^2 + 27n + 5)a_n - n^3 a_{n-1}$$

R. Apéry : $a_n = \sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2$

F.Beukers: proof using modular forms

$$\Gamma_0(6)^+ = \Gamma_0(6) \cup \Gamma_0(6)W \qquad W = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & -1 \\ 6 & 0 \end{pmatrix}$$

$$t(\tau) \;=\; \left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)^{12} \;=\; q - 12q^2 + 66q^3 - 220q^4 + \ldots$$

$$f(\tau) \;=\; \frac{(\eta(2\tau)\eta(3\tau))^7}{(\eta(\tau)\eta(6\tau))^5} \;=\; 1 + 5q + 13q^2 + 23q^3 + 29q^4 + \ldots$$

$$\Big[\big(t\frac{d}{dt}\big)^3 \;-\; t\Big(34\big(t\frac{d}{dt}\big)^3 + 51\big(t\frac{d}{dt}\big)^2 + 27\big(t\frac{d}{dt}\big) + 5\Big) + t^2\big(t\frac{d}{dt} + 1\big)^3\Big]f \;=\; 0$$

$$f(\tau) = 1 + 5t(\tau) + 73t(\tau)^2 + \ldots \;=\; \sum_{n=0}^{\infty} a_n t(\tau)^n$$

$$\begin{cases} f(q) \in 1 + q\mathbb{Z}[[q]] \\ t(q) \in q + q^2\mathbb{Z}[[q]] \end{cases} \Rightarrow \; a_n \in \mathbb{Z}$$

$$D^3 \;-\; t(34D^3 + 51D^2 + 27D + 5) \;+\; t^2(D+1)^3 \quad D = t\frac{d}{dt}$$

$$D^3 \;-\; t\Big(D + \frac{1}{2}\Big)\Big(34(D^2 + D) + 10\Big) \;+\; t^2(D+1)^3$$

$$\downarrow$$

$$D^3 \;+\; t\Big(D + \frac{1}{2}\Big)\Big(\alpha(D^2 + D) + \beta\Big) \;+\; \gamma\, t^2(D+1)^3$$

Problem. Find values of parameters α, β, γ such that the corresponding differential operator has modular parametrisation.

Definition. Apéry's operator is a differential operator of the form

$$\begin{aligned} D^3 \;&+\; t\big(D+\tfrac{1}{2}\big)\big(\alpha_3(D^2+D) \;+\; \beta_1\big) \\ &+\; t^2(D+1)\big(\alpha_2(D+1)^2 \;+\; \beta_0\big) \\ &+\; \alpha_1 t^3\,(D+2)\big(D+\tfrac{3}{2}\big)(D+1) \\ &+\; \alpha_0 t^4\,(D+3)(D+2)(D+1) \end{aligned}$$

where $D = t\frac{d}{dt}$ and parameters $\alpha_3, \alpha_2, \alpha_1, \alpha_0, \beta_1, \beta_0 \in \mathbb{Q}$.

V. Golyshev, J.Stienstra, "Fuchsian equations of type DN", 2007
V. Golyshev, "Classification problems and mirror duality", 2005

Remark: With $(\alpha_3, \alpha_2, \alpha_1, \alpha_0, \beta_1, \beta_0) = (\alpha, \gamma, 0, 0, \beta, \gamma)$ we obtain the previous operator.

$$
\begin{aligned}
D^3 \;+\; & t\big(D+\frac{1}{2}\big)\big(\alpha_3(D^2+D) \;+\; \beta_1\big) \\
& +\; t^2(D+1)\big(\alpha_2(D+1)^2 \;+\; \beta_0\big) \\
& +\; \alpha_1\, t^3(D+2)\big(D+\frac{3}{2}\big)(D+1) \\
& +\; \alpha_0\, t^4(D+3)(D+2)(D+1) \\
& =\; (1+\alpha_3 t+\alpha_2 t^2+\alpha_1 t^3+\alpha_0 t^4)\, D^3 \;+\; \dots
\end{aligned}
$$

To determine singular points we rewrite the coefficient near D^3 as

$$
1+\alpha_3 t+\alpha_2 t^2+\alpha_1 t^3+\alpha_0 t^4 \;=\; \prod_{i=1}^{4}\Big(1-\frac{t}{\lambda_i}\Big)\,.
$$

Definition. Apéry's operator is called non-degenerate if all λ_i are different.

Theorem.(Golyshev, Stienstra) A non-degenerate Apéry's operator has regular singularities located at $t=0,\lambda_1,\lambda_2,\lambda_3,\lambda_4$. The local monodromy around $t=0$ is maximally unipotent.

Consider the Frobenius basis near the point of maximal unipotent monodromy $t = 0$:

$$\begin{aligned}
\phi_0(t) &= 1 + u_1\, t + u_2\, t^2 + u_3\, t^3 + \dots \\
\phi_1(t) &= \log t\, \phi_0(t) + v_1\, t + v_2\, t^2 + v_3\, t^3 + \dots \\
\phi_2(t) &= (\log t)^2\, \phi_0(t) + \dots
\end{aligned}$$

In case our equation has modular parametrisation by $t(\tau), f(\tau)$ we must have

$$\phi_0(t(\tau)) = f(\tau) \quad \phi_1(t(\tau)) = 2\pi i\tau f(\tau) \quad \phi_2(t(\tau)) = (2\pi i\tau)^2 f(\tau)$$

$$\begin{aligned}
q = \exp(2\pi i\tau) &= \exp\Big(\frac{\phi_1(t)}{\phi_0(t)}\Big) = t\exp\Big(\frac{v_1 t + v_2 t^2 + \dots}{1 + u_1 t + \dots}\Big) \\
&= t + v_1 t^2 + (\frac{1}{2}v_1^2 - u_1 v_1 + v_2)t^3 + \dots
\end{aligned}$$

$$\rightsquigarrow \quad t(q) = q - v_1 q^2 + \dots$$

$$\phi_0(t) = 1 - \frac{1}{2}\beta_1 t + \Big(\frac{3}{16}\alpha_3\beta_1 + \frac{3}{32}\beta_1^2 - \frac{1}{8}\alpha_2 - \frac{1}{8}\beta_0\Big)t^2 + \dots$$

$$\phi_1(t) = \log t\,\phi_0(t) + \psi(t) = \log t\,\phi_0(t) + \Big(-\frac{1}{2}\alpha_3 + \frac{1}{2}\beta_1\Big)t + \dots$$

$$q = \exp\Big(\frac{\phi_1(t)}{\phi_0(t)}\Big) = t + \Big(-\frac{1}{2}\alpha_3 + \frac{1}{2}\beta_1\Big)t^2 + \dots$$

$$t = q + \Big(\frac{1}{2}\alpha_3 - \frac{1}{2}\beta_1\Big)q^2 + \dots$$

$$f = \phi_0(t(\tau)) = 1 - \frac{1}{2}\beta_1 q + \dots$$

to find modular Apéry operators $\Leftrightarrow$ to determine α_i's and β_j's such that these are expansions of a modular function and a modular form of weight 2

Definition. The spectral elliptic curve of a non-degenerate Apéry's operator is (the resolution of the singularity at ∞ of)

$$y^2 = z^4 + \alpha_3 z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0 .$$

Theorem.(M.V., V.Golyshev) Suppose a non-degenerate Apéry's operator has modular parametrisation and the modular form of weight 2

$$t\,\phi_0(t) = \sum_{n=1}^{\infty} c_n q^n , \qquad q = \exp\Big(\frac{\phi_1(t)}{\phi_0(t)}\Big)$$

is in addition a newform. Then

$$\sum_{n=1}^{\infty} \frac{c_n}{n^s}$$

is the L-function of the spectral elliptic curve.

Idea of proof:

First we show that

$$\sum c_n q^n \, \frac{dq}{q} \;=\; t\phi_0(t)\, d\, \frac{\phi_1(t)}{\phi_0(t)} \;=\; t\,\frac{\phi_1'(t)\phi_0(t) - \phi_1(t)\phi_0(t)}{\phi_0(t)}\, dt$$

pulls back to a holomorphic differential on the spectral elliptic curve $\mathcal{E}$ under the map to $\mathbb{P}^1$ given by $(y, z) \mapsto t = 1/z$.

t (and hence q) can be considered as a local parameter on $\mathcal{E}$ near the origin.

Theorem (Atkin & Swinnerton-Dyer)
Let $p \neq 2, 3$, and let $y^2 = z^3 + Bz + C$ be an elliptic curve over $\mathbb{Z}_p$ with good reduction. Choose a local parameter at 0 so that $z = \xi^{-2} + \sum_{n=-1}^{\infty} d_n \xi^n$ and $y = \xi^{-3} + \ldots$ are the respective expansions , and write

$$-\frac{1}{2}\frac{dz}{y} \;=\; \Big(\sum_{n=1}^{\infty} c_n \xi^n\Big)\frac{d\xi}{\xi}\,.$$

If B, C, d_n, c_n are p-adic integers, then

$$c_{np} \,-\, a_p\, c_n \,+\, p\, c_{\frac{n}{p}} \;\equiv\; 0 \mod p^{\mathrm{ord}_p(n)+1}$$

where

$$a_p \;=\; -\sum_{m=0}^{p-1}\Big(\frac{m^3 + Bm + C}{p}\Big)\,.$$

In our case

$$-\frac{1}{2}\frac{dz}{y} \;=\; \sum_{n=1}^{\infty} c_n q^n \,\frac{dq}{q}$$

and q is a local parameter on $\mathcal{E}$ near the origin.

Therefore for almost all p we have Atkin & Swinnerton-Dyer congruences

$$c_{np} \;-\; a_p\, c_n \;+\; p\, c_{\frac{n}{p}} \;\equiv\; 0 \quad \mathrm{mod}\ p^{\mathrm{ord}_p(n)+1}$$

where a_p are the coefficients of the L-function of this elliptic curve

$$L(s) = \sum_n \frac{a_n}{n^s}\,.$$

modularity $\Rightarrow \quad c_n, a_n = o(n) \quad \Rightarrow \quad c_n = a_n$ for almost all n
multiplicity one theorem for newforms $\Rightarrow \quad c_n = a_n$ for all n (End)

Corollary: one has $c_{mn} = c_n \cdot c_m$ whenever $(m, n) = 1$.

Solving these equations

$$\begin{aligned} c_6 &= c_2 \cdot c_3 \\ c_{10} &= c_2 \cdot c_5 \\ c_{15} &= c_3 \cdot c_5 \\ &\cdots \end{aligned}$$

w.r.t. to the parameters $\vec{\alpha}, \vec{\beta}$ we obtain the list of all cases when the Theorem is satisfied. The list appears to be finite modulo 1-parametric transformations

$$t \mapsto \frac{t}{1 - \varepsilon t} \qquad f \mapsto f \cdot (1 + \varepsilon t)$$

which act on $\vec{\alpha}$ and $\vec{\beta}$ as

$$(\alpha_3, \alpha_2, \alpha_1, \alpha_0) \mapsto = \ (\alpha_3 - 4\varepsilon, \alpha_2 - 3\varepsilon\alpha_3 + 6\varepsilon^2,$$
$$\alpha_1 - 2\varepsilon\alpha_2 + 3\varepsilon^2\alpha_3 - 4\varepsilon^3, \alpha_0 - \varepsilon\alpha_1 + \varepsilon^2\alpha_2 - \varepsilon^3\alpha_3 + \varepsilon^4)$$
$$(\beta_1, \beta_0) \mapsto (\beta_1 - 2\varepsilon, \beta_0 - \beta_1\varepsilon + \varepsilon^2)$$

All non-degenerate Apéry's operators with multiplicative $c_n(\vec{\alpha}, \vec{\beta})$:

α_3	α_2	α_1	α_0	$j(\mathcal{E})$	$N(\mathcal{E})$	$g_{\mathcal{E}}(\tau)$	β_0
0	-44	0	-16	$-\frac{20720464}{15625}$	20	$\mathbf{2^2 \cdot 10^2}$	-4
0	44	0	-16		80	$\mathbf{4^6 \cdot 20^6 / 2^2 \cdot 8^2 \cdot 10^2 \cdot 40^2}$	4
0	-28	0	-128	$\frac{207646}{6561}$	24	$\mathbf{2 \cdot 4 \cdot 6 \cdot 12}$	-4
0	28	0	-128		48	$\mathbf{4^4 \cdot 12^4 / 2 \cdot 6 \cdot 8 \cdot 24}$	4
0	-40	0	144	$\frac{35152}{9}$	24	$\mathbf{2 \cdot 4 \cdot 6 \cdot 12}$	-8
0	40	0	144		48	$\mathbf{4^4 \cdot 12^4 / 2 \cdot 6 \cdot 8 \cdot 24}$	8
0	-68	0	1152	$\frac{3065617154}{9}$	48	$\mathbf{4^4 \cdot 12^4 / 2 \cdot 6 \cdot 8 \cdot 24}$	-28
0	68	0	1152		24	$\mathbf{2 \cdot 4 \cdot 6 \cdot 12}$	28
0	-48	0	512	287496	32	$\mathbf{4^2 \cdot 8^2}$	-16
0	48	0	512		32	$\mathbf{4^2 \cdot 8^2}$	16
0	0	0	-256	1728	32	$\mathbf{4^2 \cdot 8^2}$	0
0	0	0	256		64	$\mathbf{8^8 / 4^2 \cdot 16^2}$	0
0	-36	0	432	54000	144	$\mathbf{12^{12} / 6^4 \cdot 24^4}$	-12
0	36	0	432		36	$\mathbf{6^4}$	12
-4	-88	-300	-304	$-\frac{122023936}{161051}$	11	$\mathbf{1^2 \cdot 11^2}$	-8
0	0	-108	0	0	27	$\mathbf{3^2 \cdot 9^2}$	0
-2	-43	-156	-216	$\frac{4733169839}{3515625}$	15	$\mathbf{1 \cdot 3 \cdot 5 \cdot 15}$	-5
-2	-59	-136	-80	$\frac{4956477625}{941192}$	14	$\mathbf{1 \cdot 2 \cdot 7 \cdot 14}$	-5