Linear Mahler Measures and Double L-values of Modular Forms

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The Mahler measure of a Laurent polynomial

$$P(x_1, \dots, x_n) \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

is defined as

$$m(P) = \frac{1}{(2\pi i)^n} \int_{|x_1| = \cdots = |x_n| = 1} \log |P(x_1, \dots, x_n)| \, \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}.$$

This number is obviously a *period* in the sense of Kontsevich and Zagier.
For a monic polynomial in one variable we can compute $m(P)$ by *Jensen's formula*:

$$\frac{1}{2\pi i} \int_{|x|=1} \log |P(x)| \, \frac{dx}{x} = \sum_{\alpha : P(\alpha) = 0} \max(0, \log |\alpha|)$$

C. Smyth, $\approx$ 1980:

$$m(1+x_1+x_2) = \frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2)$$

where $L(\chi_{-3},s) = \sum_{n=1}^{\infty}\frac{\chi_{-3}(n)}{n^s} = 1-\frac{1}{2^s}+\frac{1}{4^s}-\frac{1}{5^s}+\dots$

$$m(1+x_1+x_2+x_3) = \frac{7}{2\pi^2}\zeta(3)$$

$$m(1+x_1+x_2+x_3+x_4) = ?$$

Conjecture (F. Rodriguez Villegas):

$$m(1 + x_1 + x_2 + x_3 + x_4) \stackrel{?}{=} 6\left(\frac{\sqrt{-15}}{2\pi i}\right)^5 L(f_{15}, 4)$$

where

$$f_{15} = \eta(3z)^3\eta(5z)^3 + \eta(z)^3\eta(15z)^3$$

is a CM modular form of weight 3 and level 15. This form corresponds to a Galois representation arising from the variety

$$\begin{cases} 1 + x_1 + x_2 + x_3 + x_4 = 0 \\ 1 + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} = 0 \end{cases} \Leftrightarrow (1+x_1+x_2+x_3)\left(1+\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}\right) = 1$$

which can be compactified to a $K3$ surface of Picard rank 20.

Mahler's Measures and Differential Equations

For a Laurent polynomial P the sequence

$$a_m = \text{the free coeffient of } P(x_1, \ldots, x_n)^m$$

always satisfies a recursion, which can be written as a differential equation for the generating function:

$$a(t) = \sum_{m=0}^{\infty} a_m t^m \qquad \mathcal{L}(t, t\frac{d}{dt})a(t) = 0$$

Observe that

$$m(1 + x_1 + \cdots + x_n) \;=\; \frac{1}{2} m(P_n)$$

where

$$P_n \;=\; \Big(1 + x_1 + \cdots + x_n\Big)\Big(1 + \frac{1}{x_1} + \cdots + \frac{1}{x_n}\Big)$$

and consider the sequence of the free coefficients of the powers of P_n:

$$n = 2 \qquad a_m : 1\,,\; 3\,,\; 15\,,\; 93\,,\; 639\; \ldots$$

$$n = 3 \qquad a_m : 1\,,\; 4\,,\; 28\,,\; 256\,,\; 2716\; \ldots$$

$$n = 4 \qquad a_m : 1\,,\; 5\,,\; 45\,,\; 545\,,\; 7885\; \ldots$$

The corresponding differential equations

$$\mathcal{L}_n\left(t, t\frac{d}{dt}\right) a(t) \; = \; 0$$

are given by

$$\mathcal{L}_2(t,\theta) \; = \; \theta^2 - t(10\theta^2 + 10\theta + 3) + t^2(9\theta^2 + 18\theta + 9)$$

$$\mathcal{L}_3(t,\theta) \; = \; \theta^3 - 2t(2\theta + 1)(5\theta^2 + 5\theta + 2) + 64t^2(\theta + 1)^3$$

$$\mathcal{L}_4(t,\theta) \; = \; \theta^4 - t(35\theta^4 + 70\theta^3 + 63\theta^2 + 28\theta + 5)$$
$$+ t^2(\theta + 1)^2(259\theta^2 + 518\theta + 285) - 225t^3(\theta + 1)^2(\theta + 2)^2$$

These differential equations for $n = 2, 3$ admit modular parametrization:

$$n = 2:$$

$$t = \frac{\eta(6z)^8\eta(z)^4}{\eta(3z)^4\eta(2z)^8} = q - 4q^2 + 10q^3 + \dots$$

$$a = \frac{\eta(2z)^6\eta(3z)}{\eta(z)^3\eta(6z)^2} = 1 + 3q + 3q^2 + \dots$$

$n = 3$:

$$t = -\Big(\frac{\eta(2z)\eta(6z)}{\eta(z)\eta(3z)}\Big)^6 \quad a = \frac{(\eta(z)\eta(3z))^4}{(\eta(2z)\eta(6z))^2}$$

$n = 4$:

$\mathcal{L}_4(t, t\frac{d}{dt})$ is not a symmetric power,

hence there is no modular parametrization . . .

The generating function $a(t)$ is related to the Mahler measure by the following trick due to F. Villegas:

$$\frac{1}{(2\pi i)^n}\int_{|x_i|=1}\log\left(\frac{1}{t}-P(x_1,\dots,x_n)\right)\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}$$

$$= -\log t-\sum_{m=1}^{\infty}\frac{t^m}{m}\frac{1}{(2\pi i)^n}\int_{|x_i|=1}P(x_1,\dots,x_n)^m\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}$$

$$= -\log t-\sum_{m=1}^{\infty}\frac{t^m}{m}a_m \;=\; -\left(t\frac{d}{dt}\right)^{-1}a(t)$$

hence

$$m(P) \;=\; -\mathrm{Re}\,\left(t\frac{d}{dt}\right)^{-1}a(t)\Big|_{t=\infty}$$

Suppose the differential equation has a modular parametrization, then this reduces to evaluation at the cusp where $t = \infty$ of

$$\left(t\frac{d}{dt}\right)^{-1} a(t) \;=\; \left(\frac{t}{Dt} D\right)^{-1} a \;=\; D^{-1}\left[\frac{Dt}{t} a\right],$$

where $D = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{dz}$, and we reprove C.Smyth's formulas:

$$m(1 + x_1 + x_2) \;=\; \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2)$$

$$m(1 + x_1 + x_2 + x_3) \;=\; \frac{7}{2\pi^2}\zeta(3)$$

Idea: $m(P_n)$ can be also calculated as

$$m(P_n) \;=\; -\mathrm{Re}\,\left(t\frac{d}{dt}\right)^{-1} b(t)\Big|_{t=\infty}$$

where $b(t)$ is a solution of

$$\mathcal{L}_{n-1}\left(t, t\frac{d}{dt}\right) b(t) \;=\; h(t)$$

with some simple rational function $h(t)$ in the right-hand side.

Recall that $\mathcal{L}_4$ is not modular, but $\mathcal{L}_3$ is!

Theorem 1. Consider the following analytic at $t = 0$ solutions $a(t)$ and $b(t)$ of

$$\mathcal{L}_2\left(t, t\frac{d}{dt}\right) a(t) \;=\; 0 \qquad a(t) = 1 + 3t + \dots$$

$$\mathcal{L}_2\left(t, t\frac{d}{dt}\right) b(t) \;=\; \frac{1}{1-t} \qquad b(t) = \frac{1}{9}t + \frac{2}{3}t^2 + \dots$$

Then

$$m(P_3) \;=\; -\frac{27\sqrt{3}}{8\pi^3}\left(t\frac{d}{dt}\right)^{-1}\left[\frac{\pi^2}{72}a(t) + b(t)\right]\Big|_{t=\infty}.$$

Theorem 2. Consider the solution $b(t) = \frac{4}{5} + O(t)$ of

$$\mathcal{L}_3\big(t, t\frac{d}{dt}\big)\, b(t) \;=\; h(t)$$

where

$$h(t) \;=\; -\frac{3\sqrt{5}\Omega^2}{10\pi}\frac{t(212t^2+251t-13)}{(1-t)^3} + \frac{3\sqrt{5}}{5\pi^3\Omega^2}\frac{t}{1-t},$$

and $\Omega = \frac{1}{\sqrt{30\pi}}\big(\prod_{j=1}^{14}\Gamma(j/15)^{\chi_K(j)}\big)^{1/4}$ is the Chowla-Selberg period for the field $K = Q(\sqrt{-15})$. Then

$$m(P_4) \;=\; -\big(t\frac{d}{dt}\big)^{-1}\, b(t)\Big|_{t=\infty}.$$

Proof: Consider the 1-parametric family of varieties

$$X_\lambda \; : \; P_n(x_1, \dots, x_n) \; = \; \lambda$$

and define

$$\Omega_n(\lambda) \; = \; \int_{X_\lambda \cap \{|x_i|=1\}} \omega_\lambda$$

where the $(n-1)$-form ω_λ is defined by

$$\frac{1}{(2\pi i)^n} \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_n}{x_n} \; = \; \omega_\lambda \wedge d\lambda.$$

Then $\Omega_n(\lambda)$ is a period for this family and satisfies the Picard-Fuchs differential equation

$$\widetilde{\mathcal{L}}_n(\lambda, \lambda \frac{d}{d\lambda}) \Omega_n(\lambda) \; = \; 0\,.$$

We obviously have

$$m(P_n) \;=\; \int_0^{(n+1)^2} \log(\lambda)\Omega_n(\lambda)d\lambda\,,$$

and by Jensen's formula

$$\frac{1}{(2\pi i)^{n+1}}\int_{|x_i|=1} \log|1+x_1+\cdots+x_{n+1}|\,\frac{dx_1}{x_1}\ldots\frac{dx_{n+1}}{x_{n+1}}$$
$$=\;\frac{1}{(2\pi i)^{n}}\int_{|x_i|=1,|1+x_1+\cdots+x_n|>1} \log|1+x_1+\cdots+x_n|\,\frac{dx_1}{x_1}\ldots\frac{dx_n}{x_n}$$

or

$$m(P_{n+1}) \;=\; \int_1^{(n+1)^2} \log(\lambda)\Omega_n(\lambda)d\lambda\,.$$

We observe that if $\Omega(\lambda)$ satisfies

$$\widetilde{\mathcal{L}}(\lambda, \lambda\frac{d}{d\lambda})\Omega(\lambda) \;=\; 0$$

then the generating function for the moments along an arbitrary path

$$b_n \;=\; \int_\alpha^\beta \lambda^n \Omega(\lambda) d\lambda \qquad b(t) = \sum_{n=0}^{\infty} b_n t^n$$

satisfies

$$\mathcal{L}(t, t\frac{d}{dt})\, b(t) \;=\; h_\beta(t) - h_\alpha(t)$$

where $\mathcal{L}(t,\theta) \;=\; \widetilde{\mathcal{L}}(1/t, -\theta - 1)$ and $h_\alpha(t)$ is a simple rational function which depends only on the values of Ω and its derivatives at $\lambda = \alpha$ and can have a pole only at $t = \alpha$.

In the case of $\mathcal{L}_n$ we have $h_\alpha(t) = 0$ for $\alpha = 0$ and $\alpha = (n+1)^2$.

For the modular parametrization $t(z)$ of $\mathcal{L}_3$ preimages of $t = 1$ are CM points in the field $K = Q(\sqrt{-15})$, therefore the Chowla-Selberg period for this field appears when we compute the right-hand side $h_\alpha(t)$ with $\alpha = 1$.

(The end of the proof.)

... and Double L-values of Modular Forms

To compute $m(P_3)$ and $m(P_4)$ using the above theorems we have to evaluate

$$\left(t\frac{d}{dt}\right)^{-1}a(t)\Big|_{t=\infty} \quad \text{and} \quad \left(t\frac{d}{dt}\right)^{-1}b(t)\Big|_{t=\infty}$$

where $a(t)$, $b(t)$ are solutions of

$$\mathcal{L}\left(t, t\frac{d}{dt}\right)a(t) = 0 \quad \text{and} \quad \mathcal{L}\left(t, t\frac{d}{dt}\right)b(t) = h(t)$$

and $\mathcal{L}$ has a modular parametrization, i.e. we are given a modular function $t(q)$ and a modular form $a(q)$ of weight k, where $k+1$ is the degree of the differential operator $\mathcal{L}$.

Let $D = q\frac{d}{dq}$, and suppose our modular parametrization is such that $t = \infty$ corresponds to $q = 1$. Then

$$\left(t\frac{d}{dt}\right)^{-1} a(t)\Big|_{t=\infty} = \left(\frac{t}{Dt}D\right)^{-1} a\Big|_{q=1} = D^{-1}f\Big|_{q=1} = L(f,1)$$

for the modular form $f = Dt \cdot a/t$ of weight $k+2$. Here for $f = \sum_{m=0}^{\infty} a_m q^m$ the L-function is defined by

$$L(f,s) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}$$

for sufficiently large $\mathrm{Re}(s)$ and by analytic continuation otherwise.

Analogously, since the pull-back of $\mathcal{L}$ under the modular parametrization is given (up to a contant multiplier) by

$$\mathcal{L}\left(t, t\frac{d}{dt}\right) \doteq \frac{1}{Dt \cdot a} D^{k+1} \frac{1}{a}$$

we have $D^{k+1}[b/a] = Dt \cdot a \cdot h(t)$. In other words, b/a is an Eichler integral of a modular form of weight $k+2$. Finally,

$$\left(t\frac{d}{dt}\right)^{-1} b(t)\Big|_{t=\infty} = D^{-1}[f \cdot D^{-(k+1)} g]\Big|_{q=1} = L(g, f, k+1, 1)$$

where $f = Dt \cdot a/t$ and $g = Dt \cdot a \cdot h(t)$ are both of weight $k+2$.

For two forms of weight k

$$g = \sum_{n=1}^{\infty} a_n q^n \qquad f = \sum_{m=0}^{\infty} b_m q^m$$

(g is a cusp form) their L-function is defined by

$$L(g, f, s_1, s_2) \ = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n b_m}{n^{s_1}(n+m)^{s_2}} \, .$$

for $\mathrm{Re}(s_1 + s_2) > 2k$, $\mathrm{Re}(s_2) > k$ and by analytic continuation otherwise. Critical double L-values ($0 < s_1, s_2 < k$, integers) are periods in the sense of Kontsevich and Zagier.

Applying the above strategy in the known case $m(P_3) = \frac{7\zeta(3)}{\pi^2}$ we get the following equality:

$$-\frac{14\pi}{3\sqrt{3}}\zeta(3) + \frac{\pi^2}{4}L'(\chi_{-3}, -1) \ = \ L(g, f, 2, 1)$$

or

$$-\frac{2\pi^3}{3\sqrt{3}}m(P_3) \ + \ \frac{\pi^2}{8}m(P_2) \ = \ L(g, f, 2, 1)$$

where the forms g, f of weight 3 are given by

$$\begin{aligned} g \ &= \ q + 4q^2 + q^3 - 16q^4 + \ldots \ = \ E(z) + 7E(2z) - 8E(4z) \\ f \ &= \ 1 + q - 5q^2 + q^3 + \ldots \qquad = \ E(z) - 2E(2z) - 8E(4z) \end{aligned}$$

with $E(z) \ = \ -\frac{1}{9} + \sum_{n\geq 1}\sum_{d|n}\chi_{-3}(d)d^2q^n \in M_3(\Gamma_0(3), \chi_{-3})$.

Analogously, the number

$$m(P_4) \quad \left(\overset{?}{=} \ 12 \left(\frac{\sqrt{-15}}{2\pi i} \right)^5 L(f_{15}, 4) \right)$$

from Villegas' conjecture is an expression involving π, the Chowla-Selberg period Ω_K and double L-values of meromorphic modular forms with poles at CM points in the same field $K = \mathbb{Q}(\sqrt{-15})$.