Higher Hasse–Witt matrices

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Motivation: zeta functions and periods

- $X/\mathbb{F}_q$ smooth projective variety, $q = p^a$

zeta function of X:

$$Z(X/\mathbb{F}_q;T) \;=\; \exp\Big(\sum_{m=1}^{\infty} \frac{\#X(\mathbb{F}_{q^m})}{m} T^m\Big) \;=\; \prod_{i=0}^{2\dim X} P_i(T)^{(-1)^{i+1}}$$

$$P_i(T) \;=\; \det(1 \;-\; T\cdot Frob_q \,|\, H^i_{crys}(X)) \in \mathbb{Z}[T]$$

- $X/\mathbb{Q}[t] \leftrightsquigarrow$ family of varieties X_t

ω_t an algebraic differential i-form on X_t, $\xi_t \in H_i(X_t;\mathbb{Z})$ a cycle
period $p(t) = \int_{\xi_t} \omega_t$ satisfies the Picard–Fuchs diff. equation

$X_t : y^2 = x(x-1)(x-t)$ The elliptic integral $p(t) \;=\; \int_1^{\infty} \frac{dx}{\sqrt{x(x-1)(x-t)}}$
satisfies $t(1-t)p''(t) + (1-2t)p'(t) - \frac{1}{4}p(t) = 0$.

- B. Dwork "Deformation theory for zeta functions" (1962):

can one use solutions to the Picard–Fuchs differential equations to give a p-adic analytic formula for $Z(X_t/\mathbb{F}_p;T)$?

Zeta function modulo p

$X/\mathbb{F}_q$ smooth projective variety, $q = p^a$

Katz's congruence formula:

$$Z(X/\mathbb{F}_q;T) \quad \equiv \quad \prod_{i=0}^{\dim X} \det(1 - T \cdot F^a \,|\, H^i(X, \mathcal{O}_X))^{(-1)^{i+1}} \quad \mod p$$

$F : H^i(X, \mathcal{O}_X) \to H^i(X, \mathcal{O}_X)$ Frobenius map

p-linear map induced by $h \mapsto h^p$ on $\mathcal{O}_X$

Hasse–Witt operation
definition and some properties

X projective hypersurface (or complete intersection)
$\dim X = n$

$$F : H^n(X, \mathcal{O}_X) \to H^n(X, \mathcal{O}_X) \qquad \text{the Hasse–Witt operation}$$

- # p-adic unit *eigenvalues* of F^a = stable rank $r(F)$
- generically $\quad r(F) = \dim H^n(X, \mathcal{O}_X)$
  (Miller, Koblitz ≈1975)

Hasse–Witt operation

computation for hypersurfaces

$$X = \{f(x_0,\dots,x_{n+1}) = 0\} \subset \mathbb{P}^{n+1}\,, \quad \deg f = d > n+1$$

$$\begin{array}{ccccccccc} 0 \to & \mathcal{O}_{\mathbb{P}^{n+1}}(-d) & \xrightarrow{f} & \mathcal{O}_{\mathbb{P}^{n+1}} & \to & \mathcal{O}_X & \to 0 \\ & \downarrow f^{p-1}F & & \downarrow F & & \downarrow F & \\ 0 \to & \mathcal{O}_{\mathbb{P}^{n+1}}(-d) & \xrightarrow{f} & \mathcal{O}_{\mathbb{P}^{n+1}} & \to & \mathcal{O}_X & \to 0 \end{array}$$

$$H^n(X,\mathcal{O}_X) \quad \xrightarrow{\sim} \quad H^{n+1}(\mathbb{P}^{n+1},\mathcal{O}_{\mathbb{P}^{n+1}}(-d))$$

basis: $x^{-u} = x_0^{-u_0}\dots x_{n+1}^{-u_{n+1}}$, $u \in U \;=\; \{\sum_{i=0}^{n+1} u_i = d\,, u_i \in \mathbb{Z}_{\geq 1}\}$
Hasse–Witt matrix:

$$\big(F\big)_{u,v\in U} \;=\; \text{the coefficient of } x^{pv-u} \text{ in } f(x)^{p-1}$$

Main result

R commutative ring
$f = \sum_u a_u x^u \in R[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$

$\Delta(f) \subset \mathbb{R}^N$ Newton polytope of f = convex hull $\{u : a_u \neq 0\}$

$J = \Delta(f)^\circ \cap \mathbb{Z}^N$ internal integral points, $g = \#J$

$$(\beta_m)_{u,v \in J} \;=\; \text{the coefficient of } x^{(m+1)v-u} \text{ in } f(x)^m$$
$$\beta_m \in \mathrm{Mat}_{g\times g}(R)\,, \quad m \geq 0$$

p prime
$\alpha_s := \beta_{p^s-1}$, $s \geq 0$
$\alpha_0 \;=\; \beta_0 \;=\; Id_{g\times g}$
$\alpha_1 \;=\; \beta_{p-1} \pmod p$ = the "Hasse–Witt matrix")

Main result

Theorem 1. Assume there exists $\sigma \in \mathrm{End}(R)$, a lift of Frobenius on R/pR (i.e. $\sigma(a) \equiv a^p \mod p$ for any $a \in R$). Then:

(i) $\alpha_s \;\equiv\; \alpha_1 \cdot \sigma(\alpha_1) \cdot \ldots \cdot \sigma^{s-1}(\alpha_1) \mod p$

If all α_s are invertible over $\widehat{R} = \varprojlim R/p^s R$ then:

(ii) $\alpha_{s+1} \cdot \sigma(\alpha_s)^{-1} \;\equiv\; \alpha_s \cdot \sigma(\alpha_{s-1})^{-1} \mod p^s$

(iii) for any derivation $D : R \to R$

$$D(\alpha_{s+1}) \cdot \alpha_{s+1}^{-1} \;\equiv\; D(\alpha_s) \cdot \alpha_s^{-1} \mod p^s$$

Idea of proof

R ring with a pth power Frobenius lift $\sigma \in \mathrm{End(R)}$: $\sigma(a) \equiv a^p \mod p$ for any $a \in R$

For $a \in R$ define recursively $\delta_s(a) \in R$:

$$a^{p^s-1} \;=\; \delta_1(a) \cdot \sigma(a^{p^{s-1}-1}) \,+\, \delta_2(a) \cdot \sigma^2(a^{p^{s-2}-1}) \,+\, \ldots \,+\, \delta_s(a)\,.$$

$\delta_1(a) = a^{p-1}$
$\delta_2(a) = a^{p^2-1} - a^{p-1}\sigma(a^{p-1}) \;=\; a^{p-1}(a^{(p-1)p} - \sigma(a^{p-1})) \in pR$
Lemma. $\delta_s(a) \in p^{s-1}R$
apply in $R' = R[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$, define matrices

$$(\gamma_s)_{u,v \in J} \;=\; \text{the coefficient of } x^{p^s v - u} \text{ in } \delta_s(f) \in p^{s-1}\mathrm{Mat}_{\mathrm{g}\times\mathrm{g}}(R)$$

Lemma. $\alpha_s \;=\; \gamma_1 \cdot \sigma(\alpha_{s-1}) \,+\, \gamma_2 \cdot \sigma^2(\alpha_{s-2}) \,+\, \ldots \,+\, \gamma_s$

Corollary

Theorem 1 $\Rightarrow$ when the "Hasse–Witt matrix" $\alpha_1 \mod p$ is invertible,

$$\exists \quad F = \lim_{s\to\infty} \alpha_{s+1}\cdot\sigma(\alpha_s)^{-1} \in \mathrm{Mat}_{g\times g}(\widehat{R})$$

$$\nabla(D) = \lim_{s\to\infty} D(\alpha_s)\cdot\alpha_s^{-1} \in \mathrm{Mat}_{g\times g}(\widehat{R})$$

for every derivation $D \in Der(R)$

Observations:

- $F \equiv \alpha_1 \mod p$
- F is invertible

The rest of the talk: meaning of the limiting matrices ?

Example (elliptic curve)

$$E \; : \; y^2 \; = \; x^3 + ax + b \,, \quad a, b \in R$$

$$\frac{1}{2}\frac{dx}{y} \; = \; \Big(1 + 2a\,T^4 + 3b\,T^6 + 6a^2\,T^8 + 20ab\,T^{10} + \dots\Big)dT$$

$$= \; \Big(\sum_{m=1}^{\infty} c_m T^m\Big)\frac{dT}{T}\,, \quad T \; = \; -x/y$$

$$c_m \; = \; \begin{cases} \text{the coefficient of } x^{m-1} \text{ in } (x^3 + ax + b)^{(m-1)/2}\,, & m \text{ odd}, \\ \quad 0\,, & m \text{ even}. \end{cases}$$

$R = \mathbb{Z}$ or $\mathbb{Z}_p \Rightarrow$ Atkin and Swinnerton-Dyer congruences

$$c_m + a_p\,c_{m/p} \; + \; p\,c_{m/p^2} \equiv 0 \mod p^{\mathrm{ord}_p(m)}$$

$$a_p \; = \; p + 1 - \#E(\mathbb{F}_p)$$

Example (elliptic curve)

$$(\mathrm{ASD})\quad c_m + a_p\, c_{m/p} + p\, c_{m/p^2} \equiv 0 \mod p^{\mathrm{ord}_p(m)}$$

$$a_p = p + 1 - \#E(\mathbb{F}_p)$$

$$\beta_m = \text{the coefficient of } x^m y^m \text{ in } (x^3 + ax + b - y^2)^m$$

$$= (-1)^{m/2} \binom{m}{m/2} c_{m+1}\,; \quad \alpha_s = \beta_{p^s-1}$$

Theorem 1 $\Rightarrow$ if $p \not| \alpha_1 (\Leftrightarrow p \not| a_p \Leftrightarrow$ ordinary curve)

$$\exists \lim_{s\to\infty} \frac{\alpha_s}{\alpha_{s-1}} = \lambda \in \mathbb{Z}_p^\times \left(= \lim_{s\to\infty} c_{p^s}/c_{p^{s-1}} \right)$$

$$\mathrm{ASD} \quad \Rightarrow \quad c_{p^s} + a_p\, c_{p^{s-1}} + p\, c_{p^{s-2}} \equiv 0 \mod p^s$$

$$\lambda^2 + a_p\,\lambda + p = 0$$

$$\lambda = \text{the } p\text{-adic unit eigenvalue of Frobenius}$$

Proposition. When $R/pR = \mathbb{F}_p$ and

$$X_0 = X \times_{Spec(R)} Spec(R/pR) \;=\; \{\overline{f}(x) = 0\}$$

is a smooth projective variety over $\mathbb{F}_p$, then the eigenvalues of the limiting matrix $F = \lim_{s\to\infty} \alpha_{s+1} \cdot \sigma(\alpha_s)^{-1}$ are p-adic unit eigenvalues of the Frobenius operator on $H^n_{crys}(X_0)$.

$$\mathbb{F}_p \rightsquigarrow \mathbb{F}_q \qquad F \rightsquigarrow F \cdot \sigma(F) \cdot \ldots \sigma^{a-1}(F) \;=\; \lim_{s\to\infty} \alpha_{s+a} \cdot \alpha_s^{-1}$$

$$q \;=\; p^a$$

Strategy of proof: formal group laws & generalized ASD due to Stienstra

Formal group laws

A commutative *formal group law* of dimension g over a ring R is a tuple of g power series

$$G = (G_1, \ldots, G_g), \quad G_i = G_i(x,y) \in R[\![x,y]\!]$$
$$x = (x_1, \ldots, x_g), \; y = (y_1, \ldots, y_g)$$

satisfying

- $G(x,0) = G(0,x) = x$
- $G(x, G(y,z)) = G(G(x,y),z)$
- $G(x,y) = G(y,x)$

If $R \hookrightarrow R \otimes \mathbb{Q}$ (*characteristic* 0 *ring*), there exists the *logarithm*:

$$\log_G(x) := \ell(x) = x + \ldots \in (R \otimes \mathbb{Q})[\![x]\!]$$
$$G(x,y) = \ell^{-1}(\ell(x) + \ell(y))$$

Main result 2: formal group laws from a polynomial

R characteristic 0 ring; $f \in R[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$
$J = \Delta(f)^\circ \cap \mathbb{Z}^N$ or $\Delta(f) \cap \mathbb{Z}^N$
$g = \#J > 0$
$(\beta_m)_{u,v \in J}$ = the coefficient of $x^{(m+1)v-u}$ in $f(x)^m$

$$\ell(x) = (\ell_u(x))_{u \in J} := \sum_{m=1}^{\infty} \frac{1}{m} \beta_{m-1} x^m \qquad x = (x_v)_{v \in J}$$

$$G_f(x,y) := \ell^{-1}(\ell(x) + \ell(y)) \in (R \otimes \mathbb{Q})[\![x,y]\!]$$

Theorem 2. If R can be endowed with a pth power Frobenius endomorphism $\sigma \in \mathrm{End(R)}$, then G_f actually has coefficients in $R \otimes \mathbb{Z}_{(p)}$.

$$\mathbb{Z}_{(p)} = \left\{ \frac{m}{k} \; : \; p \nmid k \right\} \subset \mathbb{Q}\,.$$

Artin-Mazur formal groups and generalized ASD
J. Stienstra (1987)

$X = \{f(x) = 0\} \subset \mathbb{P}_R^{n+1}$ smooth projective

R ring of characteristic 0
$G_f(x, y)$ is a coordinalization of the Artin-Mazur formal group $H^n(X, \widehat{\mathbb{G}}_{m,X})$

$R = \mathbb{Z}$
$\det(1 - T \cdot F | H^n_{crys}(X_0)) \;=\; 1 + c_1 T + \ldots + c_k T^k \in \mathbb{Z}[T]$

$(ASD) \quad \alpha_s + c_1\,\alpha_{s-1} + c_2\,\alpha_{s-2} + \ldots + c_k\,\alpha_{s-k} \equiv 0 \mod p^s$

$\Rightarrow \qquad F^k + c_1 F^{k-1} + \ldots + c_{k-1} F + c_k = 0$

Example: hyperelliptic curve $y^2 = x^5 + 2x^2 + x + 1$

$$\beta_n = \text{ the coefficients of } \begin{pmatrix} x^n y^n & x^{2n+1}y^n \\ x^{n-1}y^n & x^{2n}y^n \end{pmatrix} \text{ in } (y^2 - x^5 - 2x^2 - x - 1)^n$$

$p = 11$:

s	0	1	2	3
$\alpha_s = \beta_{p^s - 1}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -81144 & -1260 \\ -81900 & -1260 \end{pmatrix}$	...	...
$\mathrm{tr}(\alpha_s \cdot \alpha_{s-1}^{-1}) \mod p^s$		$8 + O(11)$	$8 + 11 + O(11^2)$	$8 + 11 + 11^2 + ($
$\det(\alpha_s \cdot \alpha_{s-1}^{-1}) \mod p^s$		$7 + O(11)$	$7 + 6 \cdot 11 + O(11^2)$	$7 + 6 \cdot 11 + 3 \cdot 1$

$$\det\left(1 - T \cdot F \mid H^1_{crys}(C_0)\right) = 1 + 3T + 18T^2 + 3 \cdot 11T^3 + 11^2T^4$$

$$= (1 + 4T + 11T^2)(1 - T + 11T^2)$$

$$\lambda_{1,2} = -2 \pm \sqrt{-7}, \qquad \lambda_{3,4} = \frac{1 \pm \sqrt{-43}}{2}.$$

$$\lambda_1 = 7 + 2 \cdot 11 + 2 \cdot 11^2 + O(11^3)$$

$$\lambda_3 = 1 + 10 \cdot 11 + 9 \cdot 11^2 + O(11^3)$$

$$\lambda_1 + \lambda_3 = 8 + 11 + 11^2 + O(11^3)$$

$$\lambda_1 \cdot \lambda_3 = 7 + 6 \cdot 11 + 3 \cdot 11^2 + O(11^3)$$

Conjecture

Assume:

- $X = \{f(x_0, \dots, x_{n+1}) = 0\} \subset \mathbb{P}_R^{n+1}$ smooth
- R smooth algebra over $\mathbb{Z}_p$
- for every point of $Spec(R/pR)$, the Hasse-Witt operation for the respective fibre of X_0 is invertible

Then matrices

$$F \ = \ \lim_{s\to\infty} \alpha_{s+1} {\cdot} \sigma(\alpha_s)^{-1}, \quad \nabla(D) \ = \ \lim_{s\to\infty} D(\alpha_s) {\cdot} \alpha_s^{-1} \text{ for } D \in Der(R)$$

describe respectively the Frobenius operator and the Gauss–Manin connection on the *unit-root F-crystal* of X in the basis $[\omega_u] \in H^0(X, \Omega^n_{X/S})$, $u \in J = \Delta(f)^\circ \cap \mathbb{Z}^{n+2}$, where

$$\omega_u \ = \ res_X\Big(\frac{x^u}{f(x)} \sum_{i=0}^{n+1} (-1)^i \frac{dx_0}{x_0} \wedge \dots \widehat{\frac{dx_i}{x_i}} \dots \wedge \frac{dx_{n+1}}{x_{n+1}}\Big).$$

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Crystals

R smooth algebra over $\mathbb{Z}_p$, $\widehat{R} = \lim\limits_{\leftarrow} R/p^s R$

An *F-crystal* (H, ∇, F) over R is

- locally free $\widehat{R}$-module H
- integrable connection $\nabla : H \mapsto H \otimes \Omega_{\widehat{R}/\mathbb{Z}_p}$

$$\Leftrightarrow \qquad \nabla(D) : H \to H \qquad \text{for each } D \in Der(\widehat{R})$$

- for every lift of Frobenius $\sigma : \widehat{R} \to \widehat{R}$, a horizontal morphism
$$F(\sigma) : \sigma^* H \to H$$

$X/Spec(R)$ smooth projective; $X_0 = X \times_{Spec(R)} Spec(R/pR)$

$$H^{\cdot}_{dR}(X/Spec(R)) \otimes_R \widehat{R} \simeq H^{\cdot}_{crys}(X_0) \text{ is an F-crystal}$$

Example: variation of hypersurfaces

$$f_\Lambda(x_0,\ldots,x_{n+1}) \;=\; \textstyle\sum_{k=1}^{M} \Lambda_k\, x^{a_k}\,, \quad R \;=\; \mathbb{Z}[\Lambda_1,\ldots,\Lambda_M]$$
$$a_k = (a_{0k},\ldots,a_{(n+1)k}) \in \mathbb{Z}^{n+2} \text{ with } \textstyle\sum_{i=0}^{n+1} a_{ik} = d$$

$$X \;=\; \{f_\Lambda(x)=0\} \subset \mathbb{P}^{n+1}_R$$

$$\omega_u \;=\; res_X\Big(\frac{x^u}{f(x)}\sum_{i=0}^{n+1}(-1)^i\frac{dx_0}{x_0}\wedge\ldots\widehat{\frac{dx_i}{x_i}}\ldots\wedge\frac{dx_{n+1}}{x_{n+1}}\Big)$$

$$\nabla(\tfrac{\partial}{\partial\Lambda_i}) : H^n(X) \to H^n(X)$$

Question: construct differential operators that anihilate classes $[\omega_u] \in H^0(X,\Omega^n_X)$

Example: variation of hypersurfaces

A-hypergeometric system of PDEs

$$\mathcal{A} \;=\; \{a_1, \ldots, a_M\} \subset \mathbb{Z}^{n+2}\,, \quad a_k^+ = (a_{0k}, \ldots, a_{(n+1)k}, 1) \in \mathbb{Z}^{n+3}$$

$$L \;=\; \Big\{ l = (l_1, \ldots, l_M) \in \mathbb{Z}^M \mid \sum_{k=1}^{M} l_k a_k^+ = 0 \Big\} \quad \textit{lattice of relations}$$

The *A-hypergeometric system* of partial differential equations attached to the set $\mathcal{A} = \{a_k^+\}_{k=1}^M$ with the set of parameters $\mu = (\mu_0, \ldots, \mu_{n+2}) \in \mathbb{C}^{n+3}$ consists of *box operators*

$$\Box_l \;=\; \prod_{l_k>0} \Big(\frac{\partial}{\partial \Lambda_k}\Big)^{l_k} - \prod_{l_k<0} \Big(\frac{\partial}{\partial \Lambda_k}\Big)^{-l_k}\,, \quad l \in L$$

and *Euler (or homogeneity) operators*

$$Z_i \;=\; \sum_{k=1}^{M} a_{ik}\,\Lambda_k\,\frac{\partial}{\partial \Lambda_k} - \mu_i\,, \quad i = 0, \ldots, n+3\,.$$

Example: variation of hypersurfaces

$f_\Lambda(x) \;=\; \sum_{k=1}^{M} \Lambda_k\, x^{a_k}$, $\deg f_\Lambda = d$
$\Delta \;=\; \mathrm{span}\{a_1,\dots,a_M\} \subset \mathbb{R}^{n+2}$; $J = \Delta^\circ \cap \mathbb{Z}^{n+2}$

Proposition. (A. Adolphson & S. Sperber) Assume that
$J = \{u = (u_0,\dots,u_{n+1}) \;:\; u_i \in \mathbb{Z}_{\geq 1}\,, \quad \sum_{i=0}^{n+1} u_i = d\}$.
Then for each $u \in J$ the class $[\omega_u] \in H^n(X)$ is anihilated by the A-hypergeometric system of PDEs attached to the set $\mathcal{A} = \{a_k^+\}_{k=1}^M$ with the parameter $\mu = -u^+$.

Proposition. For any $u, v \in J$ the matrix entry

$$(\alpha_s)_{u,v}(\Lambda) \;=\; \text{the coefficient of } \; x^{p^s v - u} \;\text{ in }\; f_\Lambda(x)^{p^s-1}$$

is anihilated modulo p^s by the same A-hypergeometric system with $\mu = -u^+$.

Example: the Legendre family $y^2 = x(x-1)(x-t)$

the period integral

$$\omega(t) = \frac{1}{\pi}\int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-t)}} = 1 + \frac{1}{4}t + \frac{9}{64}t^2 + \ldots$$
$$= \sum_{k=0}^{\infty} \binom{2k}{k}^2 \left(\frac{t}{16}\right)^k$$

is anihilated by the hypergeometric differential operator

$$t(1-t)\frac{d^2}{dt^2} + (1-2t)\frac{d}{dt} - \frac{1}{4}$$

Example: the Legendre family $y^2 = x(x-1)(x-t)$

$$R = \mathbb{Z}[t],\ \sigma(h(t)) = h(t^p),\ \widehat{R} = \mathbb{Z}_p[\![t]\!]$$

$$\beta_m = \beta_m(t) = \text{the coefficient of } x^m y^m \text{ in } (y^2 - x(x-1)(x-t))^m$$
$$= \begin{cases} 0\,, & m \text{ odd,} \\ \binom{m}{m/2} \cdot \sum_{k=0}^{m/2} \binom{m/2}{k}^2 t^k\,, & m \text{ even.} \end{cases}$$

- $\left(t(1-t)\frac{d^2}{dt^2} + (1-2t)\frac{d}{dt} - \frac{1}{4}\right)\alpha_s(t) \equiv 0 \mod p^s$
- $\alpha_1(t) = \beta_{p-1}(t) = \binom{p-1}{(p-1)/2} + O(t)$ is invertible in $\widehat{R}$ when $p \neq 2$, and the limits from Theorem 1 are given by

$$F = (-1)^{\frac{p-1}{2}} \frac{\omega(t)}{\omega(t^p)}\,, \qquad \nabla\left(\frac{d}{dt}\right) = \frac{\omega(t)'}{\omega(t)}$$

Thank you!