The Riemann-Roch theorem for curves and Poisson summation

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These notes have been prepared for a seminar on Tates thesis at MPIM Bonn. Their aim is to give a short introduction to the Riemann-Roch theorem as well as to explain the connection with the Poisson summation formula as it appears in Tates thesis.

There is no doubt that the Riemann-Roch theory is of utmost importance in Mathematics and plays a central role not only in Algebraic Geomerty. It seems only logical that its importance reflects in the fact that the inital observations by Riemann about the existence of functions on Riemann surfaces having prescribed poles or zero opened the door to a huge theory generalizing the results by Riemann and Roch.

Unfortunately, it is impossible for me to cover in one session any of the interesting results going significantly further than the ideas carried by the initial Riemann-Roch theorem for complex curves as e.g. the work of Hirzebruch, Grothendieck, Faltings, Atiyah and Singer. My appologies for that.

I will omit citations in the rest of these notes. It has been made extensive use of the following sources: The proof of the Riemann-Roch theorem is taken from [Serre88] chapter 1, where one can find the case that k is algebraically closed. For the proof of the Serre-duality theorem in the non-closed case using residues the very nice article [Tate68] can be recommended. The proof over finite fields using Poisson summation is from [Rama99] chapter 7.2.

1. Riemann-Roch

Let X be an algebraic curve over a field k. By an algebraic curve we mean a regular, smooth, geometrically connected projective algebraic variety of dimension 1 over a field k, i.e. X is a one dimensional variety admitting an embedding into $\mathbb{P}_k^3$ such that the topological space $X \otimes_k \bar{k}$ is reduced, the local rings are regular and the sheaf of differentials is locally free of rank 1.

A divisor D of X is an element of the free Abelian group on the points of X, i.e.

$$D = \sum_{P \in X} n_P P.$$

To each function $f \in k(X)$ we associate the divisor $(f) := \sum_{P \in X} v_P(f) P$. Such divisors are called principal and we say two divisors D and D' are linearly equivalent if they differ by a principal divisor. The Riemann Roch theorem provides us with information about the dimensions of the spaces of functions having prescribed zeros and poles by a divisor in the following sense. For a Divisor $D = \sum n_P P$ of X denote by $\mathcal{L}(D)$ the sheaf of functions having poles no worse than these of D, i.e.:

$$\mathcal{L}(D)(U) := \{f \in k(X) \mid \forall P \in X \cap U : \; v_x(f) \geq -n_P\}$$

We are interested in the dimension of the global sections $\dim_k \mathcal{L}(D)(X)$. Clearly, if $D = D' + (f)$ for $f \in k(X)$ then multiplication with f defines an isomorphism between $\mathcal{L}(D)$ and $\mathcal{L}(D')$ and whence $\dim_k \mathcal{L}([D])(X)$ can be defined for divisor classes $[D]$.

For instance, consider $X = \mathbb{P}^1_k$. Here a divisor $D = \sum n_p P$ is equivalent to $\deg(D)\infty$ where

$$\deg(D) := \sum_{P \in X} n_P [\kappa(P) : k]$$

denotes the degree of a Divisor D and $\kappa(P)$ denotes the residue field of X at P. Clearly, here $\dim_k \mathcal{L}(D)(X)$ depends only on $\deg(D)$. In fact, one obtains

$$\dim_k \mathcal{L}(D)(\mathbb{P}^1_k) = \deg(D) + 1$$

for a divisor of $\deg(D) \geq 0$. However, for a curve whose Picard group, i.e. the group of divisors modulo linear equivalence, is not isomorphic to $\mathbb{Z}$, things get more complicated. Here the Riemann-Roch theorem will help.

We introduce the Euler-charateristic χ of a Divisor D by

$$\chi(D) := l(D) - i(D) := \dim_k H^0(X, \mathcal{L}(D)) - \dim_k H^1(X, \mathcal{L}(D))$$

and call

$$g := \dim_k H^1(X, \mathcal{O})$$

the genus g of X. With this terminlogy we can state a first version of the Riemann-Roch theorem:

Theorem 1. *Let D be a divisor on X. Then $\chi(D)$ as well as g is finite and one has:*

$$\chi(D) = \deg(D) + \chi(0) = deg(D) + 1 - g$$

Proof. The proof consists of two steps: First we establish the results for a particular divisor $D = 0$ and than we will use algebraic induction to finish the proof.

The sheaf $\mathcal{L}(0)$ is the structure sheaf $\mathcal{O}$ of X and therefore one has

$$l(0) = \dim_k H^0(X, \mathcal{L}(0)) = \dim_k H^0(X, \mathcal{O}) = 1$$

and

$$i(0) = \dim_k H^1(X, \mathcal{L}(0)) = \dim_k H^1(X, \mathcal{O}) < \infty.$$

Since, moreover, $\deg(0) = 0$ it is clear that the statement is true for $D = 0$.

Therefore, in the second step it remains to show that for every point $P \in X$ the assertions are true for $D + P$ if and only if they are true for D, since one can by definition write any divisor as a finite sum of $\sum (-1)^{\epsilon_i} P_i$ with $\epsilon_i \in \{0, 1\}$.

But this follows directly from the long exact cohomology sequence associated to the short exact sequence

$$0 \to \mathcal{L}(D) \to \mathcal{L}(D + P) \to \mathcal{Q} \to 0$$

Namely, one sees that the quotient sheaf $\mathcal{Q}$ is the skyscraper sheaf at P with $\mathcal{Q}_P \cong \kappa(P)$ whose cohomology on X can easly be computed via a Čech resolution to be $H^0(X, \mathcal{Q}) = \kappa(P)$ and $H^1(X, \mathcal{Q}) = 0$. And whence the long exact cohomology sequence shows that

$$0 \to \mathcal{L}(D) \to \mathcal{L}(D+P) \to \kappa(P) \to H^1(X, \mathcal{L}(D)) \to H^1(X, \mathcal{L}(D+P)) \to 0$$

is exact and therefore establishes the formulae

$$0 \leq l(D+P) - l(D) \leq [\kappa(P) : k]$$
$$0 \leq i(D) - i(D+P) \leq [\kappa(P) : k]$$
$$\chi(D+P) = \chi(D) + [\kappa(P) : k]$$

where the last formula follows in case one has shown that $\chi(D)$ and $\chi(D+P)$ is well defined, i.e. in case $i(D) < \infty$ or $l(D) < \infty$ and $i(D+P) < \infty$ or $l(D+P) < \infty$. Whence these formulas do now in turn imply

$$l(D), i(D) < \infty \wedge \chi(D) = \deg(D) + \chi(0)$$
$$\Leftrightarrow l(D+P), i(D+P) < \infty \wedge \chi(D+P) = \deg(D+P) + \chi(0)$$

□

This formula, however, still does not give us the insight in the dimension of the space of meromorphic functions with prescibed poles in the way the classical Riemann-Roch does. To obtain the classical Riemann-Roch theorem we need to apply Serre duality. Namely, in the next chapter we will show how to identifiy the dual of $H^1(X, \mathcal{L}(D))$ with $\Omega(D)$ and therefore prove $l(D) = \dim \Omega(D)$, since multiplication by ω for some $\omega \in \Omega$ defines an isomorphism of the $\mathcal{O}_X$ modules $\mathcal{L}(\mathrm{div}(\omega) - D)$ and $\Omega(D)$, one finaly obtains the classical Riemann Roch theorem.

2. Adèles, differential Forms, Residues and Duality

In what follows we introduce a version of Serre duality setting stage for the classical formulation of the Riemann-Roch theorem.

Let us start in reformulating $H^1(X, \mathcal{L}(D))$ using the adèlic language introduced by Weil: The language of rèpartitions. For this we define

$$R := \{\{r_P\}_{P \in X} \,|\, \forall P : r_P \in k(X), \text{f.a.a.}\, P : r_P \in \mathcal{O}_P\}$$
$$R(D) := \{\{r_P\} \in R \,|\, \forall P : v_P(r_P) \geq -v_P(D)\}$$

Now one has the injection of $k(X)$ onto the diagonal inside R and we henceforth will identify $k(X)$ with its image inside R. The link to $H^1(X, \mathcal{L}(D))$ is the following:

Proposition 1. *For every Divisor D on X there is a canonical isomorphism $H^1(X, \mathcal{L}(D)) \tilde{\to} R/(R(D) + k(X))$.*

Proof. This follows directly from the investigation of the long exact cohomology sequence associated to the short exact sequence

$$0 \to \mathcal{L}(D) \to k(X) \to k(X)/\mathcal{L}(D) \to 0.$$

Namely, since one can easily compute the cohomology of the constant sheaf $k(X)$ via Čech cohomology to be $H^0(X, k(X)) = k(X)$ and $H^1(X, k(X)) = 0$, one obtains the exact sequence

$$k(X) \to H^0(X, k(X)/\mathcal{L}(D)) \to H^1(X, \mathcal{L}(D)) \to 0$$

and therfore

$$H^1(X, \mathcal{L}(D)) \cong \frac{\oplus(k(X)/\mathcal{L}(D))_P}{k(X)} \cong \frac{R}{R(D) + k(X)}.$$

Here we made use of the fact that for the sheaf $k(X)/\mathcal{L}(D)$ the canonical injection

$$i: \ H^0(X, k(X)/\mathcal{L}(D)) \hookrightarrow \oplus(k(X)/\mathcal{L}(D))_P$$

is an isomorphism. For this we have to show that for every point $P \in X$

$$(k(X)/\mathcal{L}(D))_P \subseteq i(H^0(X, k(X)/\mathcal{L}(D))).$$

This is true since for every $t_P \in (k(X)/\mathcal{L}(D))_P = (U \mapsto k(X)/(\mathcal{L}(D)(U)))_P$ there exists an open neighborhood $U \subseteq X$ of P and a section $s \in k(X)/(\mathcal{L}(D)(U))$ such that $s_P = t_P$ and $s|_{U-P} = 0$. Now s defines a section in $(k(X)/\mathcal{L}(D))(U)$ which we can extend by zero outside U since $s|_{U-P} = 0$ to a section $\tilde{s} \in (k(X)/\mathcal{L}(D))(X)$ to obtain $i(\tilde{s})_P = t_P$ and $i(\tilde{s})_Q = 0$ for $Q \neq P$. ☐

Now, we are left investigating the k-vector space $R/(R(D)+k(X))$ or to be precise $\dim_k R/(R(D)+k(X))$. We might as well investigate its dual $\mathrm{Hom}_k(R/(R(D)+k(X)), k)$. We will see shortly that this dual is isomorphic to the vector space of differentials of X having a certain prescribed behaviour related to D. For this first let us recall the notion of differentials.

The module of differentials of a k-algebra R is defined to be the R-module

$$\Omega_{R/k} := (\bigoplus_{r \in R} \mathrm{d}rR)/ < \mathrm{d}(r+r') - \mathrm{d}r - \mathrm{d}r', \mathrm{d}(rr') - r\mathrm{d}r' - r'\mathrm{d}r, \mathrm{d}k \,|\, r \in R, k \in K > .$$

For any multiplicatively closed subset $S \subset R$ one has

$$S^{-1}R \otimes_R \Omega_{R/k} \cong \Omega_{S^{-1}R/k}.$$

Whence

$$\underline{\Omega}(U) := \Omega_{\mathcal{O}(U)/k}$$

defines a coherent sheaf on X whose stalks are given by

$$\underline{\Omega}_P = \Omega_{\mathcal{O}_P/k}.$$

Furthermore, tensoring with rational functions gives differentials with rational coefficients, i.e.:

$$\Omega_{\mathcal{O}_P/k} \otimes_{\mathcal{O}_P} k(X) = \Omega_{k(X)/k} \qquad (\star)$$

Now $\underline{\Omega}$ is a subsheaf of the constant sheaf $\underline{\Omega}_{k(X)/k}$ and since our curve X is non-singular, $\underline{\Omega}$ is a locally free $\mathcal{O}$ module of rank 1: If $t \in \mathcal{O}_P$ is a local uniformizer at $P \in X$ then $dt \in \underline{\Omega}_P$ is a basis for the $\mathcal{O}_P$-module $\underline{\Omega}_P$ and $(\star)$ implies that $\mathrm{d}t \otimes 1$ is a basis for the $k(X)$-vector space $\Omega_{k(X)/k}$. Next, of course, we have to relate the differentials to a divisor D. For this we define for each $P \in X$ a valution ν_P in the following way: We choose an $\mathcal{O}_P$-module isomorphism $i : \mathcal{O}_P \xrightarrow{\sim} \underline{\Omega}_P$ and use it to transfer the valution v_P on $k(X)$ to $\Omega_{k(X)/k}$, i.e. we set for $\omega \in \Omega_{k(X)/k}$:

$$\nu_P(\omega) := v_P(f)$$

where $\omega = fi(1)$. This definition is independent of the choosen $\mathcal{O}_P$-isomorphism. In fact, if $j : \mathcal{O}_P \xrightarrow{\sim} \underline{\Omega}_P$ denotes another isomorphism, we have $\forall n \in \mathbb{N} : \; i((P^n) = j(P^n)$ and whence the valuations do agree. We also define the divisor of a differential ω to be

$$(\omega) := \sum_{P \in X} \nu_P(\omega) P$$

and write $\Omega(D)$ for the vector space of the differentials such that $(\omega) \geq -D$ together with 0.

We want to obtain a pairing between répartitions and differential forms. For this we define a second invariant, the residue $\mathrm{res}_P(\omega)$ of a differential ω in the following way: We choose a local uniformizer t at P and if $\kappa(P)|\, k$ is seperable we define

$$\mathrm{res}_P : \Omega_{k(X)/k} \xrightarrow{l \otimes 1} \kappa(P)[[T]] \otimes_{\mathcal{O}_P} k(X) = \kappa(P)((T)) \xrightarrow{\pi_{-1}} \kappa(P) \xrightarrow{tr} k$$

where

$$l : \underline{\Omega}_P \xrightarrow[dt \mapsto 1]{\sim} \mathcal{O}_P \hookrightarrow \widehat{\mathcal{O}}_P \xrightarrow[t \mapsto T]{\sim} \kappa[[T]]$$

and π_{-1} the k-homomorphism sending $\sum_{i=-\infty}^{\infty} a_i T^i \in k((T))$ to a_{-1}, i.e. res_P is defined to be $\mathrm{res}_P(\omega) := tr_{\kappa(P)|\, k}(a_{-1})$ where one writes $\omega \in \underline{\Omega}_P$ in the form $\omega = (\sum_{i=-\infty}^{\infty} a_i t^i)\, dt$. If $\kappa(P)|\, k$ is not a seperable extension we choose a field extension $k'|\, k$ such that all points $Q \in X \otimes_k k'$ lying over P have a seperable residue field extension $\kappa(Q)|\, k'$ and define $\mathrm{res}_P^k(\omega) := \sum_{Q \mapsto P} \mathrm{res}_Q^{k'}(\omega)$. Now, of course, we need:

Proposition 2. *The above definition of* res_P *is independent of the choosen uniformizer and, in the non-closed case, as well independent of the choosen field extension. Furthermore, one has for all* $\omega \in \Omega_{k(X)|\, k} : \sum \mathrm{res}_P(\omega) = 0$.

Proof. If $k = \mathbb{C}$, i.e. in the Riemann surface case, one has $\mathrm{res}_P(\omega) = \frac{1}{2\pi i} \oint_P \omega$ and Stokes formula gives $\sum \mathrm{res}_P = 0$. A proof of the residue formula for $X = \mathbb{P}^1$ in the case that for all P $\kappa(P)|\, k$ is a separable extension can be found in the Appendix. This gives a proof for a general curve if one uses the forumla

$$\mathrm{res}_Q(\mathrm{tr}(f)dt) = \mathrm{res}_P(\mathrm{tr}_{\kappa(Q)|\kappa(P)}(f)dt),$$

which e.g. is proven in [Serre88, Chapter 1, Lemma 5]. The general case is a little intricate, the most conceptual exposition I am aware of can be found in [Tate68]. □

Now we can define a parinig between differentials and répartitions by

$$\langle \omega, r \rangle := \sum_{P \in X} \text{res}_P(r_P \omega)$$

to obtain the $k(X)$-homomorphism

$$\theta : \Omega_{k(X)/k} \longrightarrow (R/k(X))^\star := \lim_{D}(R/(R(D) + k(X)))^\star$$

sending ω to $\theta(\omega) := \langle \omega, \cdot \rangle$ where the $k(X)$ action on $(R/(R(D) + k(X)))^\star$ is defined by $r\langle \alpha, \cdot \rangle := \langle \alpha, r\cdot \rangle$.

Theorem 2. *θ induces an isomorphism*

$$\Omega(-D) \to (R/(R(D) + k(X))^\star.$$

Proof. First, note that $\theta^{-1}((R/(R(D)+k(X)))^\star) \subseteq \Omega(-D)$. Indeed if $\omega \notin \Omega(-D)$ we can find a répartition r such that $\text{res}_P(r_P\omega) \neq 0$ but for $Q \neq P$ one has $r_Q\omega \in \underline{\Omega}_P$ and yet for all $P : -v_P(r_P) \leq v_P(-D)$, i.e. $v_P(r_P) \geq v_P(D)$. But then $\langle \omega, r \rangle \neq 0$ and whence $\theta(\omega) \notin (R/(R(D) + k(X)))^\star$.

This immediately implies that θ must be injective, since $\theta(\omega) = 0$ shows that $\omega \in \Omega(D)$ holds for all D and therefore $\omega = 0$. Also, since θ is a non-trivial $k(X)$-homomorphism from the one-dimensional space $\Omega_{k(X)/k}$ to $(R/(R(D) + k(X)))^\star$ it has to be surjective, if one can show that the dimension of this space over $k(X)$ is at most 1. On the other hand an $\omega \in \Omega_{k(X)/k}$ such that $\theta(\omega)$ vanishes on $R(D)$ has to be in $\Omega(-D)$ after what has just been said. It therefore suffices to show that $\dim_{k(X)}(R/(R(D) + k(X)))^\star \leq 1$ to finish the proof.

As for this we assume there does exist two in $\Omega_{k(X)/k}$ linearly independent $\alpha, \alpha' \in \Omega(D)$ for some D. Observe that this gives rise to an injection

$$\mathcal{L}(\Delta_n)(X)^2 \hookrightarrow (R/(R(D - \Delta_n) + k(X)))^\star$$

for every divisor Δ_n by sending (f, h) to $f\alpha + h\alpha'$. This, in particular, shows that

$$2l(\Delta_n) \leq i(D - \Delta_n).$$

Now, let Δ_n be a divisor of degree n, then by the version of Riemann-Roch stated above we obtain for large n:

$$2(n + 1 - g) \leq n + 1 - g - \deg(D).$$

Letting n go to ∞ leads a contradiction.

□

After having established this duality result we finally arrived at the classical formulation of Riemann-Roch.

3. Classical Riemann-Roch

To each function $f \in k(X)$ we associate the divisor $(f) := \sum_{P \in X} v_P(f)P$. Such divisors are called principal and we say two divisors D and D' are linearly equivalent if they differ by a principal divisor.

Now, since $\Omega_{k(X)/k}$ has dimension 1 all divisors (ω) for $\omega \in \Omega_{k(X)/k}$ are linearly equivalent and choosing an $\omega \neq 0$ gives us an $\mathcal{O}$-module isomorphism

$$\mathcal{L}(K-D) \xrightarrow{\sim} \Omega(-D)$$

by sending 1 to ω, resp. 0 to 0 if $\mathcal{L}(K-D) = 0$.

We call $K := [(\omega)]$ the canonical divisor class. After what has been said in the previous paragraph we now obtain

$$i(D) = \dim_k H^1(X, \mathcal{L}(D))^\star = \dim_k \Omega(D) = \dim_k H^0(X, \mathcal{L}(K-D)) = l(K-D).$$

Combining this with the version of Riemann-Roch as stated in the first paragraph yields

Theorem 3. *For every divisor D one has*

$$\chi(D) = l(D) - l(K-D) = \deg(D) + 1 - g.$$

Putting $D = K$ gives us the degree of the cannonical divisor

$$\deg(K) = 2g - 2$$

and whence if Δ_n denotes a divisor of degree $n \geq 2g-1$ one obtains

$$l(\Delta_n) = n + 1 - g.$$

since for any $f \in k(X)$ one has $\deg((f)) = 0$. Whence, in particular, if $X = \mathbb{P}^1_k$ we have $g = \dim_k H^1(X, \mathcal{O}) = 0$ and obtain the formula

$$\dim_k \mathcal{L}(D)(\mathbb{P}^1_k) = \deg(D) + 1$$

as established in the first paragraph. So things worked out well.

4. Poisson summation and Riemann-Roch over finite fields

Here we want to obtain a second proof for the Riemann-Roch theorem under the assumption that k is a finite field. In this case we will see that the theorem is implied by the Poisson summation formula illustrating the strength of this formula. First we have to establish the Poisson summation for function fields.

So, let k be a finite field with q elements and X be a smooth, projective curve over k. Let $\mathbb{A}_{k(X)}$ denote the adèles of the global field $k(X)$. This is a locally compact group. In what follows we will fix character ψ of the adèles which is trivial on $k(X)$ but non-trivial on any P-component. We then have the topological isomorphism

$$\mathbb{A}_{k(X)} \xrightarrow{\sim} \widehat{\mathbb{A}_{k(X)}}$$

defined by $\alpha \mapsto \psi(\alpha \,\cdot)$. A construction of such a character is given in the appendix.

Henceforth we will identify $\widehat{\mathbb{A}_{k(X)}}$ with the adèles themselves via this isomorphism. Furthermore, for any haar measure μ on $\mathbb{A}_{k(X)}$ we will use this identification to obtain a haar measure, also called μ on the character group. Next, we define the Fourier transform of a function $f \in L^1(\mathbb{A}_{k(X)})$ to be the function

$$\hat{f} := \int\limits_{\mathbb{A}_{k(X)}} f(\eta)\overline{\psi(\eta\,\cdot)}d\eta$$

Now, we know by the theory of Fourier Analysis on locally compact groups that there does exist a constant $c \in \mathbb{R}^\star$ such that for every function $f \in L^1(\mathbb{A}_{k(X)})$ satisfying $\hat{f} \in L^1(\mathbb{A}_{k(X)})$ on has $\hat{\hat{f}} = cf$. In order to compute c consider $f = 1_C$ for the compact set $C := \prod_P \widehat{\mathcal{O}}_P$:

$$\hat{f}(\xi) = \int\limits_C \overline{\psi(\xi\eta)}d\eta = 1_{\psi(\xi C)=1}(\xi)\ \mu(C) = 1_{\mathfrak{D}}(\xi)\ \mu(C)$$

where $\mathfrak{D} := \psi^{-1}(1)$ denotes the conductor of ψ. Furthermore, we obtain:

$$cf(x) = \hat{\hat{f}}(x) = \int\limits_{\mathfrak{D}} \mu(C)\overline{\psi(x\xi)}d\xi = 1_{\psi(x\mathfrak{D})=1}(x)\mu(C)\mu(\mathfrak{D}) = \mu(C)\mu(\mathfrak{D})f(x)$$

Therefore, we will choose μ relative to ψ to be the haar measure such that $\mu(C) = \mu(\mathfrak{D})^{-1}$ so that we obtain $f = \hat{\hat{f}}$. For the rest of these notes ψ and μ will be fixed according to these choices. We compute

$$\mu(C)^{-1} = \mu(\mathfrak{D}) = \mu(C)\mathfrak{N}(\mathfrak{D})$$

and whence $\mu(C) = \mathfrak{N}(\mathfrak{D})^{-1/2}$.

Now we are set to establish the Poisson summation formula:

Theorem 4. *Let $f \in L_1(\mathbb{A}_{k(X)})$ such that $\sum_{\xi\in k(X)} |f(x+\xi)|$ is uniformly convergent on compact subsets and $\sum_{\xi\in k(X)} |\hat{f}(\xi)|$ is convergent. Then*

$$\sum_{\xi\in k(X)} f(\xi) = \sum_{\xi\in k(X)} \hat{f}(\xi).$$

Proof. In order to apply Theorem 1 (the general Poisson summation formula) of Mariya's notes from the last talk we need to find a fundamental domain $F \subset \mathbb{A}_{k(X)}$ for the subgroup $\Gamma := k(X)$, show that Γ is discrete and F relatively compact. Furthermore, we need to show that under the isomorphism

$$i : \mathbb{A}_{k(X)} \xrightarrow{\sim} \widehat{\mathbb{A}_{k(X)}}$$

defined by $\alpha \mapsto \psi(\alpha\,\cdot)$ the subgroup $k(X)$ maps onto $\widehat{\mathbb{A}_{k(X)}/k(X)} = k(X)^\perp$. This just follows as in Theorem 2 from last talk: One has $i(k(X)) \subset k(X)^\perp$, which is discrete, since $\mathbb{A}_{k(X)}/k(X)$ is compact. Therefore $k(X)^\perp/i(k(X)) \subset$

$\mathbb{A}_{k(X)}/i(k(X))$ is discrete and compact, whence finite. But $k^{\perp}$ is now an $i(k(X))$-vectorspace which has finite index as a group over $i(k(X))$. Whence $i(k(X)) = k^{\perp}$.

As for the existence of a relatively compact fundamental domain F one first observes that for $X = \mathbb{P}^1$ the compact set $\prod \hat{\mathcal{O}}_P$ must allready contain such a domain: For any adèle $\mathfrak{a} \in \mathbb{A}_{k(t)}$ let S denote the set consiting of the finitely many finite places such that $v_P(\mathfrak{a}) < 0$. One chooses an $f \in k[t]$ such that for all finite places $v_P(f\mathfrak{a}) \geq 0$ and then an $\alpha \in k[t]$ such that for all $P \in S$ one has $v_P(f\mathfrak{a} - \alpha) = v_P(f)$ and such that its degree is minimal with this property. Then one obtains for all places $v_P(\mathfrak{a} - \alpha/f) \geq 0$ as one computes easily for the place at ∞. Alternatively, one does not care about the degree of α and substracts the polynomial part of $\mathfrak{a}_\infty - \alpha/f$ in a second step to obtain the result. Now, for a general function field $k(X)$ one obtains that every adèle $\mathfrak{a} \in \mathbb{A}_{k(X)}$ can be written as $\mathfrak{a} = f + \mathfrak{a}_F$ with $f \in k(X)$ and $\mathrm{tr}(\mathfrak{a}_F) \in F$. Whence one obtains that the compact set $\mathrm{tr}^{-1}(F)$ contains already a fundamental domain.

Furthermore $k(X)$ is discrete inside $\mathbb{A}_{k(X)}$, since the functions satisfying for all places $v_P(f) \geq 0$ are the constants by the product formula for the norm in global fields and whence the only function in the open set $\prod \hat{\mathcal{O}}_P \times q\hat{\mathcal{O}}_Q$ for some place Q is the zero function.

□

As in the case of differential forms we will define the divisor associated to a character ψ which vanishes on $k(X)$ by $(\psi) := \sum_{P \in X} -n_P P$ where $n_P = \nu_P(\psi)$ is the order of vanishing of ψ at P, i.e. $\psi_P(\mathfrak{p}^{n_p}) = 1$ and $\psi_P(\mathfrak{p}^{n_p - 1}) \neq 1$. Again if ψ' is another character vanishing on $k(X)$ then $\psi' = \psi(\alpha\cdot)$ for some $\alpha \in k(X)^\star$ and whence the class is independent of the choosen character and we will write $K = [(\psi)]$. That the sum is actually finite is clear by examining the character constructed in the Appendix and the fact that all other characters differ just by an element in $k(X)$.

Next let $f = \otimes_P 1_{\mathcal{O}_P}$ be the function on the adèles whose components are the characteristic functions of $\mathcal{O}_P$. For every divisor D let $x(D)$ be an idèle such that for all P one has $v_P(x(D)_P) = n_P$. Then $f(\cdot\, x(D))|_{k(X)} = 1_{\mathcal{L}(D)(X)}$. Since our haar meassure is adjusted such that the inversion formula holds the ring of integers $\mathcal{O}_P$ volume $\mathfrak{p}^{n/2}$ and we therefore obtain

$$\hat{f}_P = q^{[\kappa(P):\,k]\nu_P(\psi)/2} \cdot 1_{\mathfrak{p}^{\nu_P(\psi)}}$$

where $\mathfrak{p}$ is the prime ideal associated to P. Putting $h(y) := f(yx(D))$ Poisson summation shows that

$$\sum_{y \in k(X)} f(yx) = \sum_{y \in k(X)} \hat{h}(y) = \frac{1}{|x(D)|} \sum_{y \in k(X)} \hat{f}(yx(D)^{-1})$$

since

$$\hat{h}(y) = \int_{\mathbb{A}_{k(X)}} f(zx)\psi(yz)dz = \frac{1}{|x(D)|} \int_{\mathbb{A}_{k(X)}} f(z)\psi(yzx^{-1})dz = \frac{1}{|x(D)|}\hat{f}(yx^{-1})$$

Now we just have to compute the two sums

$$\sum_{y\in k(X)} f(yx(D)) = \sum_{x\in \mathcal{L}(D)(X)} 1 = q^{l(D)}$$

$$\sum_{y\in k(X)} \hat{f}(yx(D)^{-1}) = \sum_{P\in \mathcal{L}(K-D)} q^{\sum[\kappa(P):k]\,\nu_P(\psi)/2} = q^{l(K-D)}q^{\deg(\psi)/2}$$

and observe that

$$\frac{1}{|x(D)|} = \prod_{P\in X} q^{[\kappa(P):k]v_P(D)} = q^{\deg D}$$

to obtain

$$l(D) - l(K-D) = \deg(D) + \deg(\psi)/2 =: \deg(D) + 1 - g.$$

5. Appendix

For the whole theory of fourier analysis on the adèles it is cruzial to identify the adèles with their Pontryagin dual in such a way that $k(X)^{\perp}\cong k(X)$. For this, as we have shown, it is neccesary to establish the existence of a non-trivial character on the adèles which is trivial on $k(X)$. In the case of number fields such a character has already been explicitely constructed. We will do this in what follows for function fields.

For this we will construct an obviously non-trivial character $\chi = \chi_{\mathbb{A}_{k(X)}}$ on $\mathbb{A}_{k(X)}$ for $X = \mathbb{P}^1_{\mathbb{F}_p(t)}$ and show that it is trivial on $k(X)$. Then for an arbitrary curve Y we can define

$$\chi_{\mathbb{A}_{k(Y)}} : \mathbb{A}_{k(Y)} \xrightarrow{\mathrm{tr}} \mathbb{A}_{k(X)} \xrightarrow{\chi_{\mathbb{A}_{k(X)}}} \mathbb{C}$$

where the trace map tr on the adèles is defined by

$$\mathrm{tr}((r_Q)_{Q\in Y}) = (\sum_{Q|P} \mathrm{tr}_{k(Y)_Q|\,k(X)_P}(r_Q))_{P\in X}.$$

In this way since $k(Y)|\,k(X)$ is per assumption separable one gets a non-trivial character on the adèles of $k(Y)$ which, moreover, is trivial on $k(Y)$, as is clear from the fact that for all $P \in X$ we have:

$$\mathrm{tr}_{k(Y)|\,k(X)}(f) = \sum_{\sigma\in \mathrm{Gal}(k(Y)|\,k(X))} \sigma(f) = \sum_{Q|P}\sum_{\sigma\in \mathrm{Gal}(Q|\,P)} \sigma(f) = \sum_{Q|P} \mathrm{tr}_{k(Y)_Q|\,k(X)_P}(f)$$

So what is left is to construct a non-trivial character on $\mathbb{A}_{\mathbb{F}_p(t)}$ which is trivial on $\mathbb{F}_p(t)$. For this we will use the isomorphism

$$\Omega_{\mathbb{F}_p(t)/\mathbb{F}_p} \xrightarrow{\sim} (\mathbb{A}_{\mathbb{F}_p(t)}/\mathbb{F}_p(t))^{\star}$$

established in the second paragraph which sends ω to $((r_P) \mapsto \sum_P \mathrm{res}_P(r_P\omega))$. Recall that we defined $\mathrm{res}_P(\sum a_i\pi^i d\pi) := \mathrm{tr}_{\kappa(P)|\,k}(a_{-1})$ where π is a local uniformizer at P.

Using the residue formula as established in the second paragraph this yields what we want. Since we did not give a proof for the residue theorem, we will prove what is needed here.

Therefore, choosing the differential $\omega = dt$ and choosing for each finite place P the canonical uniformizer $\pi_P \in k[t]$ which is an irreducible, monic polynomial as well as the uniformizer $\pi_\infty := 1/t$ at ∞ we define our character

$$\chi((r_P)) := e^{2\pi i \Lambda(\cdot)}$$

with

$$\Lambda((r_P)) := \frac{1}{p}\sum_P tr_{\kappa(P)|\,k}(a_{P,-1}) \in \mathbb{Q}/\mathbb{Z},$$

where for each P we write:

$$r_P dt = \frac{r_P}{\pi'} d\pi = \sum a_{P,i}\, \pi_P^i \, d\pi_P$$

With this definition we obtain:

Proposition 3. *χ is a non-trivial, continous, unitary charcater on $\mathbb{A}_{\mathbb{F}_p(t)}$ which is trivial on $\mathbb{F}_p(t)$.*

Proof. It is clear that the sum actually is finite and that Λ is a non-trivial group homomorphism. Furthermore Λ is continous, since for every adèle (r_P) the set

$$U := \prod_{P: r_P \in \hat{\mathcal{O}}_P} \hat{\mathcal{O}}_P \times \prod_{P: r_P \notin \hat{\mathcal{O}}_P} \{\sum a_i \pi_P^i \in \kappa((t)) \,|\, a_{-1} = a_{P,-1}\}$$

is open, contains (r_P) and satisfies $\Lambda(U) = \Lambda((r_P))$. It therefore remains to show, that Λ is trivial on $\mathbb{F}_p(t)$.

First, note that each $f \in \mathbb{F}_p(t)$ can be written as a finite sum

$$f = p + \sum_{1\le i\le N} a_i \frac{t^{m_i}}{p_i^{n_i}}$$

with $p, p_i \in \mathbb{F}_p[t]$, $a_i \in \mathbb{F}_p$, $m_i < \deg(p_i)$ and where the p_i are monic and irreducible. To see this, first observe that $\mathbb{F}_p[t]$ is a principal ideal domain and whence one can write $f = \sum r_i/l_i^{n_i}$ where the l_i are irreducible. Then using the Euclidean algorithm one archives the desired shape.

Now by additivity and $\mathbb{F}_p$-linearity of Λ what is left is to show that the sum of the resiudes of a differential of the form $t^m/p^n\, dt$ is 0 with p monic, irreducible and $m < \deg(p)$ or with $p = 1$. As for the second case, i.e. $\omega = t^m dt$ it is clear that the residues at all finite places P are zero, since we have $t^m dt = t^m/\pi'_P\, d\pi_P$ with $t^m/\pi'_P \in \mathcal{O}_P$ and so is the residue at ∞ since we have $t^m dt = -t^{m+2} d\pi_\infty = -\pi_\infty^{-m-2} d\pi_\infty$. Similarly, for p an irreducible polynomial, we find that the residue at all finite places P with $(\pi_P, p) = \mathbb{F}_p[t]$ is zero, since we have $\omega = t^m/p^n\, dt = t^m/(p^n \pi'_P)\, d\pi_P$ with $t^m/(p^n\pi'_P) \in \mathcal{O}_P$.

It remains to show that $\operatorname{res}_P(t^m/p^n\,dt) = -\operatorname{res}_\infty(t^m/p^n\,dt)$ where P is the place defined by p. Whence we write $p^n = t^k + a_{k-1}t^{k-1} + \cdots + a_0$ and using $d\pi_\infty = -\pi_\infty^2 dt$ we compute:

$$\begin{aligned}\operatorname{res}_\infty(\frac{t^m}{p^n}\,dt) &= \operatorname{res}_\infty(-\pi_\infty^{-m-2}(\pi_\infty^{-k} + a_{k-1}\pi_\infty^{k-1} + \cdots + a_0)^{-1}\,d\pi_\infty) \\ &= \operatorname{res}_\infty(-\pi_\infty^{k-m-2}(1 + a_{k-1}\pi_\infty + \cdots + a_0\pi_\infty^k)^{-1}\,d\pi_\infty) = -1\end{aligned}$$

if $m = k-1 = \deg p - 1$ and 0 otherwise, since $m < \deg p$. Next, we compute that

$$\operatorname{res}_P(\frac{t^m}{p^n}\,dt) = \operatorname{res}_P(\frac{t^m}{p^n p'}d\pi_P) = 1_{n=1}1_{m=\deg p-1}$$

completing the proof, where the computation of the trace of t^m/p' follows from the computation of the dual basis of the $k((\pi))$-basis (t^i) of $\kappa((\pi))$ with respect to tr and minimal polynomial $p(X)-\pi$ as it can be found in the next proposition. Namely we obtain $\operatorname{tr}_{\kappa((\pi))|\,k((\pi))}(t^m/p') = 1_{m=\deg(p)-1}$. □

Proposition 4. *Let k be a field and t be separable algebraic over k with minimal polynomial $p(X)$ of degree n and $p(X)/(X-t) = b_{n-1}X^{n-1} + \cdots + b_0$. Then the dual basis to t^i with respect to* $\operatorname{tr} : k(t) \times k(t) \to k$ *is given by $b_i/(p'(t))$, i.e. one has* $\operatorname{tr}(t^i b_j/(p'(t))) = \delta_{ij}$.

Proof. Observe that for all $0 \le i \le n-1$ the polynomial

$$X^i - \sum_{\sigma\in\operatorname{Gal}(k(t)|k)} \frac{p(X)}{X-\sigma(t)}\frac{\sigma(t)^i}{p'(\sigma(t))}$$

has degree at most $n-1$ and at least the n distinct roots $\sigma(t)$. Whence it must be the 0 polynomial and we obtain:

$$\operatorname{tr}_{k(t)|k}(b_{n-1}\frac{t^i}{p'(t)})X^{n-1} + \cdots + \operatorname{tr}_{k(t)|k}(b_0\frac{t^i}{p'(t)}) = \operatorname{tr}_{k(t)|k}(\frac{p(X)}{X-t}\frac{t^i}{p'(t)}) = X^i$$

Comparing the coefficients gives the result. □

6. Bibliography

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