Lecture 2: Calabi–Yau differential operators

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Mirror theorem

enumerative geometry on X	solving differential equation
$n_d = \#$ of rational curves of degree d on X	for period integrals on X'
Gromov–Witten invariants	instanton numbers
e.g. $X \subset \mathbb{P}^4$ generic quintic	$Y(q) = \sum_{d \geq 0} n_d d^3 \frac{q^d}{1-q^d}$
$n_1 = 2875$ (H.Schubert,1886)	
$n_2 = 609250$ (S.Katz,1986)	

Theorem (Givental, Lian–Liu–Yau, mid 90's) $n_d(A) = n_d(B)$.

Beginnings of mirror symmetry

P. Candelas, X. de la Ossa, P. Green, L. Parkes, *An exactly soluble superconformal theory from a mirror pair of Calabi–Yau manifolds*, Phys. Lett. B 258 (1991), no.1–2, 118–126

$$L = \theta^4 - 5^5 t \left(\theta + \tfrac{1}{5}\right)\left(\theta + \tfrac{2}{5}\right)\left(\theta + \tfrac{3}{5}\right)\left(\theta + \tfrac{4}{5}\right), \quad \theta = t\tfrac{d}{dt}$$

The differential equation $Ly = 0$ has solutions

$$y_0(t) = \sum_{n=0}^{\infty} \frac{(5n)!}{n!^5} t^n = 1 + 120t + 113400t^2 + \ldots =: f_0(t) \in \mathbb{Z}[\![t]\!]$$

and

$$y_1(t) = f_0(t)\log(t) + f_1(t), \quad f_1(t) := \sum_{n=1}^{\infty} \frac{(5n)!}{n!^5} \left(\sum_{j=1}^{5n} \frac{5}{j}\right) t^n \in t\mathbb{Q}[\![t]\!]$$

Observation: $q(t) := \exp\left(\frac{y_1(t)}{y_0(t)}\right) = t\exp\left(\frac{f_1(t)}{f_0(t)}\right) \in t\mathbb{Z}[\![t]\!]$
(proved by B.-H.Lian and S.-T.Yau in 1996)

Canonical coordinate and Yukawa coupling

$q(t) = \exp\left(y_1(t)/y_0(t)\right) = t + 770t^2 + 1014275t^3 +$
is called the *canonical coordinate*. The *mirror map* is the inverse series $t(q) \in q + q^2\mathbb{Q}[\![q]\!]$.

Solutions to $Ly = 0$:

$$y_0(t) = f_0, \quad y_1(t) = f_0 \log(t) + f_1,$$
$$y_2(t) = f_0 \frac{\log(t)^2}{2!} + f_1 \log(t) + f_2, \quad f_2 \in t\mathbb{Q}[\![t]\!]$$

Express the ratios y_i/y_0 in terms of $q = q(t)$:

$$\frac{y_0}{y_0} = 1, \quad \frac{y_1}{y_0} = \log(q),$$
$$\frac{y_2}{y_0} = \frac{1}{2}\log(q)^2 + 575q + \frac{975375}{4}q^2 + \frac{1712915000}{9}q^3 + \dots$$

$Y(q) := \left(q\frac{d}{dq}\right)^2 \frac{y_2}{y_0} = 1 + 575q + 975375q^2 + \dots$
is called the *Yukawa coupling*.

Physics wins!

$$Y(q) = \left(q\frac{d}{dq}\right)^2 \frac{y_2}{y_0} = 1 + 575q + \ldots = \frac{1}{5}\sum_{d\geq 0} n_d d^3 \frac{q^d}{1-q^d}$$

$$n_0 = 5, \quad n_1 = 2875, \quad n_2 = 609250,$$

$$n_3 = 317206375, \quad n_4 = 242467530000, \ldots$$

are called *instanton numbers*.

Observation / prediction: The numbers n_d coincide with the numbers of degree d rational curves that lie on a generic threefold of degree 5 in $\mathbb{P}^4$.

Only the first two numbers were known at that time! In 1993 G.Ellingsrud and S.Strømme computed the number of cubic curves on the quintic threefold. Their result served as a crucial cross-check for the above physicists' prediction.

Integrality of instanton numbers

$$L = \theta^4 - 5^5 t \left(\theta + \tfrac{1}{5}\right)\left(\theta + \tfrac{2}{5}\right)\left(\theta + \tfrac{3}{5}\right)\left(\theta + \tfrac{4}{5}\right), \quad \theta = t\tfrac{d}{dt}$$

$$y_0 = f_0,\ y_1 = f_0 \log(t) + f_1,\ y_2 = f_0 \tfrac{\log(t)^2}{2!} + f_1 \log(t) + f_2$$

$$q = \exp(y_1/y_0), \quad Y(q) = \left(q\tfrac{d}{dq}\right)^2 (y_2/y_0) = \frac{1}{5}\sum_{d\geq 0} n_d d^3 \tfrac{q^d}{1-q^d}$$

Observation / prediction: $n_d \in \mathbb{Z}$ for every d.

Theorem (MV–Frits Beukers, 2020)[1] For the quintic case, the denominators of instanton numbers n_d can only have prime divisors 2, 3, 5.

[1] Our proof is essentially elementary, we will stay on B-side. An alternative proof is possible on A-side, via the mirror theorem. Around 1998 R.Gopakumar and C.Vafa introduced the *BPS-numbers* for Calabi–Yau threefolds, which include the Gromov–Witten invariants as $g = 0$ case. E.N. Ionel and T.H. Parker proved that BPS-numbers are integers by using methods from symplectic topology in *The Gopakumar –Vafa formula for symplectic manifolds*, Annals of Math. 187 (2018), 1–64.

Calabi–Yau differential operators

In 2003 Gert Almkvist wrote to Duco van Straten asking if he knows more operators *like the one for the quintic*.[2] In subsequent years many similar examples were constructed by Gert Almkvist, Christian van Enckevort, Duco van Straten and Wadim Zudulin.

A 4th order differential operator

$$L = \theta^4 + \sum_{j=1}^{4} a_j(t)\theta^{4-j}, \quad \theta = t\frac{d}{dt}, \quad a_j \in \mathbb{Q}(t),\ 1 \le j \le 4$$

is called a *Calabi–Yau operator* if:

- its singularities are regular
- $t = 0$ is a point of maximally unipotent monodromy (MUM), that is $a_j(0) = 0,\ 1 \le j \le 4$
- it is self-dual
- it satisfies the integrality conditions:
  - the holomorphic solution $y_0(t) \in \mathbb{Z}[\![t]\!]$
  - the canonical coordinate $q = \exp(y_1/y_0) \in \mathbb{Z}[\![t]\!]$
  - the instanton numbers $n_d \in \mathbb{Z}$

[2] D. van Straten, *Calabi–Yau operators* in Adv. Lect. Math. 42 (2018), p. 7

Calabi–Yau differential operators

If one allows N-integrality instead of integrality, about 500 such operators were found *experimentally* in *Tables of Calabi–Yau operators* by G. Almkvist, C. van Enckevort, D. van Straten, W. Zudilin, (arXiv:math/0507430) *"AESZ tables"* (2010).

In some cases the power series solution to L can be written as a period function of a family of toric hypersurfaces[3]:

$$y_0(t) = \frac{1}{(2\pi i)^n} \oint \cdots \oint \frac{1}{1 - tg(\mathbf{x})} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

for a Laurent polynomial $g(\mathbf{x}) \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. This fact then explains integrality of the analytic solution $y_0(t) = \sum_{k=0}^{\infty} c_k t^k$ where c_k is the constant term of $g(\mathbf{x})^k$.

[3]When the Newton polytope Δ of $g(\mathbf{x})$ is *reflexive* then the hypersurfaces $1 - tg(\mathbf{x}) = 0$ can be compactified to Calabi–Yau hypersurfaces (V. Batyrev).

Calabi–Yau differential operators

$AESZ\#1 \qquad L = \theta^4 - 5^5 t^5 (\theta+1)(\theta+2)(\theta+4)(\theta+5)$

$$g(\mathbf{x}) = x_1 + x_2 + x_3 + x_4 + \frac{1}{x_1 x_2 x_3 x_4}$$

$AESZ\#8 \qquad L = \theta^4 - 108^2 t^6 (\theta+1)(\theta+2)(\theta+4)(\theta+5)$

$$g(\mathbf{x}) = x_1 + x_2 + x_3 + x_4 + \frac{1}{x_1^2 x_2 x_3 x_4}$$

(operators up to AESZ#14 are hypergeometric)

$AESZ\#15 \qquad L = \theta^4 - 3^3 t^3 (\theta+1)(\theta+2)(7\theta^2 + 21\theta + 18)$
$\qquad\qquad + 18^3 t^6 (\theta+1)(\theta+2)(\theta+4)(\theta+5)$

$$g(\mathbf{x}) = x_1 + x_2 + x_3 + x_4 + \frac{1}{x_1 x_2} + \frac{1}{x_3 x_4}$$

$AESZ\#16 \qquad L = (1024t^4 - 80t^2 + 1)\theta^4 + 64(128t^4 - 5t^2)\theta^3$
$\quad + 16(1472t^4 - 33t^2)\theta^2 + 32(896t^4 - 13t^2)\theta + 128(96t^4 - t^2)$

$$g(\mathbf{x}) = x_1 + \frac{1}{x_1} + x_2 + \frac{1}{x_2} + x_3 + \frac{1}{x_3} + x_4 + \frac{1}{x_4}$$

Towards the proof of integrality of instanton numbers

Lemma. For a power series $Y(q) \in \mathbb{Q}[\![q]\!]$, consider the Lambert expansion

$$Y(q) = \sum_{d \geq 0} a_d \frac{q^d}{1 - q^d}.$$

Take a prime number p. Suppose $\exists\, \phi \in \mathbb{Z}_p[\![q]\!]$ such that

$$Y(q^p) - Y(q) = \left(q \frac{d}{dq}\right)^s \phi(q).$$

Then $a_d / d^s \in \mathbb{Z}_p$ for all $d \geq 1$.

Towards the proof of integrality of instanton numbers

Take $s = 3$ and write the respective $\phi \in \mathbb{Q}[\![q]\!]$ explicitly:

$$\sum_{d\geq 1} n_d d^3 \frac{q^d}{1-q^d} = \left(q\frac{d}{dq}\right)^3 Z, \quad Z(q) = \sum_{d\geq 1} n_d Li_3(q^d) \in \mathbb{Q}[\![q]\!]$$

$$Li_3(x) = \sum_{m\geq 1} \frac{x^m}{m^3}, \quad \left(x\frac{d}{dx}\right)^3 Li_3(x) = \frac{x}{1-x}$$

$$\phi := p^{-3} Z(q^p) - Z(q) \overset{??}{\in} \mathbb{Z}_p[\![q]\!]$$

J. Stienstra, *Ordinary Calabi–Yau–3 Crystals*, Fields Inst. Commun., 38 (2003): one can prove p-integrality of ϕ by relating it to a matrix coefficient of the p-adic *Frobenius structure* for the differential operator L

M. Kontsevich, A. Schwarz, V. Vologodsky, *Integrality of instanton numbers and p-adic B-model*, Phys. Lett. B 637 (2006), no. 1–2

V. Vologodsky, *On the N-integrality of instanton numbers*, arXiv:0707.4617

Frobenius structure (after Dwork)

A p-adic Frobenius structure is an equivalence between the differential system corresponding to L and its pullback under the change of variable $t \mapsto t^p$, over the field $E_p = \widehat{\mathbb{Q}(t)}$ of p-adic analytic functions.

$$L = \theta^4 + \sum_{j=1}^{4} a_j(t)\theta^{4-j} \qquad \text{with MUM point at } t = 0$$

$$y_0 = f_0,\ y_1 = f_0 \log(t) + f_1, y_2 = f_0 \frac{\log(t)^2}{2!} + f_1 \log(t) + f_2$$

$$y_3 = f_0 \tfrac{\log(t)^3}{3!} + f_1 \tfrac{\log(t)^2}{2!} + f_2 \log(t) + f_3,\ f_i \in \mathbb{Q}[\![t]\!]$$

$$U = \left(\theta^i y_j\right)_{i,j=0}^{3} \text{ fundamental solution matrix}$$

Are there constants $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{Q}_p$ such that

$$\Phi(t) = U(t) \begin{pmatrix} \alpha_0 & p\alpha_1 & p^2\alpha_2 & p^3\alpha_3 \\ 0 & p\alpha_0 & p^2\alpha_1 & p^3\alpha_2 \\ 0 & 0 & p^2\alpha_0 & p^3\alpha_1 \\ 0 & 0 & 0 & p^3\alpha_0 \end{pmatrix} U(t^p)^{-1} \in E_p^{4\times 4} \quad ?$$

Frobenius structure: definition adapted to our problem

$$U = \left(\theta^i y_j\right)_{i,j=0}^{3} \text{ fundamental solution matrix for } L$$

$$\Phi(t) = U(t) \begin{pmatrix} \alpha_0 & p\alpha_1 & p^2\alpha_2 & p^3\alpha_3 \\ 0 & p\alpha_0 & p^2\alpha_1 & p^3\alpha_2 \\ 0 & 0 & p^2\alpha_0 & p^3\alpha_1 \\ 0 & 0 & 0 & p^3\alpha_0 \end{pmatrix} U(t^p)^{-1} \in \mathbb{Q}[\![t]\!]^{4\times 4}$$

Definition. We say that L has a p-adic Frobenius structure if there exist p-adic constants $\alpha_0 = 1, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}_p$ such that

$$\Phi_{ij} \in p^j \mathbb{Z}_p[\![t]\!], \quad 0 \le i, j \le 3.$$

Conjecture. [4] Calabi-Yau differential operators have p-adic Frobenius structure for almost all p. Moreover, $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = r\zeta_p(3)$, where $r \in \mathbb{Q}$ is independent of p and can be expressed via geometric invariants of the mirror manifold.

[4] P. Candelas, X. de la Ossa, D. van Straten, *Local Zeta Functions From Calabi-Yau Differential Equations*, arXiv:2104.07816 [hep-th], §4.4

p-Integrality of instanton numbers

$L = \theta^4 + a_1(t)\theta^3 + a_2(t)\theta^2 + a_3(t)\theta + a_4(t)$
$a_i(0) = 0, i = 1, \ldots, 4 \quad$ (MUM point at $t = 0$)

Theorem (MV-Frits Beukers, 2020). Suppose that a p-adic Frobenius structure exists for L. Then

- the analytic solution is p-integral: $y_0 \in \mathbb{Z}_p[\![t]\!]$
- the canonical coordinate is p-integral: $q = \exp(y_1/y_0) \in \mathbb{Z}_p[\![t]\!]$
- if in addition L is self-dual and $\alpha_1 = 0$, then the instanton numbers of L are p-integral: $n_d \in \mathbb{Z}_p$ for all $d \geq 1$

In the latter case, the series ϕ such that $Y(q^p) - Y(q) = (q\frac{d}{dq})^3\phi$ is basically given by the top right Frobenius matrix entry: $\phi \approx p^{-3}\Phi_{03}$.

The hard part: existence of Φ with required properties

Given $L = \theta^r + \ldots$, we would like to construct the Frobenius structure matrix Φ and show that $\alpha_1 = 0$. We need a geometric model, a family of hypersurfaces whose periods are solutions of L.

- Find $g(\mathbf{x}) \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ such that

$$y_0(t) = \frac{1}{(2\pi i)^n} \oint \ldots \oint \frac{1}{1 - tg(\mathbf{x})} \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n}$$

e.g. $n = 4$, $g(\mathbf{x}) = x_1 + x_2 + x_3 + x_4 + \frac{1}{x_1 x_2 x_3 x_4}$
$L = \theta^4 - (5t)^5 (\theta + 1)(\theta + 2)(\theta + 3)(\theta + 4)$

More generally, consider a Laurent polynomial $f(\mathbf{x})$ with coefficients in $\mathbb{Z}[t]$ and let $X_f = \{f(\mathbf{x}) = 0\} \subset \mathbb{T}^n$ be the toric hypersurface of its zeroes. Assume that the cohomology class

$$\omega = \frac{1}{f(\mathbf{x})} \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n} \in H^n(\mathbb{T}^n \setminus X_f)$$

is annihilated by L.

- In the above example, take $f(\mathbf{x}) = 1 - tg(\mathbf{x})$.

Cohomology and differential forms

$f(\mathbf{x}) \in R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, R is a localization of $\mathbb{Z}[t]$,
$X_f = \{f(\mathbf{x}) = 0\} \subset \mathbb{T}^n$, $\Delta \subset \mathbb{R}^n$ Newton polytope of $f(\mathbf{x})$

$$\Omega_f = \left\{ \frac{h(\mathbf{x})}{f(\mathbf{x})^m} \;\middle|\; \begin{array}{l} m \geq 1, h \in R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \\ supp(h) \subset m\Delta \end{array} \right\} \quad R\text{-module}$$

$\cup$

$$d\Omega_f = R\text{-module generated by } x_i \frac{\partial \nu}{\partial x_i}, \nu \in \Omega_f, i = 1, \ldots, n$$

$$\Omega_f / d\Omega_f \cong H^n_{DR}(\mathbb{T}^n \setminus X_f) \quad \text{(Griffiths, Batyrev)}$$

$$\Omega_f \ni \frac{h(\mathbf{x})}{f(\mathbf{x})^m} \mapsto \frac{h(\mathbf{x})}{f(\mathbf{x})^m} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

$$d\Omega_f \leftrightarrow \text{ exact forms}$$

$$\Omega_f(\cdot) = \{m \leq \cdot\} \leftrightarrow \text{ Hodge filtration}$$

p-adic Cartier operation

fix p prime

$$\mathcal{C}_p : \frac{h(\mathbf{x})}{f(\mathbf{x})^m} = \sum_{\mathbf{u}\in\mathbb{Z}^n} a_{\mathbf{u}}(t)\mathbf{x}^{\mathbf{u}} \mapsto \sum_{\mathbf{u}\in\mathbb{Z}^n} a_{p\mathbf{u}}(t)\mathbf{x}^{\mathbf{u}} \quad \notin \Omega_f$$

$$\uparrow$$

$$\text{formal expansion, e.g.} \frac{1}{1-tg(\mathbf{x})} = \sum_{k\geq 0} t^k g(\mathbf{x})^k = \sum_{\mathbf{u}\in\mathbb{Z}^n} a_{\mathbf{u}}(t)\mathbf{x}^{\mathbf{u}}$$

Lemma. For $\frac{h(\mathbf{x})}{f(\mathbf{x})^m} = \sum a_{\mathbf{u}}(t)\mathbf{x}^{\mathbf{u}}$, the series $\sum a_{p\mathbf{u}}(t)\mathbf{x}^{\mathbf{u}}$ can be approximated p-adically by rational functions with powers of $f^\sigma(\mathbf{x})$ in the denominator.

Here f^σ is f with t substituted by t^p, e.g. for $f(\mathbf{x}) = 1 - tg(\mathbf{x})$ one has $f^\sigma(x) = 1 - t^p g(\mathbf{x})$. We thus have

$$\mathcal{C}_p(\Omega_f) \subset \widehat{\Omega}_{f^\sigma} = \text{ p-adic completion of } \Omega_{f^\sigma}$$

From Cartier operation to Frobenius structure

The R-linear operation

$$\mathcal{C}_p : \widehat{\Omega}_f \to \widehat{\Omega}_{f^\sigma}, \quad \sum a_{\mathbf{u}}(t)\mathbf{x}^{\mathbf{u}} \mapsto \sum a_{p\mathbf{u}}(t)\mathbf{x}^{\mathbf{u}}$$

- descends to cohomology:

$$\mathcal{C}_p \circ x_i \frac{\partial}{\partial x_i} = p\, x_i \frac{\partial}{\partial x_i} \circ \mathcal{C}_p \Rightarrow \quad \mathcal{C}_p(d\widehat{\Omega}_f) \subset d\widehat{\Omega}_{f^\sigma},$$

$$\mathcal{C}_p : \widehat{\Omega}_f / d\widehat{\Omega}_f \to \widehat{\Omega}_{f^\sigma} / d\widehat{\Omega}_{f^\sigma},$$

- commutes with derivations $\theta : R \to R$, e.g. $\theta = t\frac{d}{dt}$,

$$\mathcal{C}_p \circ \theta = \theta \circ \mathcal{C}_p.$$

Matrix of $\mathcal{C}_p$ on the cyclic submodule generated by $\omega = 1/f(\mathbf{x})$ yields the Frobenius structure for the differential operator L:

$$\mathcal{C}_p(1/f) = \sum_{j=0}^{r-1} \Phi_{0j}(t) \left(\theta^j \frac{1}{f}\right)^\sigma \mod d\widehat{\Omega}_{f^\sigma}.$$

Supercongruences

Theorem (MV-Frits Beukers, Dwork crystals III).[5] Let $1 \leq k < p$. Assume that R is p-adically complete and the k'th *Hasse–Witt condition* is satisfied. Then

$$\widehat{\Omega}_f = \Omega_f(k) \oplus \mathcal{F}_k,$$

where

$$\Omega_f(k) = \text{free } R\text{-module generated by } \tfrac{\mathbf{x}^{\mathbf{u}}}{f(\mathbf{x})^k},\ \mathbf{u} \in k\Delta \cap \mathbb{Z}^n$$

and

$$\begin{aligned}\mathcal{F}_k &= \left\{ \omega = \sum a_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} \in \widehat{\Omega}_f \;\middle|\; \forall \mathbf{u} \quad a_{\mathbf{u}} \in g.c.d.(u_1, \ldots, u_n)^k R \right\} \\ &= \left\{ \omega = \sum a_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} \in \widehat{\Omega}_f \;\middle|\; \forall s \geq 1 \quad \mathcal{C}_p^s(\omega) \in p^{ks} \widehat{\Omega}_{f^{\sigma^s}} \right\} \\ &= \widehat{\Omega}_f \ \cap\ R\text{-module generated by } x_{i_1} \frac{\partial}{\partial x_{i_1}} \ldots x_{i_k} \frac{\partial}{\partial x_{i_k}} \sum b_{\mathbf{u}} \mathbf{x}^{\mathbf{u}}\end{aligned}$$

is the submodule of *formal kth partial derivatives*.

[5] For $k = 1$ this result is a version of N. Katz's *Internal reconstruction of unit-root F-crystals via expansion coefficients* (1985).

Supercongruences and vanishing of $\Phi_{01}(0) = p\alpha_1$

$$\Omega_f(\Delta^\circ) = \left\{ \frac{h(\mathbf{x})}{f(\mathbf{x})^m} \;\middle|\; m \geq 1,\; supp(h) \subset m\Delta^\circ \right\}$$

$G \subset GL_n(\mathbb{Z})$ group of symmetries of $f(\mathbf{x})$

$M = \Omega_f(\Delta^\circ)^G / d\Omega_f \cong \oplus_{j=0}^{3} R\, \theta^j(1/f), \quad \mathcal{C}_p : M \to M^\sigma$

$$\begin{array}{l} d\Omega_f = \{ \text{ partial derivatives } \} \subset \mathcal{F}_1 = \{ \text{ formal partial derivatives } \} \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \cup \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \mathcal{F}_2 = \{ \text{ formal 2nd partial derivatives } \} \end{array}$$

In the quintic case and several other cases which have geometric models with sufficiently large symmetry group G, one has

$$\{ \text{ partial derivatives } \} \cap \Omega_f(\Delta^\circ)^G \subset \mathcal{F}_2.$$

Supercongruences and vanishing of $\Phi_{01}(0) = p\alpha_1$

$$M = \Omega_f(\Delta^\circ)^G / d\Omega_f \cong \oplus_{j=0}^3 R\,\theta^j(1/f), \quad \mathcal{C}_p : M \to M^\sigma$$

$$d\Omega_f \cap \Omega_f(\Delta^\circ)^G \subset \mathcal{F}_2, \quad M/\mathcal{F}_2 = R\,1/f + R\,\theta(1/f)$$

$\rightsquigarrow$

$$\mathcal{C}_p(1/f) = \sum_{j=0}^{3} \Phi_{0j}(t)\theta^j(1/f)^\sigma \qquad \bmod d\widehat{\Omega}_{f^\sigma}$$

$$= \mu_0(t)1/f^\sigma + \mu_1(t)\theta(1/f)^\sigma \quad \bmod \mathcal{F}_2$$

$$\mu_0(0) = \Phi_{00}(0), \quad \mu_1(0) = \Phi_{01}(0)$$

For the expansion coefficients $\frac{1}{f(\mathbf{x})} = \sum a_{\mathbf{u}}(t)\mathbf{x}^{\mathbf{u}}$ this yields congruences

$$a_{p^{s+1}\mathbf{u}}(t) \equiv \mu_0(t)a_{p^s\mathbf{u}}(t^p) + \mu_1(t)(\theta a_{p^s\mathbf{u}})(t^p) \quad \bmod p^{2s}.$$

These explicit congruences allow us to check the vanishing of $\mu_1(0) = p\alpha_1$, which is the crusial step in establishing integrality of instanton numbers.

F. Beukers, M. Vlasenko, *On p-integrality of instanton numbers*, Pure and Applied Mathematics Quarterly, vol. ?

Work in progress:

$$M = \Omega_f(\Delta^\circ)^G / d\Omega_f \cong \oplus_{j=0}^{3} R\, \theta^j(1/f), \quad \mathcal{C}_p : M \to M^\sigma$$

$$\mathcal{C}_p(1/f) = \sum_{j=0}^{3} \Phi_{0j}(t)\theta^j(1/f)^\sigma \qquad \text{mod } d\widehat{\Omega}_{f^\sigma}$$

Considering this identity modulo $\mathcal{F}_3$, we can solve the respective supercongruences to check the vanishing of $\Phi_{02}(0) = p^2\alpha_2$. Similarly, working modulo $\mathcal{F}_4$ we can compute the value of α_3 and check the conjecture of Candelas, de la Ossa and van Straten.

Thank you!