Motivic gamma functions

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work in progress
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(1)

$$L = p_0(t)\frac{d^N}{dt^N} + p_1(t)\frac{d^{N-1}}{dt^{N-1}} + \dots + p_N(t) \qquad p_j \in \mathbb{C}(t)$$

$\downarrow$ entire functions

$$\underset{(\sigma,\varphi)}{\Gamma}(s) = \int_\sigma t^s \varphi(t)\frac{dt}{t}$$

σ closed path avoiding $0, \infty$ and singularities of L, contractible in $\mathbb{C}^*$

φ ~~so~~lution to L with trivial monodro[my] around σ

A gamma function is <u>motivic</u> when L is of Picard-Fuchs type and $\varphi(t)$ is a period function

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ U/K \end{array} \qquad \varphi(t) = \int_{\gamma(t)} \omega(t)$$

<u>Example</u> $L = (1-t)\frac{d}{dt} - \frac{1}{2} \qquad \varphi(t) = (1-t)^{-1/2}$

$$\underset{(\sigma,\varphi)}{\Gamma}(s) = \int_\sigma t^{s-1}(1-t)^{-1/2}\,dt$$

$$= \int_0^1 - \int_1^0 - e^{-2\pi i s}\int_0^1 + e^{-2\pi i s}\int_1^0$$

$$= 2\left(1-e^{-2\pi i s}\right)\frac{\Gamma(s)\Gamma(1/2)}{\Gamma(s+1/2)} \qquad \begin{array}{l}\text{entire}\\ \text{motivic}\end{array}$$

More generally:

$$\Gamma_{\xi}(s) = \sum_m e^{2\pi i m s} \int_{\sigma_m} \varphi_m(t)\, t^{s-1}\, dt \qquad (*)$$

<u>Claim</u> For given L, gamma functions form a finitely generated $K[e^{\pm 2\pi i s}]$-module.

Why? $U = \mathbb{P}^1(\mathbb{C}) \setminus \{0, \infty, \text{singular pts of } L\}$

$Sol(L)$ local system of K-vector spaces on U

$K \subseteq \mathbb{C}$

$$\xi = \left[\sum e^{2\pi i m s} \; \varphi_m \big|_{\sigma_m} \right] \in H_1(U, Sol(L) \otimes t^s)$$

$$t^s = Sol\left(t\frac{d}{dt} - s\right)$$

$(*)$ depends only on the homological class

and

$H_1(U, Sol(L) \otimes t^s)$ is a f.g. $K[e^{\pm 2\pi i s}]$-module

② Monodromy

$t=0$ regular singular point for L

$\rho_1, \dots, \rho_N$ local exponents

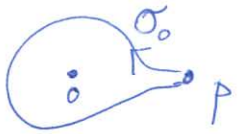
solutions $\quad \varphi_0(t) = t^\rho(1 + a_1 t + a_2 t^2 + \dots) = t^\rho \varphi_0^{an}(t)$

$$\varphi_1(t) = t^\rho(\varphi_0^{an}(t) \cdot \log t + \varphi_1^{an}(t))$$

$$\varphi_2(t) = t^\rho\left(\varphi_0^{an}(t)\frac{(\log t)^2}{2!} + \varphi_1^{an}(t)\log t + \varphi_2^{an}(t)\right)$$

$\dots$

<u>Frobenius basis</u> $\quad \varphi_j \in Sol_p(L) \otimes_K \mathbb{C}$



$$[\sigma_0]\varphi_0 = e^{2\pi i \rho}\varphi_0$$

$$[\sigma_0]\varphi_1 = e^{2\pi i \rho}(\varphi_1 + 2\pi i \varphi_0)$$

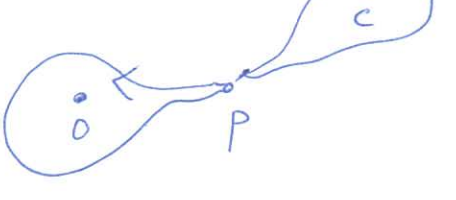
$$[\sigma_0]\varphi_2 = e^{2\pi i \rho}\left(\varphi_2 + 2\pi i \varphi_1 + \frac{(2\pi i)^2}{2!}\varphi_0\right)$$

$\dots$

when L is geometric:

- all singularities are regular
- local monodromies quasi-unipotent ($\rho \in \mathbb{Q}$)
- Frobenius basis spans de Rham structure of the limiting MHS

zoom out



$$\varphi_k^{(c)} = \sum C_{jk}\, \varphi_j^{(0)}$$

connection matrix

Special case:

(C) $\dim \operatorname{Im}([\sigma_c]-1) = 1$

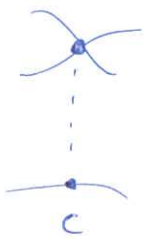
fix $\delta \in Sol_p(L)$ s.t. $\operatorname{Im}([\sigma_c]-1) = K\delta$

$$([\sigma_c]-1)\,\varphi_j^{(0)} = \varkappa_j\,\delta$$

$\varkappa_0, \dots, \varkappa_{N-1}$ <u>Apéry constants</u>

Example: conifold point

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ \mathcal{U} \end{array}$$



$$([\sigma_c]-1)\,\varphi = \pm\langle \varphi, \delta\rangle\,\delta$$

intersection pairing

vanishing cycle at c

Picard-Lefschetz theorem

$$\varkappa_j = \pm\langle \varphi_j^{(0)}, \delta\rangle$$

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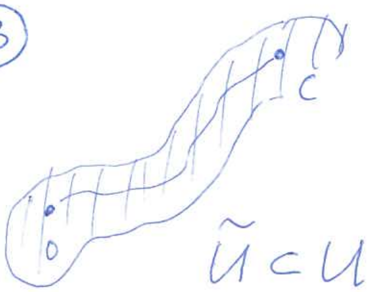

$\tilde{U} \subset U$

Theorem Under assumption (C)

$\Gamma_\xi(s)$, $\xi \in H_1(\tilde{U}, \mathcal{S}ol(L) \otimes t^s)$ is a free $K[e^{\pm 2\pi i s}]$-module of rank ≤ 1.

If $[\sigma_0]$ has a unique eigenvector (all local exps $= \rho$)

then the generator satisfies $\Gamma_\xi(-\rho) \neq 0$

and

$$\left(\frac{2\pi i s}{e^{2\pi i s} - 1}\right)^N \Gamma_\xi(s - \rho) = \varkappa(s) = \sum_{j=0}^{\infty} \varkappa_j s^j$$

$\varkappa_0, \ldots, \varkappa_{N-1}$ are Apéry constants for the adjoint differential operator $L^\vee$

and $\left(t\frac{d}{dt} - \rho\right)^k L^\vee \quad k \geq 1$

$$L^\vee \varphi_{N+k} = t^{-\rho} \frac{(\log t)^k}{k!} \quad k = 0, 1, 2, \ldots$$

$$([\sigma_c] - 1)\varphi_{N+k} = \varkappa_{N+k}\,\delta$$ higher Apéry constants (Golyshev–Zagier)

Corollary When L is geometric all $\varkappa_j$ are periods

iterated integrals

$$\Gamma(s) = \int_\sigma t^s \varphi(t) \frac{dt}{t}$$

$$\Gamma^{(j)}(0) = \int_\sigma (\log t)^j \varphi(t) \frac{dt}{t}$$

$$\rightsquigarrow \int_\sigma \underbrace{\frac{dt}{t} \cdots \frac{dt}{t}}_{j} \varphi(t) \frac{dt}{t} \qquad \text{Chen}$$

<u>Example</u> $L = (1-t)\frac{d}{dt} - \frac{1}{2}$

$$\mathcal{H}(s) = \frac{s\,\Gamma(s)\Gamma(1/2)}{\Gamma(s+1/2)} = 2^{2s} \frac{\Gamma(1+s)^2}{\Gamma(1+2s)}$$

$$= \exp\left(2 \log 2\, s + \sum_{j=2}^{\infty} \frac{\zeta(j)}{j} (-s)^j (2-2^j) \right)$$