Gamma functions, monodromy and Apéry constants

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work in progress
joint with Spencer Bloch and Francis Brown

① L diff. operator on $U = \mathbb{G}_m \setminus S$

$\rightsquigarrow$ entire functions

$$\Gamma_{(\sigma,\varphi)}(s) = \int_\sigma t^s \varphi(t) \frac{dt}{t}$$

σ oriented closed path in U contractible in $\mathbb{G}_m$

φ holomorphic solution to $L\varphi = 0$ defined in a neighbourhood of σ and having trivial monodromy around σ $[\sigma]\varphi = \varphi$

A gamma function is called <u>motivic</u>

when L is of Picard-Fuchs type and $\varphi(t)$ is a period function

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ U/K \end{array} \qquad \varphi(t) = \int_{\delta(t)} \omega(t)$$

$\omega(t)$ ↳ de Rham form on $\mathcal{X}_t$

$\delta(t)$ ↳ Betti cycle

<u>Example</u> $L = (1-t)\frac{d}{dt} - \frac{1}{2}$ $\qquad \varphi(t) = (1-t)^{-1/2}$

$$\Gamma_{(\sigma,\varphi)}(s) = \int_\sigma \frac{t^s}{\sqrt{1-t}} \frac{dt}{t}$$

$$= \int_0^1 - \int_1^0 - e^{-2\pi i s}\int_0^1 + e^{-2\pi i s}\int_1^0 = 2(1-e^{-2\pi i s}) \int_0^1 \frac{t^s}{\sqrt{1-t}}\frac{dt}{t}$$

$$= 2(1-e^{-2\pi i s}) \frac{\Gamma(s)\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+s)}$$

entire
motivic

More generally:

$Sol(L)$ local system of K-vector spaces $K \subseteq \mathbb{C}$

$$\xi \in H_1(U, Sol(L) \otimes t^s)$$

$$t^s = Sol\left(t\frac{d}{dt} - s\right)$$

$$= H_1\left(\pi_1(U,p), Sol_p(L) \underset{K}{\otimes} K[e^{\pm 2\pi i s}]\right)$$

$\pi_1(U,p)$ acts by monodromy of x^s

$$\xi \sim \sum_m \sigma_m \otimes \varphi_m \otimes e^{2\pi i m s}$$

$$\Gamma_\xi(s) = \sum e^{2\pi i m s} \int_{\sigma_m^{-1}} \varphi_m(t)\, t^s \frac{dt}{t}$$ well defined

<u>Proposition</u> $H_1(U, Sol(L) \otimes t^s)$ is a module of finite rank over $K[e^{\pm 2\pi i s}]$

<u>Remark</u> Application to interpolation of recurrences

$$L = p_0(D) + t\, p_1(D) + \dots + t^r p_r(D) \qquad D = t\frac{d}{dt}$$

$$\varphi = \sum a_m t^m$$

$$L\varphi = 0 \iff \sum_{j=0}^{r} p_j(m-j)\, a_{m-j} = 0$$

$a_s = \Gamma_\xi(-s)$ satisfies

$$\sum_{j=0}^{r} p_j(s-j)\, a_{s-j} = 0$$

② Local monodromy $\quad t=0$

$$t^{\alpha} L = D^N + q_0(t) D^{N-1} + \dots + q_N(t) \qquad D = t\frac{d}{dt}$$

$t=0$ is a regular singularity $\iff$ all $q_i(t)$ holomorphic at $t=0$

local exponents $\rho_1, \dots, \rho_N$ = solutions to

$$\rho^N + q_0(0)\rho^{N-1} + \dots + q_N(0) = 0$$

$$[\sigma_0] \sim \exp\left(2\pi i \begin{pmatrix} \rho & 1 & & 0 \\ & \rho & \ddots & \\ & & \ddots & 1 \\ 0 & & & \rho \end{pmatrix}\right)$$

$\uparrow$

in the Frobenius basis

$$\varphi_0(t) = t^{\rho}\left(1 + a_1 t + a_2 t^2 + \dots\right) = t^{\rho} \varphi_0^{an}(t)$$

$$\mathrm{Sol}_p(L) \otimes_K \mathbb{C} \ni \quad \varphi_1(t) = t^{\rho}\left(\varphi_0^{an}(t)\log t + \varphi_1^{an}(t)\right)$$

$$\varphi_2(t) = t^{\rho}\left(\varphi_0^{an}(t)\frac{\log^2 t}{2!} + \varphi_1^{an}(t)\log t + \varphi_2^{an}\right)$$

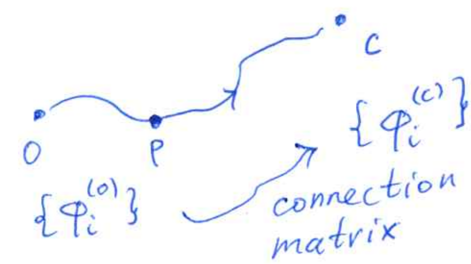
$$\dots$$

when L is geometric

- all singularities are regular
- local monodromies quasi-unipotent
- Frobenius basis spans the Betti structure of the limiting MHS

③ Apéry constants

zoom out



Assumption (C)

$$\dim \operatorname{Im}([\sigma_c]-1) = 1$$

fix $\delta \in \mathrm{Sol}_p(L)$

$$\operatorname{Im}([\sigma_c]-1) = K\delta$$

Example (when (C) holds)

$\mathcal{X}$ ⤫ c conifold point

$\downarrow$

U —•— c

$$([\sigma_c]-1)\varepsilon = \pm\langle \varepsilon, \delta\rangle\, \delta$$

δ ← vanishing cycle at c

$\langle \varepsilon, \delta\rangle$ ← cup product

Picard-Lefschetz theorem

$\{\varphi_i^{(0)}\}$

Frobenius basis at another singularity

Def $\varkappa_0, \dots, \varkappa_{N-1} \in \mathbb{C}$

$$([\sigma_c]-1)\varphi_i^{(0)} = \varkappa_i\, \delta$$

Apéry constants

Example $L = D^3 - t(2D+1)(17D^2+17D+5) + t^2(D+1)^3$

$$= t^3(1-34t+t^2)\frac{d^3}{dt^3} + \dots$$

$t=0$ maximally unipotent $\quad [\sigma_0] \sim \exp\left(2\pi i \begin{pmatrix} 0&1&0\\0&0&1\\0&0&0\end{pmatrix}\right)$

$t = c = 17-\sqrt{17^2-1} = 0.0294\ldots$ conifold $\quad [\sigma_c] \sim \begin{pmatrix} -1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

$\mathcal{X}$

$\downarrow$

$U = \mathbb{C}^\times \setminus \{c, \frac{1}{c}\}$

family of K3 surfaces

$$1 - tF(x_1,x_2,x_3) = 0$$

$$\varphi_0^{(0)}(t) = \frac{1}{(2\pi i)^3}\iiint_{|x_i|=1} \frac{1}{1-tF(x)}\frac{dx_1}{x_1}\frac{dx_2}{x_2}\frac{dx_3}{x_3}$$

$$= 1 + 5t + 73t^2 + 1445t^3 + \dots$$

$\varkappa_0 = 1 \qquad \varkappa_1 = 0 \qquad \varkappa_2 = -\frac{\pi^2}{3}$

$DL = D^4 + \dots$

$t=0$ is (MUM) $\qquad L\varphi_3 = 1$

$t=c$ is (C) $\qquad ([\sigma_c]-1)\varphi_3 = \varkappa_3\,\delta$

$$\varkappa_3 = \frac{17}{6}\zeta(3)$$

Golyshev–Zagier: one can continue…

Proposition

$t=0$ (MUM) for $L \Rightarrow$ (MUM) for $(D-\rho)^m L \qquad m=1,2,3,\dots$

$t=c$ (C) for $L \Rightarrow$ (C)

$$\underbrace{\varphi_0,\dots,\varphi_{N-1}}_{\text{Frob. basis at } t=0 \text{ solutions for } L},\ \underbrace{\varphi_N,\ \varphi_{N+1},\ \dots}_{\text{inhomogeneous solutions}} \qquad L\varphi_{N+m} = t^{\rho}\frac{(\log t)^m}{m!}$$

$$\varkappa_N \quad \varkappa_{N+1} \quad \dots$$

$$\varkappa(s) = \sum_{n=0}^{\infty} \varkappa_n s^n$$

Apéry example (experimentally):

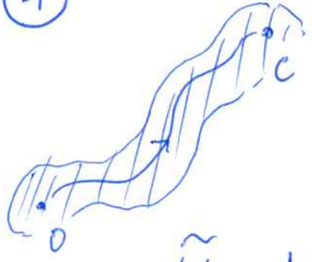
$\varkappa_4 = \frac{\pi^4}{45} = \frac{4}{5}\zeta(2)^2 = 2\zeta(4)$

$\varkappa_5 = \frac{7}{3}\zeta(5) - \frac{17}{3}\zeta(3)\zeta(2)$

…

$\varkappa_{11}$ involves MZVs $= \frac{2}{3}\zeta(3,5,3) + \dots$

Note: there is no apparent reason for $(D-\rho)^m L$ to be geometric when L was…



$\tilde{U} \subset U$

Theorem Under assumption (C)

$\Gamma_\xi(s)$, $\xi \in H_1(\tilde{U}, Sol(L) \otimes t^s)$

is a free $K[e^{\pm 2\pi i s}]$-module of rank ≤ 1.

If $t=0$ is (MUM) then the generator of this module satisfies

$$(\mathcal{H}_0 \neq 0) \qquad \left(\frac{2\pi i s}{e^{2\pi i s} - 1}\right)^N \Gamma_\xi(s+\rho) = \mathcal{H}(s)$$

$$(\mathcal{H} = O(s^m)) \qquad \left(\frac{2\pi i s}{e^{2\pi i s} - 1}\right)^{N-m} \Gamma_\xi(s+\rho) = \frac{\mathcal{H}(s)}{(2\pi i s)^m}$$

$0 \leq m \leq N-1$

Apéry constants for the adjoint $L^\vee = \ldots$

Corollary When L is geometric all $\mathcal{H}_n$ are periods / iterated integrals

$$\Gamma(s) = \int_\sigma t^s \varphi(t) \frac{dt}{t} \qquad \Gamma^{(n)}(0) = \int_\sigma (\log t)^n \varphi(t) \frac{dt}{t}$$

$$\rightsquigarrow \int_\sigma \underbrace{\frac{dt}{t} \cdots \frac{dt}{t}}_{n} \varphi(t) \frac{dt}{t}$$

$\gamma : [0,1] \to \sigma \qquad \omega_1, \ldots, \omega_k$ 1-forms

Chen $\displaystyle\int_\sigma \omega_1 \ldots \omega_k = \int_{0 \leq u_1 \leq \ldots \leq u_k \leq 1} \!\!\cdots\!\! \int \gamma^* \omega_1(u_1) \ldots \gamma^* \omega_k(u_k)$

Example

$$\mathcal{L} = (1-t)\, t \frac{d}{dt} - \frac{t}{2} = D - t\left(D + \frac{1}{2}\right)$$

$$\mathcal{H}(s) = \frac{s\,\Gamma(s)\,\Gamma(1/2)}{\Gamma(s+1/2)} = 2^{2s} \frac{\Gamma(1+s)^2}{\Gamma(1+2s)}$$

$$= \exp\left(2 \log 2\, s + \sum_{k=2}^{\infty} \frac{\zeta(k)}{k} (-s)^k (2 - 2^k) \right)$$

(*) $\mathcal{L} = q_0(t) D^N + q_1(t) D^{N-1} + \dots + q_N(t)$

adjoint

$$\mathcal{L}^\vee = -(-D)^N q_0(t) - (-D^{N-1}) q_1(t) \dots - q_N(t)$$