Appendix:
families of elliptic curves

May 25, 2022

Elliptic curves and complex tori

$$E_{a,b} \ : \ y^2 = x^3 + ax + b, \qquad 4a^3 + 27b^2 \neq 0$$

$$j(E_{a,b}) = 1728 \, \frac{4a^3}{4a^3 + 27b^2}$$

Theorem. The map

$$\mathbb{C}/\Lambda \to E_{a,b}, \qquad z \mapsto [\wp(z) : \tfrac{1}{2}\wp'(z) : 1]$$

where

$$a = -15 G_4(\Lambda), \quad b = -35 G_6(\Lambda),$$

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

is an isomorphism of Riemann surfaces and a homomorphism of abelian groups.

$$j\left(\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})\right) = j(\tau) = \frac{1}{q} + 744 + \dots, \quad q = e^{2\pi i \tau}$$

Families of elliptic curves: an example

$$E_t \; : \; y^2 \; = \; x(x-1)(x-t) \qquad t \neq 0, 1$$
$$= \; x^3 - (1+t)x^2 + tx$$
$$= \; \left(x - \tfrac{1+t}{3}\right)^3 \; + \; a\left(x - \tfrac{1+t}{3}\right) + b$$
$$a = -\frac{1}{3}(t^2 - t + 1), \quad b = -\frac{1}{27}(t-2)(t+1)(2t-1)$$
$$j(t) \; = 1728\frac{4a^3}{4a^3 + 27b^2} = 256\,\frac{(t^2 - t + 1)^3}{t^2(t-1)^3}$$
$$256(t^2 - t + 1)^3 = j(t) \cdot t^2(t-1)^2$$

$\Rightarrow j(t)$ is generically 6 : 1

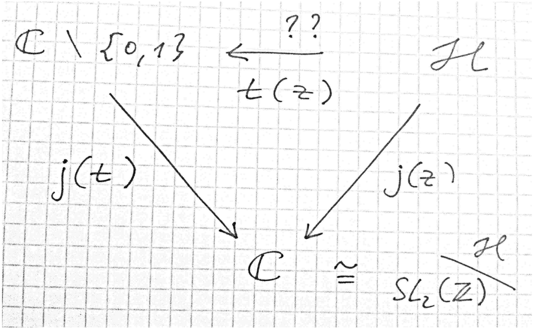
each isomorphism class of elliptic curves over $\mathbb{C}$

occurs in this family exactly 6 times, with a few exceptions

An example: Legendre family

$$E_t \ : \ y^2 \ = \ x(x-1)(x-t) \qquad t \in \mathbb{C}\backslash\{0,1\}$$

$$j(t) \ = 256\,\frac{(t^2-t+1)^3}{t^2(t-1)^2}$$



Problem: find a modular function $t(z)$ such that $j(t(z)) = j(z)$.

We expect $t(z)$ to be modular on a subgroup of index 6.

An example: Legendre family

$$256(t^2 - t + 1)^3 = j \cdot t^2(t-1)^2$$

$$t(z) \;=\; \frac{a}{q^\alpha} + \dots \qquad a, \alpha?$$

$$256\frac{a^6}{q^{6\alpha}} + \dots \;=\; \left(\frac{1}{q} + 744 + \dots\right)\left(\frac{a^4}{q^{4\alpha}} + \dots\right)$$

$$6\alpha = 4\alpha + 1 \;\Rightarrow\; \alpha = \frac{1}{2}$$

$$256a^2 = 1 \;\Rightarrow\; a = \pm\frac{1}{16}$$

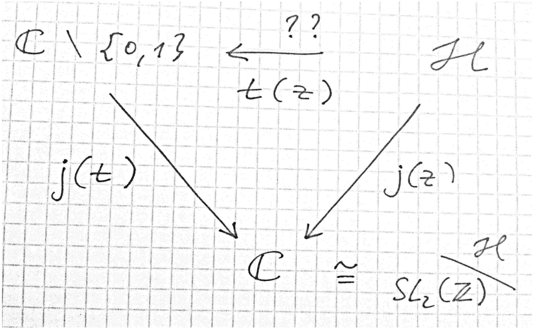
$$t(z) \;=\; \tfrac{1}{16}q^{-\frac{1}{2}} + c + \dots \qquad c?$$

$$\dots$$

$$t(z) \;=\; \tfrac{1}{16}q^{-\frac{1}{2}} + \tfrac{1}{2} + \tfrac{5}{4}q^{\frac{1}{2}} - \tfrac{31}{8}q^{\frac{3}{2}} + O(q^{\frac{5}{2}})$$

An example: Legendre family

$$E_t \ : \ y^2 \ = \ x(x-1)(x-t) \qquad t \in \mathbb{C}\backslash\{0,1\}$$
$$t(z) \ = \ \tfrac{1}{16}q^{-\frac{1}{2}} + \tfrac{1}{2} + \tfrac{5}{4}q^{\frac{1}{2}} - \tfrac{31}{8}q^{\frac{3}{2}} + O(q^{\frac{5}{2}})$$



- We expect $t(z)$ to be modular on a subgroup of index 6.
- $t(z) \neq 0$ for $z \in \mathcal{H}$. Try to guess it as an η-product?

An example: Legendre family

$$E_t \; : \; y^2 \; = \; x(x-1)(x-t) \qquad t \in \mathbb{C}\backslash\{0,1\} = \mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$$

$$\begin{aligned} t(z) \; &= \; \tfrac{1}{16} q^{-\frac{1}{2}} + \tfrac{1}{2} + \tfrac{5}{4} q^{\frac{1}{2}} - \tfrac{31}{8} q^{\frac{3}{2}} + O(q^{\frac{5}{2}}) \\ &= \; \tfrac{1}{16} q^{-\frac{1}{2}} \left(1 + 8q^{\frac{1}{2}} + 20q - 62q^2 + O(q^3)\right) \\ &\overset{?}{=} \; \tfrac{1}{16} \frac{\eta(z)^{24}}{\eta(\frac{z}{2})^8 \eta(2z)^{16}} \end{aligned}$$

We can use *PARI/GP* to confirm this guess: one checks that $256(t^2 - t + 1)^3 = j \cdot t^2(t-1)^2$ with any high precision $O(q^N)$.

The above given $t(z) = \frac{1}{16} \frac{\eta(z)^{24}}{\eta(\frac{z}{2})^8 \eta(2z)^{16}}$ is a modular function for $\Gamma(2)$. It plays the same role as j-invariant does for $SL_2(\mathbb{Z})$:

$$t : X(\Gamma(2)) \cong \mathbb{P}^1(\mathbb{C}), \qquad X(\Gamma) = {}_{\Gamma}\backslash \left(\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})\right)$$

$$t : {}_{\Gamma}\backslash\mathbb{P}^1(\mathbb{Q}) \cong \{0,1,\infty\} \qquad t(\infty) = \infty, t(0) = 1, t(1) = 0$$

Such a function for a subgroup of genus 0 is called a *Hauptmodul*.

Another example

Exercise: Play the above game with the family

$$E_t \;:\; y^2 \;=\; x^3 - 2x^2 + (1-t)x, \quad t \neq 0, 1.$$

You should discover a modular function $t(z)$ on a subgroup of index 3. The answer is given on the next page, don't turn over!

The answer

$$t(z) = 64q - 2560q^2 + 84736q^3 + \dots$$
$$= \frac{64\Delta(2z)}{\Delta(z) + 64\Delta(2z)}$$

is a Hauptmodul for $\Gamma_0(2)$.