1000–1M19WFM Introduction to Modular Forms
Tutorial 8 – May 10

Written assignment: exercises marked with (H), due on May 17.

1. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index containing -1. Let $f \in M_k(\Gamma)$ be a non-zero modular form. Prove that

$$\sum_{z \in \Gamma\backslash\mathcal{H}} \frac{\nu_z(f)}{e_z} + \sum_{\alpha \in \Gamma\backslash\mathbb{P}^1(\mathbb{Q})} \nu_\alpha(f) = \frac{k}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma],$$

where $e_z = \#I_{\Gamma/\{\pm 1\}}(z)$ is the order of stabilizer of z in $\Gamma/\{\pm 1\}$.

Remark: for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ we proved this formula in Lecture 2. Hints for this exercise will be given in a separate document.

(H)2. a) Let $\theta(z) = \sum_{n\in\mathbb{Z}} q^{n^2} = 1 + 2q^2 + 2q^4 + \dots$ be the Jacobi theta function (Lecture 8). Express $\theta^4 \in M_2(\Gamma_0(4))$ as a linear combination of the following basis elements of this space:

$$f_1 = E_2(z) - 2E_2(2z)\,,\ f_2 = E_2(2z) - 2E_2(4z)$$

(see Exercise 5 of Assignment 7). Comparing the coefficients near q^n, find the formula for the number of representations of n as a sum of four squares.

b) Prove the theorem of Lagrange: every positive integer is a sum of four squares.

(H)3. Use the Theorem of Hecke and Schoenberg to show that

$$\Theta(z) = \sum_{m,n\in\mathbb{Z}} q^{m^2+mn+n^2} \in M_1(\Gamma_0(3), \chi_{-3})$$

where

$$\chi_{-3}(n) = \left(\frac{-3}{n}\right) = \begin{cases} 0\,, 3 \mid n \\ 1\,, n \equiv 1 \mod 3 \\ -1\,, n \equiv 2 \mod 3 \end{cases}$$

is the nontrivial Dirichlet charater modulo 3.

(H)4. We continue with the notation of Exercise 3. In the same space we have the Eisenstein series

$$G_{1,\chi} = \frac{1}{6} + \sum_{n=1}^{\infty}\sum_{d|n} \chi_{-3}(d)\, q^n \in M_1(\Gamma_0(3), \chi_{-3})$$

(see Appendex 3). Use the fact that $\dim M_1(\Gamma_0(3), \chi_{-3}) = 1$ to find the number of integer solutions $(m, n) \in \mathbb{Z}^2$ to the equation

$$m^2 + mn + n^2 = 147\,.$$

5. Describe all spherical homogeneous polynomials $P(x, y)$ of degree 2 with respect to the quadratic form $Q(m, n) = m^2 + mn + n^2$. Write

$$\Theta_P(z) = \sum_{m,n \in \mathbb{Z}} P(m, n)\, q^{m^2+mn+n^2}$$

in terms of the basis

$$G_{3,\chi_{-3}} = -\frac{1}{9} + \sum_{n=1}^{\infty} \sum_{d|n} \chi_{-3}(d) d^2\, q^n ,$$

$$G'_{3,\chi} = \sum_{n=1}^{\infty} \sum_{d|n} \chi_{-3}(d) \left(\frac{n}{d}\right)^2 q^n$$

of the space $M_3(\Gamma_0(3), \chi_{-3})$.

Remark: it turns out that all $\Theta_P(z)$ in the last exercise vanish. Can you prove it directly?