1000–1M19WFM Introduction to Modular Forms
Hints for Exercise 1 in Assignment 8

Exercise. *Let* $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ *be a subgroup of finite index containing* -1. *Let* $f \in M_k(\Gamma)$ *be a non-zero modular form. Prove that*

$$(1) \qquad \sum_{z \in \Gamma\backslash\mathcal{H}} \frac{\nu_z(f)}{e_z} + \sum_{\alpha \in \Gamma\backslash\mathbb{P}^1(\mathbb{Q})} \nu_\alpha(f) \;=\; \frac{k}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma],$$

where $e_z = \#I_{\Gamma/\{\pm 1\}}(z)$ *is the order of stabilizer of* z *in* $\Gamma/\{\pm 1\}$.

Note that for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ we proved formula (1) in Lecture 2 by doing integration of df/f along a contour which tends to the boundary of the fundamental domain. That proof was already quite elaborate, and we would like to avoid performing it again for general subgroups Γ.

Approach 1. Note that k is even because the condition $-1 \in \Gamma$ implies that there are no non-zero modular forms of odd weight. In Lecture 7 we defined $\omega_f \in \Omega^{k/2}(X(\Gamma))$, a meromorphic $(k/2)$-form on the compact Riemann surface $X = X(\Gamma)$ which corresponds to f, and computed its orders of vanishing $\nu_P(\omega_f)$ for $P \in X$ in terms of $\nu_Q(f)$ for $Q \in \mathcal{H}$. Combine that computation with the fact that

$$\deg(div(\omega_f)) = \sum_{P \in X} \nu_P(\omega_f) = (2g-2)\frac{k}{2}$$

(see Exercise 2 of Assignment 7).

Approach 2. Denote $d = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$. Choose some set of right-coset represetatives $\mathrm{SL}_2(\mathbb{Z}) = \cup_{j=1}^{d} \Gamma g_j$ and consider

$$F(z) = \prod_{j=1}^{d} (f|_k g_j)(z).$$

Note that $F \in M_{dk}(\mathrm{SL}_2(\mathbb{Z}))$ and apply (1) for F, which was proved in Lecture 2.