1000–1M19WFM Introduction to Modular Forms
Tutorial 7 – April 26

Written assignment: exercises marked with (H), due on May 10.

(H)1. Let X be a compact Riemann surface.

a) Let $\omega \in \Omega(X)$ be a meromorphic 1-form. Prove that the sum of its residues vanishes: $\sum_{P \in X} Res_P(\omega) = 0$.

b) For a non-constant meromorphic function $f \in \mathcal{M}(X)$, compute residues of the form $\omega = df/f$. What does part a) imply for meromorphic functions?

c) Show that there are no non-constant holomorphic functions on X.

(H)2. Deduce the following fact from the Riemann–Roch Theorem: on a compact Riemann surface X for any non-zero k-form $0 \neq \omega \in \Omega^{\otimes k}(X)$ one has $\deg(div(\omega)) = (2g-2)k$.

Remark: We use this fact in Lecture 7 to prove dimension formulas for modular forms of even weight.

(H)3. Let $N \geq 1$ and $\mathcal{M}_N = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{Z}\,,\; ad - bc = N \right\}$ be the set of integer matrices with determinant N. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on this space by multiplication from the left. Prove that every orbit for this action has a unique representative of the form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \qquad a, d > 0, ad = N, 0 \leq b < d\,.$$

Remark: It follows that there are exactly $\sigma_1(N)$ orbits for $\mathrm{SL}_2(\mathbb{Z})$ in $\mathcal{M}_N$.

(H)4. Prove that if $f(z) \in M_k(\Gamma_0(M))$ than $f(Nz) \in M_k(\Gamma_0(MN))$.

Hint: One can use previous exercise to prove property (iii), i.e. boundness at all cusps. Namely, prove that if f is such that

$$f|_k g \;=\; O(1) \;\text{ when }\; \Im(z) \to \infty$$

for every $g \in \mathrm{SL}_2(\mathbb{Z})$, than for any $m \in \mathcal{M}_N$ the function $f|_k m$ also has this property. To get $f(Nz)$ we take $m = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}$.

5. Compute $\dim_{\mathbb{C}} M_2(\Gamma_0(4))$ and construct a basis in this space.

Hint: You can use the Eisenstein series of weight 2 and the idea from Exercise 3.