1000–1M19WFM Introduction to Modular Forms
Tutorial 3

Written assignment: exercises marked with (H), due on March 29.

1. In Exercise 6 of Assignment 1 we discovered that the Eisenstein series can be normalized to have rational Fourier coefficients:
$$E_k(z) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n.$$
Here $q = e^{2\pi i z}$, $\sigma_s(n) = \sum_{m|n} m^s$ and $B_k \in \mathbb{Q}$ are the Bernoulli numbers defined by $\frac{x}{e^x-1} = \sum_{k=0}^{\infty}\frac{B_k}{k!}x^k$. Note that the q-series E_4 and E_6 have coefficients in $\mathbb{Z}$. Using elementary arguments (e.g. Fermat's Little Theorem), show that the coefficients of $E_4^3 - E_6^2$ are divisible by 1728.

(H)2. Consider the coefficients in the expansion of the following cusp form of weight 12:
$$\Delta = \frac{E_4^3 - E_6^2}{1728} = \sum_{n=1}^{\infty}\tau(n)q^n\,.$$
By the previous exercise, we have $\tau(n) \in \mathbb{Z}$. The function $\tau : \mathbb{N} \to \mathbb{Z}$ is called *the Ramanujan tau function* after the Indian mathematician Srinivasa Ramanujan, who experimented a lot with it. In 1916 he conjectured that
$$\tau(mn) = \tau(m)\tau(n) \text{ when } m, n \text{ are coprime}$$
$$\tau(p^{r+1}) - \tau(p)\tau(p^r) + p^{11}\tau(p^{r-1}) = 0 \text{ for every } r \geq 1 \text{ and } p \text{ prime}\,.$$
We will prove this later, using Hecke operators. Ramanujan also discovered several congruence properties one of which is the subject of this exercise.

Prove that
$$\tau(n) \equiv \sigma_{11}(n) \mod 691\,.$$
Hint: Notice that $B_{12} = -\frac{691}{2730}$, so $E_{12} = 1 + \frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n$.

(H)3. Let $E_2(z) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n$ be the Eisenstein series of weight 2. In Assignment 2 we have seen that the transformation properties of this function are close to those of a modular form of weight 2.

Show that if $f \in M_k$ and δ denotes the derivation $q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{dz}$, then
$$\delta f - \frac{k}{12}E_2 f \in M_{k+2}.$$
(This modular form is sometimes called the Serre derivative of f.)

(H)4. Show that the ring $\mathbb{C}[E_2, E_4, E_6]$ is closed under derivation $\frac{d}{dz}$. More specifically, prove that
$$\delta E_2 = \frac{E_2^2 - E_4}{12}, \quad \delta E_4 = \frac{E_2E_4 - E_6}{3}, \quad \delta E_6 = \frac{E_2E_6 - E_4^2}{2}.$$
These identities were first observed by Ramanujan, who used them to prove some curious congruence properties of the partition function $p(n)$ (see our Project 1).

Elements of the ring $\widetilde{M} = \mathbb{C}[E_2, E_4, E_6]$ are called quasimodular forms. See §5.3 of [1-2-3] for their intrinsic definition.

Congruence subgroups

5. For an integer $N > 1$ we want to compute the indices of the following group inclusions:

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ : \ a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$
$$\cup$$
$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ : \ \ldots, \ c \equiv 0 \mod N \right\}$$
$$\cup$$
$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ : \ \ldots, \ c \equiv 0, a, d \equiv 1 \mod N \right\}$$
$$\cup$$
$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ : \ \ldots, \ b, c \equiv 0, a, d \equiv 1 \mod N \right\}$$

The index is mutiplicative: for $H_1 \subset H_2 \subset G$, one has $[G : H_1] = [G : H_2] \cdot [H_2 : H_1]$.

Since $\Gamma(N) = Ker\Big(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})\Big)$, $\Gamma(N)$ is a normal subgroup and $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)] = \#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.

a) Let p be a prime and let e be a positive integer. Show that $\#\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z}) = p^{3e}(1 - 1/p^2)$.
Hint: For $e = 1$, $\#\mathrm{GL}_2(\mathbb{F}_p)$ is the number of bases of $\mathbb{F}_p^2$, and $\mathrm{SL}_2(\mathbb{F}_p)$ is the kernel of the surjective determinant map to $\mathbb{F}_p^\times$. For the induction, count $Ker\Big(\mathrm{SL}_2(\mathbb{Z}/p^{e+1}\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})\Big)$. This map surjects since $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$ does.

b) Show that $\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) = N^3 \prod_{p|N}(1 - 1/p^2)$.

c) Show that the map $\Gamma_1(N) \to \mathbb{Z}/N\mathbb{Z}$ given by
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto b \mod N$$
surjects and has kernel $\Gamma(N)$. Compute $[\Gamma_1(N) : \Gamma(N)]$.

d) Show that the map $\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times$ given by
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \mod N$$
surjects and has kernel $\Gamma_1(N)$. Compute $[\Gamma_0(N) : \Gamma_1(N)]$.

e) Show that $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N}(1 + 1/p)$.