1000–1M19WFM Introduction to Modular Forms
Tutorial 10 - May 24

Written assignment: exercises marked with (H), due on May 31.

(H)1. Let $g(x) \in \mathbb{C}[x]$ be a polynomial of degree ≥ 1. Check that the plane curve

$$C = \{(x, y) \in \mathbb{C}^2 \; : \; y^2 = g(x)\}$$

is non-singular if and only if g has no multiple roots.

Remark: *Let $f \in \mathbb{C}[x, y]$ be a two variable polynomial and (x_0, y_0) be a point on the curve $\{f(x, y) = 0\}$. By the Implicit Function Theorem (see Appendix 5) when a partial derivative in some variable is non-vanishing, say $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$, then (x_0, y_0) is a non-singular point and the other variable (in our case this is x) can be taken as a local coordinate on the curve near (x_0, y_0).*

Note that for the curve $C = \{y^2 = g(x)\}$ this implies that x is a local coordinate near (x_0, y_0) whenever $g(x_0) \neq 0$, and otherwise $g'(x_0) \neq 0$ (by the no-multiple-roots condition) and y is a local coordinate.

2. In the notation of Exercise 1, consider the case of $\deg g(x) = 3$, g has no multiple roots. The projectivization of

$$C \quad : \quad y^2 = a_0x^3 + a_1x^2 + a_2x + a_3$$

is given by

$$\overline{C} \quad : \quad Y^2Z \; = \; a_0X^3 + a_1X^2Z + a_2XZ^2 + a_3Z^3 \, .$$

a) Show that $\overline{C} \backslash C \; = \; \{[0 : 1 : 0]\}$ in the projective coordinates $[X : Y : Z]$.
b) Show that $\overline{C}$ is a non-singular projective curve. Check that the rational function $x/y = X/Y$ is a local coordinate near the point $O = [0 : 1 : 0]$.
c) Find a linear change of coordinates in $\mathbb{P}^2(\mathbb{C})$ that brings the equation of $\overline{C}$ into the form

$$Y^2Z \; = \; X^3 + aXZ^2 + bZ^3.$$

What are the new coordinates of the point O?

(H)3. Let $g(x) = \sum_{i=0}^{n} a_i x^{n-i} \in \mathbb{C}[x]$ be a polynomial of degree n. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ be its roots: $g(x) = a_0 \prod_{i=1}^{n}(x - \alpha_i)$. The discriminant of g is defined as

$$\Delta_g = a_0^{2n-2} \prod_{i<j} (\alpha_j - \alpha_i)^2 \, .$$

Clearly, $\Delta_g \neq 0$ if and only if all roots of g are distinct. As a function of a_i's, Δ_g is a homogeneous polynomial of degree $2n-2$ with rational coefficients.

a) Check that the discriminant of a cubic polynomial $g(x) = cx^3 + ax + b$ is given by $\Delta_g = -4a^3c - 27b^2c^2$.

b) Let $z \in \mathcal{H}$. As we know (Assignment 10 and Lecture 11), the complex torus $\mathbb{C}/(\mathbb{Z}z + \mathbb{Z})$ is isomorphic to the elliptic curve

$$y^2 \;=\; x^3 - 15\, G_4(z)\, x - 35\, G_6(z),$$

where $G_k(z) = \sum_{(m,n)\neq(0,0)} \frac{1}{(mz+n)^k}$ are the Eisenstein series. In particular, the cubic polynomial on the right has distinct roots. Compute its discriminant as a function of z. Do you recognize this modular form?

4. As we know (Appendix 5), points of an elliptic curve form an abelian group. Compute the addition law on the curve

$$y^2 = x^3 + ax + b, \qquad 4a^3 + 27b^2 \neq 0.$$

Namely, check that $-(x, y) = (x, -y)$ and, when $x_1 \neq x_2$, one has

$$(x_1, y_1) \oplus (x_2, y_2) = (x_3, y_3),$$

with

$$x_3 = \left(\frac{y_1 - y_2}{x_1 - x_2}\right)^2 - x_1 - x_2,$$

$$y_3 = -\frac{y_1 - y_2}{x_1 - x_2}(x_3 - x_1) - y_1.$$

(H)5. We define a function F on pairs (Λ, S) where $\Lambda \subset \mathbb{C}$ is a lattice and $S \subset \mathbb{C}/\Lambda$ is a subgroup of order 2, by the formula

$$F(\Lambda, S) \;=\; \wp_\Lambda(z_S)$$

where $\wp_\Lambda : \mathbb{C} \setminus \Lambda \to \mathbb{C}$ is the Weierstrass p-function of Λ and z_S is a preimage in $\mathbb{C}$ of the unique nonzero element in S.

a) Check that F is a homogeneous function and determine its weight.

b) By construction, $f(z) = F(\mathbb{Z}z + \mathbb{Z}, \langle \frac{1}{2} \rangle)$ should transform like a modular form on $\Gamma_0(2)$. Identify this modular form.