FINITENESS OF THE HOMOTOPY TYPES OF RIGHT ORBITS OF MORSE FUNCTIONS ON SURFACES

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Abstract. Let M be a connected orientable surface, P be either the real line $\mathbb{R}$ or the circle S^1, and $f : M \to P$ be a Morse map. Denote by $\mathcal{D}_{\mathrm{id}}$ the group of diffeomorphisms of M isotopic to the identity. This group acts from the right on the space of smooth maps $C^\infty(M,P)$ and one can define the stabilizer $\mathcal{S} = \{h \in \mathcal{D}_{\mathrm{id}} \mid f \circ h = f\}$ and the orbit $\mathcal{O} = \{f \circ h \mid h \in \mathcal{D}_{\mathrm{id}}\}$ of f with respect to that action.

In turn, the stabilizer $\mathcal{S}$ acts on the Kronrod-Reeb graph Γ of f. Denote by G the group of all automorphisms of Γ induced by elements from $\mathcal{S}$.

It is proved that if M is distinct from the 2-sphere and 2-torus then there exists a free action of G on a p-dimensional torus T^p for some $p \geq 0$ such that the orbit $\mathcal{O}$ (endowed with C^∞ topology) is homotopy equivalent to the factor space T^p/G.

1. Introduction

Let M be a compact connected surface and P be either the real line $\mathbb{R}$ or the circle S^1. For a closed subset $V \subset M$ denote by $\mathcal{D}(M,V)$ the group of all C^∞ diffeomorphisms of M fixed on V. This group acts from the right on the space $C^\infty(M,P)$ by the following rule: if $h \in \mathcal{D}(M,V)$ and $f \in C^\infty(M,P)$, then the result of the action of h on f is the composition map $f \circ h : M \to P$.

For $f \in C^\infty(M,P)$ let

$$\mathcal{S}(f,V) = \{h \in \mathcal{D}(M,V) \mid f \circ h = f\}, \qquad \mathcal{O}(f,V) = \{f \circ h \mid h \in \mathcal{D}(M,V)\}$$

be respectively the *stabilizer* and the *orbit* of f under that action. Endow these spaces with C^∞ topologies and denote by $\mathcal{D}_{\mathrm{id}}(M,V)$ and $\mathcal{S}_{\mathrm{id}}(f,V)$ the corresponding path components of id_M in $\mathcal{D}(M,V)$ and $\mathcal{S}(f,V)$, and by $\mathcal{O}_f(f,V)$ the path component of $\mathcal{O}(f,V)$ containing f. We will omit V from notation whenever it is empty.

Definition 1.1. A smooth map $f : M \to P$ will be called *Morse* if

- all critical points of f are non-degenerate and belong to the interior of M;
- the restriction of f to each connected component of ∂M is a constant map.

In a series of papers [4, 5, 7, 8] the author studied the homotopy types of $\mathcal{S}_{\mathrm{id}}(f)$ and $\mathcal{O}_f(f)$ for certain classes of smooth mappings $f : M \to P$. The results concerning Morse maps can be summarized as follows.

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Theorem 1.2. [4, 5, 7, 8]. *Let $f : M \to P$ be a Morse map with n critical points, and V be a finite (possibly empty) union of regular (that is containing no critical points) connected components of certain level sets of f. Then the following statements hold.*

(1) *$\mathcal{O}_f(f,V) = \mathcal{O}_f(f, V \cup \partial M)$ and this space* ***has a homotopy type of some (possibly non-compact)*** $(2n-1)$***-dimensional CW-complex***.

(2) *The map $\sigma : \mathcal{D}(M,V) \longrightarrow \mathcal{O}(f,V)$ defined by $\sigma(h) = f \circ h$ is a Serre fibration with fiber $\mathcal{S}(f,V)$, i.e. it has a homotopy lifting property for CW-complexes.*

Suppose either f has a critical point of index 1 or M is non-orientable. Then $\mathcal{S}_{\mathrm{id}}(f)$ is contractible, $\pi_k\mathcal{O}(f) = \pi_k M$ for $k \geq 3$, $\pi_2\mathcal{O}(f) = 0$, and for $\pi_1\mathcal{O}(f)$ we have the following short exact sequence of fibration σ:

$$1 \longrightarrow \pi_1\mathcal{D}(M) \xrightarrow{\sigma} \pi_1\mathcal{O}(f) \xrightarrow{\partial} \pi_0\mathcal{S}'(f) \longrightarrow 1, \tag{1.1}$$

where $\mathcal{S}'(f) = \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(M)$. If f is ***generic****, i.e. each critical level set of f contains exactly one critical point, then $\mathcal{O}_f(f)$ is* ***homotopy equivalent to some*** p***-dimensional torus*** T^p.

(3) *The restriction $\sigma|_{\mathcal{D}_{\mathrm{id}}(M,V)} : \mathcal{D}_{\mathrm{id}}(M,V) \longrightarrow \mathcal{O}_f(f,V)$ is also a Serre fibration with fiber $\mathcal{S}'(f,V) = \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(M,V)$.*

Suppose either $\chi(M) < 0$ or $V \neq \varnothing$. Then $\mathcal{D}_{\mathrm{id}}(M,V)$ and $\mathcal{S}_{\mathrm{id}}(f,V)$ are contractible and from the exact sequence of homotopy groups of the fibration $\sigma|_{\mathcal{D}_{\mathrm{id}}(M,V)}$ we get that $\pi_k\mathcal{O}(f,V) = 0$ for $k \geq 2$ and the boundary map

$$\partial : \pi_1\mathcal{O}(f,V) \longrightarrow \pi_0\mathcal{S}'(f,V) \tag{1.2}$$

is an isomorphism. □

In the present note we obtain exact description of the homotopy types of $\mathcal{O}_f(f)$ for all Morse maps $f : M \to P$ for the case when M is orientable and distinct from S^2 and T^2, see Theorem 1.3 below.

Let $f : M \to P$ be a Morse map. Consider the partition of M into connected components of level-sets $f^{-1}(c)$, $c \in P$. Then the corresponding factor space has a structure of a finite one-dimensional CW-complex and is called the *Kronrod-Reeb* graph of f, see e.g. [1, 12]. We will denote this graph by Γ. Its vertices correspond to *critical* (i.e. containing critical points) connected components of level-sets of f and to the connected components of ∂M.

Recall, [4, §3.1], that $\mathcal{S}(f)$ naturally acts on Γ by the following rule. Let $h \in \mathcal{S}(f)$. Then $f \circ h = f$, and so $h(f^{-1}(c)) = f^{-1}(c)$ for all $c \in P$. Hence h permutes connected components of $f^{-1}(c)$ being *points* of Γ. In other words, h yields a bijection of Γ which is in fact homeomorphism. Therefore we get an action homomorphism $\lambda : \mathcal{S}(f) \to \mathrm{Aut}(\Gamma)$ into the group of all homeomorphisms of Γ. The image of λ is finite. Let

$$G := \lambda\big(\mathcal{S}'(f)\big)$$

be the group of all automorphisms of Γ induced by isotopic to the identity diffeomorphisms from $\mathcal{S}(f)$. Our main result is the following theorem.

Theorem 1.3. *Suppose M is orientable and distinct from 2-sphere S^2 and 2-torus T^2. Then for each Morse map $f : M \to P$ there exists a free action of the group $G = \lambda(\mathcal{S}'(f))$ on a p-dimensional torus T^p for some $p \geq 0$ such that $\mathcal{O}_f(f)$ is homotopy equivalent to the factor space T^p/G.*

This result refines statements (1) and (2) of Theorem 1.2. A sketch of a proof of Theorem 1.3 will be given in §3 and §4. The proof is constructive in the sense that for every Morse map $f : M \to P$ one can explicitly describe the corresponding free action of G on T^p which gives a homotopy equivalence $T^p/G \simeq \mathcal{O}_f(f)$.

If $M = S^2$, then by (2) of Theorem 1.2 $\mathcal{O}_f(f)$ is not aspherical, and so it is not homotopy equivalent to T^p/G for any free G-action on any torus T^p. On the other hand there are some partial results supporting the conjecture that Theorem 1.3 might hold for $M = T^2$, see [11].

In fact, Theorem 1.3 is true for a larger class of maps f having singularities smoothly equivalent to homogeneous polynomials $\mathbb{R}^2 \to \mathbb{R}$ without multiple factors. But in this case one should extend the notion of Kronrod-Reeb graph for such maps. This version of Theorem 1.3 together with detailed proofs will appear elsewhere.

2. Preliminaries

2.1. **Wreath products.** Let G and H be two groups. Then the set G^H of all maps $H \to G$ (not necessarily homomorphisms) is a group with respect to the point-wise multiplication. Moreover, the group H acts on G^H from the right by the following rule: if $\alpha : H \to G$, and $h \in H$, then the result α^h of the action of h on α is defined by the formula: $\alpha^h(s) = \alpha(sh)$, for all $s \in H$. The semidirect product $G^H \rtimes H$ associated with this action is called the *wreath product* of G and H and is denoted by $G \wr H$.

More generally, let K be another group and $\mu : K \to H$ be a homomorphism. This homomorphism induces a natural right action of K on H, and so one can define the corresponding semidirect product $G^H \rtimes_\mu K$ which will be called the *wreath product of G and K over μ* and denoted by $G \wr_\mu K$. Evidently, if $\mu = \mathrm{id}_H : H \to H$ is the identity isomorphism, then $G \wr_\mu H$ is the same as $G \wr H$.

In particular, for $m \geq 2$ let $\mu : \mathbb{Z} \to \mathbb{Z}_m$ be the natural mod m homomorphism. Then for every group $\mathcal{S}$ the wreath product $\mathcal{S} \wr_\mu \mathbb{Z}$ will be denoted by

$$\mathcal{S} \underset{\mathbb{Z}_m}{\wr} \mathbb{Z}.$$

Thus $\mathcal{S} \underset{\mathbb{Z}_m}{\wr} \mathbb{Z}$ is the set $\mathcal{S}^{\mathbb{Z}_m} \times \mathbb{Z}$ with the multiplication defined in the following way: if $(\alpha, a), (\beta, b) \in \mathcal{S}^{\mathbb{Z}_m} \times \mathbb{Z}$, then $(\alpha, a)(\beta, b) = (\gamma, a + b)$, where $\gamma : \mathbb{Z}_m \to \mathcal{S}$ is given by $\gamma(s) = \alpha(s + b \bmod m)\beta(s)$ for $s \in \mathbb{Z}_m$.

2.2. **Free actions on tori.** For $p \geq 0$ let $T^p = \mathbb{R}^p/\mathbb{Z}^p$ be a p-dimensional torus which can also be regarded as a topological product of p circles. We also assume that T^0 is a point.

Suppose a finite group G freely acts on T^p. Then the factor map $q : T^p \to T^p/G$ is a covering map. This implies that T^p/G is aspherical and we have the following short exact sequence:

$$1 \longrightarrow \pi_1 T^p \xrightarrow{q} \pi_1(T^p/G) \xrightarrow{\delta} G \longrightarrow 1,$$

where δ is the corresponding boundary homomorphism.

Definition 2.1. Let $\lambda : \mathcal{S} \to G$ be a group homomorphism. We will say that λ *arises from a free action on a torus* if there exists a free action of G on T^p for some $p \geq 0$ and the following commutative diagram:

$$\begin{array}{ccc} \pi_1(T^p/G) & \xrightarrow[\cong]{\alpha} & \mathcal{S} \\ \delta \downarrow & & \downarrow \lambda \\ G & \xrightarrow[\cong]{\beta} & G \end{array}$$

in which α and β are isomorphisms.

Lemma 2.2. (1) *A trivial homomorphism $e_n : \mathbb{Z}^n \to \{1\}$, $n \geq 0$, arises from the free trivial action of the unit group $\{1\}$ on T^n.*

(2) *For each $m \geq 1$ the canonical* (mod m)*-epimorphism $\mu : \mathbb{Z} \to \mathbb{Z}_m$ arises from the free action of $\mathbb{Z}_m$ on T^1 defined as follows: if $x \in T^1 = \mathbb{R}/\mathbb{Z}$ and $k \in \mathbb{Z}_m$, then* $z \cdot k = z + \frac{k}{m} \bmod 1$.

(3) *Suppose a homomorphism $\lambda_i : \mathcal{S}_i \to G_i$ arises from a free action of G_i on T^{p_i} for $i = 1, \ldots, n$. Denote $p = \sum_{i=1}^{n} p_i$ Then the product homomorphism*

$$\mathop{\times}_{i=1}^{n} \lambda_i : \mathop{\times}_{i=1}^{n} \mathcal{S}_i \longrightarrow \mathop{\times}_{i=1}^{n} G_i$$

arises from the free action of $\mathop{\times}\limits_{i=1}^{n} G_i$ on $T^p = T^{p_1} \times \cdots \times T^{p_n}$ defined as follows: if $x_i \in T^{p_i}$ and $g \in G_i$, then $(x_1, \ldots, x_n) \cdot (g_1, \ldots, g_n) = (x_1\, g_1, \ldots, x_n\, g_n)$.

(4) *Suppose $\lambda : \mathcal{S} \to G$ arises from a free action of G on T^p and let $n \geq 0$. Then the homomorphism $\bar{\lambda} : \mathcal{S} \times \mathbb{Z}^n \to G$ defined by $\bar{\lambda}(s, z) = \lambda(s)$ arises from the free G-action on $T^{p+n} = T^p \times T^n$ defined as follows: if $x \in T^p$, $y \in T^n$, and $g \in G$, then $(x, y) \cdot g = (xg, y)$.*

(5) *Suppose $\lambda : \mathcal{S} \to G$ arises from a free action of G on T^p and let $m \geq 2$. Define the following homomorphism $\bar{\lambda} : \mathcal{S} \underset{\mathbb{Z}_m}{\wr} \mathbb{Z} \longrightarrow G \wr \mathbb{Z}_m$ by*

$$\bar{\lambda}(\xi, k) = \big(\lambda \circ \xi,\ k \bmod m\big), \tag{2.1}$$

for $\big(\xi : \mathbb{Z}_m \to \mathcal{S},\ k\big) \in \mathcal{S} \underset{\mathbb{Z}_m}{\wr} \mathbb{Z}$. Then $\bar{\lambda}$ arises from the free action of $G \wr \mathbb{Z}_m$ on $T^{pm+1} = \underbrace{T^p \times \cdots \times T^p}_{m} \times T^1$ defined as follows: if $x_i \in T^p$, $i = 0, \ldots, m-1$, $y \in T^1$, *and* $\big(\alpha : \mathbb{Z}_m \to G,\ k\big) \in G \wr \mathbb{Z}_m$, *then*

$$(x_0, x_1, \ldots, x_{m-1}, y) \cdot (\alpha, k) = \Big(x_k\,\alpha(0), x_{k+1}\,\alpha(1), \ldots, y + \tfrac{k}{m} \bmod 1\Big),$$

where all indices are taken modulo m.

Proof. Statements (1)-(4) are easy. The proof of (5) is technical and will be published elsewhere. □

3. Proof of Theorem 1.3

Suppose M is orientable and distinct from S^2 and T^2. Let also $f : M \to P$ be a Morse map, $\lambda : \mathcal{S}(f) \to \mathrm{Aut}(\Gamma)$ be the action homomorphism, $G = \lambda(\mathcal{S}'(f))$, and V be any (possibly empty) collection of connected components of ∂M.

Notice that each $h \in \mathcal{S}'(f)$ preserves every connected component of ∂M with its orientation, and so it is isotopic in $\mathcal{S}'(f)$ to a diffeomorphism h_1 fixed on ∂M, see e.g. [8, Lemma 6.1]. In particular, we obtain that $\lambda(h) = \lambda(h_1)$, whence

$$G = \lambda(\mathcal{S}'(f)) = \lambda\big(\mathcal{S}'(f, V)\big) = \lambda\big(\mathcal{S}'(f, \partial M)\big).$$

Moreover, it is easy to see that for each $h \in \mathcal{S}'(f, V)$ its image $\lambda(h)$ depends only on the isotopy class of $[h]$ in $\mathcal{S}'(f, V)$, and so λ induces the following epimorphism

$$\lambda : \pi_0\mathcal{S}'(f, V) \to G. \tag{3.1}$$

Proposition 3.1. *There exists a collection* V *of connected components of* ∂M *satisfying the following conditions:*

(a) *the boundary map* $\partial : \pi_1\mathcal{O}(f, V) \to \pi_0\mathcal{S}'(f, V)$ *is an isomorphism;*
(b) *the homomorphism* (3.1) *arises from a free* G*-action on some torus* T^p.

It follows from (b) that we have an isomorphism $\theta : \pi_1(T^p/G) \to \pi_0\mathcal{S}'(f, V)$, which together with (1) of Theorem 1.2 and (a) gives another isomorphism:

$$\partial^{-1} \circ \theta :\ \pi_1(T^p/G) \ \xrightarrow{\theta}\ \pi_0\mathcal{S}'(f, V) \ \xrightarrow{\partial^{-1}}\ \pi_1\mathcal{O}(f, V) \ \equiv\ \pi_1\mathcal{O}_f(f).$$

As both T^p/G and $\mathcal{O}_f(f)$ are aspherical, there exists a homotopy equivalence between these spaces such that the corresponding isomorphism of fundamental groups coincides with $\partial^{-1} \circ \theta$. This proves Theorem 1.3 modulo Proposition 3.1.

4. Proof of Proposition 3.1

Case $M = D^2$ **or** $S^1 \times I$**.** Let V be a connected component of ∂M and $\mathcal{S} = \mathcal{S}'(f, V)$. Then by (1.2) the boundary map $\partial : \pi_1\mathcal{O}_f(f, V) \to \pi_0\mathcal{S}$ is an isomorphism. Therefore it remains to show that the homomorphism $\lambda : \pi_0\mathcal{S} \to G$ arises from a certain free G-action on a torus T^p for some $p \geq 0$.

We will use an induction on the number n of critical points of f.

Notice also that $\mathcal{D}(M, V)$ is contractible, [13, 2, 3]. Hence $\mathcal{D}(M, V) = \mathcal{D}_{\mathrm{id}}(M, V)$ and so $\mathcal{S} = \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(M, V) = \mathcal{S}(f) \cap \mathcal{D}(M, V) = \mathcal{S}(f, V)$.

1) Suppose $n = 0$. This is possible only when $M = S^1 \times I$, see Figure 1. We can also assume that $V = S^1 \times 0$ and each of the circles $S^1 \times s$, $s \in I$, is a connected

FIGURE 1. Cases $n = 0, 1$

component of some level set of f. In this case Γ consists of a unique edge, and therefore $G = \lambda(\pi_0\mathcal{S}) = \{1\}$. It follows from the proof of Case 2 of Lemma 6.3 from [4, page 261] that $\pi_0\mathcal{S} = \{1\}$. Then by (1) of Lemma 2.2 $\lambda : \pi_0\mathcal{S} \to G$ arises from a trivial action of G on T^0.

2) Suppose $n = 1$. In this case $M = D^2$, $V = \partial M$, a unique critical point q of f is a local extreme, and Γ again consists of a unique edge, so $G = \{1\}$, see Figure 1. By Morse lemma we have that $f(x, y) = x^2 + y^2$ in some local representation of f at q, therefore not loosing generality one can assume that $M = \{z \in \mathbb{C} \mid |z| = 1\}$ is a unit 2-disk in the complex plane, and the level sets of f are the origin $0 \in \mathbb{C}$ and concentric circles $S_t = \{|z| = t\}$, $t \in (0, 1]$. It follows from the proof of Case 3 of Lemma 6.3 from [4, page 261] that $\pi_0\mathcal{S} = \{1\}$. Hence again by (1) of Lemma 2.2 $\lambda : \pi_0\mathcal{S} \to G$ arises from a trivial action of G on T^0.

3) Now let $n > 1$. Assume that we proved our statement for all Morse maps $D^2 \to P$ and $S^1 \times I \to P$ having less than n critical points, and let $f : M \to P$ be a Morse map with exactly n critical points.

Notice that the Kronrod-Reeb graph Γ of f is a finite tree. Let v be a vertex Γ of f corresponding to V. Then there exists a unique edge $e = (v, u)$ in Γ incident to v. Consider another vertex u of e and let U be the connected component of the level set of f corresponding to u, see Figure 2.

For $\varepsilon > 0$ let J_ε be a closed ε-neighbourhood of the point $t = f(U)$ in P and N be the connected component of $f^{-1}(J_\varepsilon)$ containing U. We can choose ε so small that $N \cap \partial M = \varnothing$ and the set $N \setminus U$ contains no critical points of f.

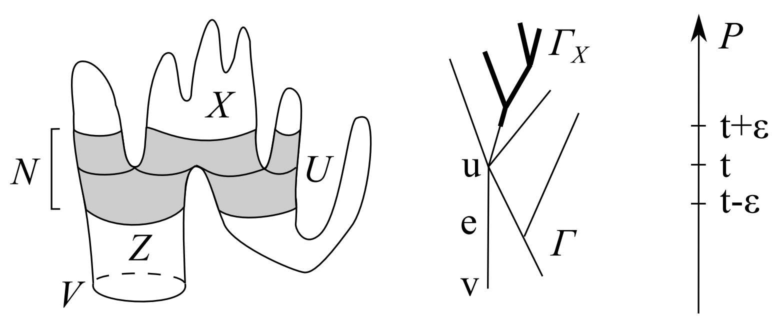

FIGURE 2

Let Z be a connected component of $\overline{M \setminus N}$ containing V. Then Z is a cylinder having no critical points of f. Let $\tau : M \to M$ be a Dehn twist preserving f and supported in the interior of Z, see [4, §6], and $\boldsymbol{\tau} \in \pi_0\mathcal{S}$ be its isotopy class. Evidently, $\lambda(\boldsymbol{\tau}) = \mathrm{id}_\Gamma$.

Lemma 4.1. [10] *There exists an epimorphism $\eta : \pi_0\mathcal{S} \to \mathbb{Z}$ having the following properties.*

(a) *Denote $m = \eta(\boldsymbol{\tau})$. Then $m \geq 1$.*
(b) *If $\boldsymbol{h} \in \pi_0\mathcal{S}$ is such that $\eta(\boldsymbol{h})$ is divided by m, then the class $\boldsymbol{h}$ contains a representative fixed on some neighbourhood of N.*
(c) *If $\eta(\boldsymbol{h}) = 0$, then $\boldsymbol{h}$ contains a representative fixed on some neighbourhood of $N \cup Z$.*

For each connected component X of $\overline{M \setminus N}$ put $\hat{X} := X \cap \partial N$. Then the pair $(X, \hat{X})$ is diffeomorphic either to $(D^2, \partial D^2)$ or to $(S^1 \times I, S^1 \times 0)$. As f is constant on connected components of ∂X, the restriction $f_X = f|_X : X \to P$ is Morse in the sense of Definition 1.1.

Let Γ_X be the Kronrod-Reeb graph of f_X. Then Γ_X can be regarded as a subtree of Γ. Denote $\mathcal{S}_X := \mathcal{S}(f_X, \hat{X})$. Let also $\lambda_X : \pi_0\mathcal{S}_X \to \mathrm{Aut}(\Gamma_X)$ be the corresponding action homomorphism, and $G_X = \lambda_X\big(\pi_0\mathcal{S}_X\big)$.

Since the number of critical points of f_X is less than n, we have by inductive assumption that $\lambda_X : \pi_0\mathcal{S}_X \to G_X$ arises from a free action on some torus.

Now the situation splits into two cases.

3a) Suppose $m = 1$. Then by (b) of Lemma 4.1 every $\boldsymbol{h} \in \pi_0\mathcal{S}$ has a representative fixed near N. Hence if we denote by $X_1, \dots, X_a$ all the connected components of $\overline{M \setminus N}$ distinct from Z, then $h(X_i) = X_i$ for all $i = 1, \dots, a$ and $h \in \mathcal{S}$.

Lemma 4.2. [9, 10] *There is the following commutative diagram:*

$$\begin{array}{ccc} \Big(\mathop{\times}\limits_{i=1}^{a} \pi_0\mathcal{S}_{X_i}\Big) \times \mathbb{Z} & \xrightarrow[\cong]{\ \alpha\ } & \pi_0\mathcal{S} \\ \Big\downarrow{\scriptstyle\bar{\lambda}} & & \Big\downarrow{\scriptstyle\lambda} \\ \mathop{\times}\limits_{i=1}^{a} G_{X_i} & \xrightarrow[\cong]{\ \beta\ } & G \end{array} \tag{4.1}$$

in which α and β are isomorphisms, and $\bar{\lambda}$ is defined by

$$\bar{\lambda}(\boldsymbol{h}_1, \dots, \boldsymbol{h}_a, k) = \big(\lambda_1(\boldsymbol{h}_1), \dots, \lambda_a(\boldsymbol{h}_a)\big)$$

for $\boldsymbol{h}_i \in \mathcal{S}_{X_i}$ and $k \in \mathbb{Z}$. Hence by (3) *and* (4) *of Lemma* 2.2 *λ arises from a free action on a torus.*

Proof. Isomorphisms α and β are constructed [10] and [9] respectively. We will just recall their definitions.

Let $\boldsymbol{h}_i \in \pi_0\mathcal{S}_{X_i}$, $i = 1, \dots, a$, and $k \in \mathbb{Z}$. Choose a representative $h_i \in \mathcal{S}_{X_i}$ of $\boldsymbol{h}_i$ such that h_i is fixed on some neighbourhood of $\hat{X}_i$, and so it extends by the identity to a diffeomorphism of M. Let $\boldsymbol{h} \in \pi_0\mathcal{S}$ be the isotopy class of the composition $h_1 \circ \cdots \circ h_a$. Put $\alpha(\boldsymbol{h}_1, \dots, \boldsymbol{h}_a, k) = \boldsymbol{h}\boldsymbol{\tau}^k$. It is shown in [10] that α is a group isomorphism.

Let $\gamma_i \in G_{X_i}$, $i = 1, \ldots, a$. Notice that Γ_{X_i} is a subtree of Γ and that γ_i extends by the identity to an automorphism of all of Γ. Let $\gamma = \gamma_1 \circ \cdots \circ \gamma_n$. Then it is easy to show that $\gamma \in G$, i.e. it is induced by some $h \in \mathcal{S}$, so we set $\beta(\gamma_1, \ldots, \gamma_n) = \gamma$. It is proven in [9] that β is a group isomorphism. It also easily follows from the definitions that the diagram (4.1) is commutative. □

3b) Suppose $m \geq 2$. Let $X_1, \ldots, X_a$ be all the connected components of $\overline{M \setminus N}$ distinct from Z and *invariant* with respect to all $h \in \mathcal{S}$, and Y be the collection of all connected components of $\overline{M \setminus N}$ being *not invariant* under $\mathcal{S}$.

Lemma 4.3. [9, 10] *There exist a subcollection $\widetilde{Y} = \{Y_1, \ldots, Y_b\} \subset Y$ and $g \in \mathcal{S}$ having the following properties:*

1. *$\eta(g) = 1$ and g is fixed near each X_i, $i = 1, \ldots, a$;*
2. *$g^i(\widetilde{Y}) \cap g^j(\widetilde{Y}) = \varnothing$ for $i \neq j \in \{0, \ldots, m-1\}$;*
3. *$Y = \bigcup\limits_{i=0}^{m-1} g^i(\widetilde{Y})$;*
4. *g^m preserves every connected component of $\overline{M \setminus N}$;*
5. *$\lambda(g^m) = \mathrm{id}_\Gamma$.*

Moreover, there is the following commutative diagram:

$$
\begin{array}{ccc}
\Big(\mathop{\times}\limits_{i=1}^{a} \pi_0\mathcal{S}_{X_i}\Big) \times \Big(\Big(\mathop{\times}\limits_{j=1}^{b} \pi_0\mathcal{S}_{Y_j}\Big) \underset{\mathbb{Z}_m}{\wr} \mathbb{Z}\Big) & \xrightarrow[\cong]{\alpha} & \pi_0\mathcal{S} \\
\Big\downarrow \bar{\lambda} & & \Big\downarrow \lambda \\
\Big(\mathop{\times}\limits_{i=1}^{a} G_{X_i}\Big) \times \Big(\Big(\mathop{\times}\limits_{j=1}^{b} G_{Y_j}\Big) \wr \mathbb{Z}_m\Big) & \xrightarrow[\cong]{\beta} & G
\end{array}
\tag{4.2}
$$

in which α and β are isomorphisms and $\bar{\lambda}$ is defined by

$$\bar{\lambda}(\boldsymbol{h}_1, \ldots, \boldsymbol{h}_a, \xi, k) = \Big(\lambda_1(\boldsymbol{h}_1), \ldots, \lambda_a(\boldsymbol{h}_a), \quad \big(\mathop{\times}\limits_{j=1}^{b} \lambda_j\big) \circ \xi, \quad k \bmod m\Big),$$

for $\boldsymbol{h} \in \pi_0\mathcal{S}_{X_i}$, a map $\xi : \mathbb{Z}_m \to \mathop{\times}\limits_{j=1}^{b} \pi_0\mathcal{S}_{Y_j}$, and $k \in \mathbb{Z}$. Hence by (3) *and* (5) *of Lemma* 2.2 *λ arises from a free action on a torus.*

Proof. Isomorphisms α and β are constructed in [10] and [9] respectively. Again we will just recall their definitions.

Let $\boldsymbol{h}_i \in \pi_0\mathcal{S}_{X_i}$, $i = 1, \ldots, a$, $\xi = (\xi_1, \ldots, \xi_b) : \mathbb{Z}_m \to \mathop{\times}\limits_{j=1}^{b} \pi_0\mathcal{S}_{Y_j}$ be any map, and $k \in \mathbb{Z}$. Choose a representative $h_i \in \mathcal{S}_{X_i}$ of $\boldsymbol{h}_i$ fixed on some neighbourhood of $\hat{X}_i$. Let also h_{js}, $1 \leq j \leq b$, $s \in \mathbb{Z}_m$, be a representative of $\xi_j(s) \in \pi_0\mathcal{S}_{Y_j}$ fixed on some neighbourhood of $\hat{Y}_j = Y_j \cap N$, and $h : M \to M$ be a map given by the formula:

$$h = \begin{cases} h_i, & \text{on } X_i,\ i = 1, \dots, a \\ g^{s+k} \circ h_{js} \circ g^{-s}, & \text{on } g^s(Y_j),\ \text{for } s \in \mathbb{Z}_m \text{ and } j = 1, \dots, b, \\ g^k, & \text{on } N \cup Z. \end{cases}$$

Then $h \in \mathcal{S}$ and we define $\alpha(\boldsymbol{h}_1, \dots, \boldsymbol{h}_a, \xi, k) \in \pi_0\mathcal{S}$ to be the isotopy class of h. It is shows in [10] that α is a group isomorphism.

Let $\gamma_i \in G_{X_i}$, $i = 1, \dots, a$, $\nu = (\nu_1, \dots, \nu_b) : \mathbb{Z}_m \to \mathop{\times}\limits_{j=1}^{b} G_{Y_j}$ be map, and $k \in \mathbb{Z}_m$. Define $\gamma : \Gamma \to \Gamma$ by the following formula:

$$\gamma = \begin{cases} \gamma_i, & \text{on } \Gamma_{X_i},\ i = 1, \dots, a \\ \lambda(g)^{s+k} \circ \nu_j(s) \circ \lambda(g)^{-s}, & \text{on } \lambda(g)^s(\Gamma_{Y_j}),\ \text{for } s \in \mathbb{Z}_m \text{ and } j = 1, \dots, b, \\ \lambda(g)^k, & \text{on the remained part of } \Gamma. \end{cases}$$

Then $\gamma \in G$ and the correspondence $\beta : (\gamma_1, \dots, \gamma_n) \longmapsto \gamma$ is a group isomorphism, see [9]. It also easily follows from the definitions that the diagram (4.2) is commutative. □

General case. Suppose M is orientable and distinct from S^2, T^2, D^2, and $S^1 \times I$. Set $V = \partial M$ and denote $\mathcal{S} = \mathcal{S}'(f, V)$. Then $\chi(M) < 0$ whence by (1.2) the boundary map $\partial : \pi_1\mathcal{O}_f(f, V) \to \pi_0\mathcal{S}$ is an isomorphism.

Lemma 4.4. [6, 9] *If $\chi(M) < 0$, then there exist mutually disjoint subsurfaces $B_1, \dots, B_n$ of M having the following properties:*

(a) *B_i is diffeomorphic either to a 2-disk D^2 or to a cylinder $S^1 \times I$, and the restriction $f_i := f|_{B_i} : B_i \to P$ is Morse for all $i = 1, \dots, n$.*

(b) *Let Γ_i be the Kronrod-Reeb graph of f_i, $\mathcal{S}_i = \mathcal{S}'(f_i, \partial B_i)$, $\lambda_i : \pi_0\mathcal{S}_i \to \mathrm{Aut}(\Gamma_i)$ be the corresponding action homomorphism, and $G_i = \lambda_i(\mathcal{S}_i)$, $i = 1, \dots, n$. Then we have the following commutative diagram:*

$$\begin{array}{ccc} \mathop{\times}\limits_{i=1}^{n} \pi_0\mathcal{S}_i & \xrightarrow[\cong]{\alpha} & \pi_0\mathcal{S} \\ \Big\downarrow {\scriptstyle \mathop{\times}\limits_{i=1}^{n} \lambda_i} & & \Big\downarrow {\scriptstyle \lambda} \\ \mathop{\times}\limits_{i=1}^{n} G_i & \xrightarrow[\cong]{\beta} & G \end{array} \tag{4.3}$$

in which α and β are isomorphisms.

Proof. Surfaces $\{B_i\}_{i=1}^n$ and the isomorphism α are defined in [6, Theorem 1.7]. The isomorphism β and diagram (4.3) are constructed in [9]. In fact α and β are similar to the corresponding isomorphisms from Lemma 4.2 and therefore we skip their descriptions. □

Since each B_i is either a 2-disk D^2 or a cylinder $S^1 \times I$, the homomorphism λ_i arises from a free action of G_i on some torus. Then by (3) of Lemma 2.2 so does $\lambda : \pi_0\mathcal{S} \to G$. Proposition 3.1 is completed.

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