On realizations and representations of $\mathfrak{sl}(2, \mathbb{C})$

Mykola Staryi
PhD Student, Institute of Mathematics, Kyiv, Ukraine

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Work objective

The purpose of our studies involves a comparison of different methods used for constructing representations of Lie algebras. The considerations will concern the examples of real and complex simple three-dimensional Lie algebras.

Subproblems that we have solved

- Construction of all inequivalent realizations (in terms of first-order differential operators) of $\mathfrak{sl}(2, \mathbb{C})$.
- Comparison of realizations derived from irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$ with the realizations of $\mathfrak{sl}(2, \mathbb{C})$ (as of now only two cases completed).

Lie Groups and Algebras

- Lie group G:
  - G is a smooth manifold;
  - For $(x, y) \in G \times G$, the map $(x, y) \mapsto xy^{-1} \in G$ is smooth.
- Left-invariant vector fields $X \in \mathrm{Vec}(G)$ on G:
  - Left translation for $x \in G$: $L_x|$, $x \in G$: $G \ni g \mapsto a \cdot g$
  - Smooth vector field A on G: $A_x = L'_x A_e$ where $L'_x : T_g G \to T_{xg} G$.
  - A is left-invariant: $L'_a X_b = L'_{ab} X_e = X_{ab}$
  - Let $\mathfrak{g}$ be a space of left-invariant vector fields. Then

$$L'_a[X, Y] = [L'_a X, L'_a Y] = [X, Y] \quad \Rightarrow \quad \mathfrak{g} \cong T_e G,$$

    so $\mathfrak{g}$ is the Lie algebra of the Lie group G.
- Left-invariant vector fields on G are in one-to-one correspondence with a transformation group ϕ_t.
- Conversely, one can obtain G from $\mathfrak{g}$ via the exponential map

$$\exp\colon \ \mathfrak{g} \to G,$$

  that maps a left-invariant vector field A to $\exp(tA) = \phi_t$.

Lie algebras

- Define a (complex) *Lie algebra* as follows:

$$\mathfrak{g} = (V, [\cdot,\cdot]), \quad [\cdot,\cdot]\colon\ V \times V \to V,$$

  - V is a linear space, $\dim_{\mathbb{C}} V = n < \infty$
  - $[\cdot,\cdot]\colon V \times V \to V$ is bilinear and satisfies the properties

$$\forall x, y, z \in \mathfrak{g} \quad [x, y] = -[y, x],$$
$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$

- Let $V = \operatorname{span}\{e_1, \ldots, e_n\}$, then $\mathfrak{g} \leftrightarrow c_{ij}^k \in \mathbb{C}$, $i, j, k = 1, \ldots, n$

$$[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k.$$

The coefficients c_{ij}^k are called *structure constants*.

Representations and realizations

- A *representation* of $\mathfrak{g}$ is a Lie algebra homomorphism

$$\phi\colon\ \mathfrak{g} \to \mathfrak{gl}(V).$$

- The *adjoint* representation: Take $V = \mathfrak{g}$ and consider a map

$$\mathrm{ad}\colon\ \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \quad \mathrm{ad}_x(y) = [x, y], \quad \forall x, y \in \mathfrak{g}.$$

- A homomorphism

$$R\colon\ \mathfrak{g} \to \mathrm{Vec}(M),$$

is called a *realization* of $\mathfrak{g}$. Here $M \subset \mathbb{C}^m$ is a manifold, $\dim_{\mathbb{C}} M = m$ and $\mathrm{Vec}(M)$ is a space of vector fields on M.

Realizations $R_1 \colon \mathfrak{g} \to \mathrm{Vec}(M_1)$ and $R_2 \colon \mathfrak{g} \to \mathrm{Vec}(M_2)$ are called *equivalent* if there is an automorphism $A \in \mathrm{Aut}(\mathfrak{g})$ and a holomorphic map $f \colon M_1 \to M_2$ with the induced $f_* \colon \mathrm{Vec}(M_2) \to \mathrm{Vec}(M_1)$ such that for all $g \in \mathfrak{g}$

$$R_1(g) = f_* R_2(Ag).$$

Inequivalent realizations of $\mathfrak{sl}(2,\mathbb{C})$

There are only four inequivalent faithful realizations of $\mathfrak{sl}(2,\mathbb{C})$:

1. $\partial_1,\ x_1\partial_1+x_2\partial_2,\ x_1^2\partial_1+2x_1x_2\partial_2+x_2\partial_3,$
2. $\partial_1,\ x_1\partial_1+x_2\partial_2,\ \left(x_1^2+x_2^2\right)\partial_1+2x_1x_2\partial_2,$
3. $\partial_1,\ x_1\partial_1+x_2\partial_2,\ x_1^2\partial_1+2x_1x_2\partial_2,$
4. $\partial_1,\ x_1\partial_1,\ x_1^2\partial_1.$

- Popovych R.O., Boyko V.M., Nesterenko M.O., Lutfullin M.W., *Realizations of real low-dimensional Lie algebras*, J. Phys. A 36 (2003), no. 26, 7337–7360, arXiv:math-ph/0301029
- Nesterenko M. O., Popovych R.O., *Realizations of real unsolvable low-dimensional Lie algebras*, in Proceedings of the Third Voronoi Conference on Analytic Number Theory and Spatial Tessellations (September 22–28, 2003, Kyiv), arXiv:math-ph/0510015.

Matrix representations and realizations

Let $\mathfrak{g}$ be a Lie algebra defined by its commutation relations

$$[e_i, e_j] = c_{ij}^k e_k.$$

Let $\mathfrak{g}$ has a matrix representation $(\phi(e_i))_\alpha^\beta = (\Phi_i)_\alpha^\beta \in \mathrm{GL(m, \mathbb{C})}$.

Then the set of first order differential operators

$$\Psi_i = \sum_{\alpha=1}^{m} \left(\sum_{\beta=1}^{m} (\Phi_i)_\alpha^\beta x_\beta \right) \partial_\alpha, \quad i = 1, \ldots, n = \dim \mathfrak{g}.$$

forms a well-defined realization of $\mathfrak{g}$.

The inverse statement is also true: a realization with linear coefficients corresponds to some matrix representation of $\mathfrak{g}$.

Irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$ (highest weight repr-s)

$\mathfrak{sl}(2,\mathbb{C})$: $[e_1,e_2]=e_1,\quad [e_2,e_3]=e_3,\quad [e_1,e_3]=2e_2.$

$$\phi_1=\begin{pmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 2 & \cdots & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & d\\ 0 & 0 & \cdots & 0 & 0\end{pmatrix},\quad \phi_2=\frac{1}{2}\begin{pmatrix}-d & 0 & \dots & 0\\ 0 & -d+2 & \dots & 0\\ \vdots & \vdots & \ddots & \ddots\\ 0 & 0 & \dots & d\end{pmatrix},$$

$$\phi_3=\begin{pmatrix}0 & 0 & \cdots & 0 & 0\\ -d & 0 & \cdots & 0 & 0\\ 0 & -d+1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & -1 & 0\end{pmatrix}.$$

Are there any realizations for $\mathfrak{sl}(2,\mathbb{C})$ which cannot be obtained from ϕ_i?

Realizations with linear coefficients

Realizations of $\mathfrak{sl}(2,\mathbb{C})$ that correspond to representation matrices ϕ_i are

$$\Psi_1 = \sum_{i=2}^{d+1}(i-1)x_{i-1}\partial_i, \quad \Psi_2 = \frac{1}{2}\sum_{i=1}^{d+1}(-d-2+2i)x_i\partial_i,$$

$$\Psi_3 = \sum_{i=1}^{d}(-d+i-1)x_i\partial_i.$$

If $d=1$ we have the realization R_I:

$$\psi_1 = x_1\partial_2, \quad \psi_2 = \frac{1}{2}\left(x_2\partial_2 - x_1\partial_1\right), \quad \psi_3 = -x_2\partial_1.$$

If $d=2$ (the case of adjoint representation) we have another realization R_{II}:

$$\psi_1 = x_1\partial_2 + 2x_2\partial_3, \quad \psi_2 = -x_3\partial_3 - x_1\partial_1, \quad \psi_3 = -2x_2\partial_1 - x_3\partial_2.$$

Transformation to the known form

- After the transformation

$$\tilde{x}_1 = \frac{x_1}{x_2}, \quad \tilde{x}_2 = \frac{1}{x_1^2},$$

the realization R_I takes the form which coincides with the case 3 in the list of inequivalent realizations:

$$\tilde{\Psi}_1 = \partial_1, \quad \tilde{\Psi}_2 = x_1\partial_1 + x_2\partial_2, \quad \tilde{\Psi}_3 = x_1^2\partial_1 + 2x_1x_2\partial_2.$$

- After the transformation

$$\tilde{x}_1 = \frac{x_2 + 1}{x_1} - x_2^2 + x_1x_3, \quad \tilde{x}_2 = x_1^{-\frac{1}{2}}\left(x_2^2 - x_1x_3\right), \quad \tilde{x}_3 = \frac{1}{x_1}.$$

the realization R_{II} takes the form which coincides with the case 2 in the list of inequivalent realizations:

$$\tilde{\Psi}_1 = \partial_1, \quad \tilde{\Psi}_2 = x_1\partial_1 + x_2\partial_2, \quad \tilde{\Psi}_3 = \left(x_1^2 + x_2^2\right)\partial_1 + 2x_1x_2\partial_2.$$

Summary

The main results of our work:

- A list of inequivalent realizations of the complex Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ was found.
- For two lowest irreducible matrix representations, change of variables was found that reduce the realizations to the known forms.

Further research:

- To complete transformations of irreducible representations.
- To consider realizations constructed by algebraic methods.
- To study real forms of $\mathfrak{sl}(2, \mathbb{C})$ and its representations.

Thank you for your attention!