Starting Road to Modular forms

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December, 2024 – January, 2025

Abstract

In this lecture we will try to give some examples that will allow us to get closer to the concept of modular form and to understand why they are so important in modern mathematics. We will start off with basic definitions in that area, like a modular group and its fundemental area.

1 First Steps

The following example is significant in both projective geometry and conformal field theory. Let f be a meromorphic function in a domain D in $\mathbb{C}$. The Schwarz derivative of f is usually defined as follows [25]:

$$\{f, z\} := (\ln f')' - \frac{1}{2}\left((\ln f')'\right)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2. \tag{1}$$

It turns out to have a remarkable connection with Fuchsian differential equations [6]

$$y'' + \frac{R(z)}{2} y = 0. \tag{2}$$

More precisely, if y_1 and y_2 are two linearly independent solutions of (2), then the function $f := y_1/y_2$ turns out to be a solution of the Schwarz equation,

$$\{f, z\} = R(z). \tag{3}$$

The converse statement is also true, so if $f(z)$ is locally univalent [8] and satisfies (3), then the functions

$$y_1 = \frac{f}{\sqrt{f'}} \quad \text{and} \quad y_2 = \frac{1}{f'}$$

are solutions of (2).

When Schwarz was alive, it was a very popular problem to study classes of equations whose solutions are algebraic. The initial impetus was the hyperbolic equation

$$u'' + \frac{\gamma - (\alpha + \beta + 1)x}{x(1-x)} u' - \frac{\alpha\beta}{x(1-x)} u = 0, \tag{4}$$

where α, β, γ are some real parameters satisfying $\alpha+\beta+\gamma=\pi$, for which the sought solution $u(x)$ should be algebraic. Following the framework, one should write the corresponding Schwarz equation, which in our case takes the form:

$$\{z,x\} = \frac{1-\lambda^2}{4x^2} + \frac{1-\mu^2}{4(1-x)^2} + \frac{1+\nu^2-\lambda^2-\mu^2}{4x^2}, \tag{5}$$
$$\lambda = 1-\gamma, \quad \mu = \gamma-\alpha-\beta, \quad \nu = \alpha-\beta,$$

where $z := u_2/u_1$ is the ratio of two independent solutions of equation (4). But then the Wronskian

$$W = u_2u_1' - u_1u_2' \sim \exp\left(-\int \frac{\gamma-(\alpha+\beta+1)x}{x(1-x)}\,\mathrm{d}\,x\right) = x^{-\gamma}(1-x)^{\gamma-\alpha-\beta-1}$$

implies that u_1 and u_2 are algebraic if $\gamma, \alpha+\beta \in \mathbb{Q}$. Moreover, it can be shown that $\alpha, \beta, \gamma \in \mathbb{Q}$.

We are rather interested in relations between different systems of ODE's. For example, a natural question arises: when some systems are *strongly equivalent*, in other words, equivalent under the non-degenerate changes $u_i \mapsto \sum_j a_{ij}u_i$ of dependent variables? For the simplest case of the general second-order ODE

$$u'' + pu' + qu = 0 \tag{6}$$

a substitution $u = kv$, where k is a nonzero function depending on x, gives the following equation:

$$v'' + \left(p + 2\frac{k'}{k}\right)v' + \left(q + p\frac{k'}{k} + \frac{k''}{k}\right)v = 0.$$

Choose k in order for the coefficient at v' to vanish. ♣In that way we obtain an invariant of this equivalence relation, namely the coefficient at v, which turns out to be the Schwarz derivative:

$$R(x) = q + p\frac{k'}{k} + \frac{k''}{k} = q - \frac{p^2}{4} - \frac{p'}{2} \Rightarrow \{z,x\} = 2\left(q - \frac{p^2}{4} - \frac{p'}{2}\right),$$

where u_1 and u_2 are linearly independent solutions of (6) and $z := u_2/u_1$. Hence we obtain a significant simplification: instead of solving the nonlinear equation involving the Schwarz derivative, it suffices to consider its linear counterpart

$$u'' + qu = 0.$$

This framework has nice geometric interpretation. Consider the map

$$s\colon x \to z(x) = \frac{u_2}{u_1},$$

whose domain is the upper half-plane $\mathrm{Im}(x) > 0$ and range is the interior of the Schwarz triangle, that is a curvilinear polygon with three angles given via parameters α, β and γ in the hyperbolic equation. The values of the three parameters are entirely independent of each other, that will be shown in the future.

♣

It can be generalized to the case of a system in n independent variables and m dependent variables with rank r. Let $u_1, \ \dots \ , u_r$ are linearly independent solutions, then a map

$$s\colon x \to u_1 \colon u_2 \colon \ \dots \ \colon u_r,$$

which acts on the Grassmanian variety $\mathrm{Gr(m,r)} = \mathrm{GL(m)} \backslash \{\mathrm{m \times r \text{ matrix of rank } m}\}$. This is the main invariant we are interested in, since two (strongly) equivalent systems define the same s-maps, up to the group of motions.

For example, when $n = m = 1$, $r = 2$ the s-map equivalence implies that

$$s_1 \sim s_2 \Rightarrow \{s_1, x\} = \{s_2, x\},$$

where $\{\cdot,\cdot\}$ denotes the Schwarz derivative in the beginning.

If $r > 1$, our problem reduces to the projective geometry, namely, we have to consider curves $x \to \mathbb{P}^{r-1}$ and invariants for them up to projective motion group PGL(r). Take $r = 3$, then a curve in the plane is given by homogeneous coordinates $u(x) = u_0(x)\colon u_1(x)\colon u_2(x)$. The differential equation for the unknown u:

$$\begin{vmatrix} u''' & u'' & u' & u \\ u_0''' & u_0'' & u_0' & u_0 \\ u_1''' & u_1'' & u_1' & u_1 \\ u_2''' & u_2'' & u_2' & u_2 \end{vmatrix} = 0, \tag{7}$$

where u_0, u_1, u_2 are taken to be its solutions. If the curve is non-degenerate, then (7) can be written as follows

$$u''' + p_1 u'' + p_2 u' + p_3 u = 0.$$

This 3-rd order linear ODE defined for a particular projective curve u is not unique: a substitution $u \to ku$, where $k(x) \neq 0$ gives an equivalent equation

$$u''' + P_1 u' + P_2 u = 0,$$

where P_1, P_2 are some rational functions in $p_i, \ i = 1, 2, 3$ chosen to make the coefficient of u' equal zero. As we know from the previous paragraph, it's helpful to consider the following Schwarz equation, the solution of which will allow us to give an equivalent equation, that is uniquely determined by a planar curve:

$$\{f(x), x\} = \frac{P_2(x)}{4}$$

Suppose $f(x)$ is a solution of that equation, then do the change of variables $x \to y = f(x)$, $w = f'u$. Hence, we get an equation

$$w''' + Rw \equiv w''' + \frac{P_3 - P_2'/2}{(f')^3} w = 0$$

Regarding the weak equivalence, one can check that R is a complete invariant of curves in that case. That implies an interesting corollary that $R = 0$ if and only if the curve w is conic. If $n \geq 3$, we have the similar condition: the image of the Schwarz map is quadric, i.e $x \to u(x) \in \mathbb{P}^{n+1}$ is in a quadratic hyperspace if and only if the coefficients in the system are expressed through Christoffel symbols and the Schouten tensor of $h_{ij} \equiv e^{\theta} g_{ij}$, where g_{ij} are taken from the system of rank $r = n+2$ on u with n independent variables $(x^1, \ \ldots \ , x^n)$

$$\frac{\partial^2 u}{\partial x_i x_j} = g_{ij} \frac{\partial^2 u}{\partial x^1 \partial x^n} + A^0_{ij} u + \sum_{k=1}^{n} A^k_{ij} \frac{\partial u}{\partial x^k}, \tag{8}$$

where g_{ij}, A^k_{ij}, A^0_{ij} are symmetric in the indices, and $A^k_{1n} = A^0_{1n} = 0$. Recall that the Christoffel symbols are defined as follows

$$\Gamma^k_{ij} = \frac{1}{2} \sum_l h^{kl} \left(\frac{\partial h_{il}}{\partial x^j} + \frac{\partial h_{jl}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^l} \right),$$

which give rise to the Riemann curvature tensor R^l_{ijk}:

$$d\pi_{ij} - \sum_k \pi^k_i \wedge \pi^j_k = \frac{1}{2} \sum_{kl} R^j_{ikl} \; dx^k \wedge dx^l,$$

$$\pi^j_i = \sum_k \Gamma^j_{ik} dx^k.$$

Finally, the Schouten tensor which often appears in conformal geometry:

$$S_{ik} = \frac{1}{n-2} \left(R_{ik} - \frac{R}{2(n-1)} h_{ik} \right),$$

given using the Ricci and the scalar curvatures:

$$R_{ij} = \sum_k R^k_{ikj},$$

$$R = \sum_{ij} h^{ij} R_{ij}.$$

Then the generalized claim states that the coefficients in (8) are expressed as follows

$$A^k_{ij} = \Gamma^k_{ij} - g_{ij} \Gamma^k_{1n},$$

$$A^0_{ij} = -S_{ij} + g_{ij} S_{1n}.$$

The case when $n = 2$ is considered in [3].

2 Disclosure

2.1 The Schwarz-Chrostoffel formula

We have already mentioned that it is important to find all possible maps between any two simply connected domains since it gives us an analytic continuation from one domain to another. In other words, for any two adjacent domains D_1 and D_2 with a common boundary a, the regular analytic functions $f_1(z)$, $f_2(z)$ in D_1 and D_2 respectively, give an analytic continuation from one domain to another. In a more rigorous form it means that the Riemann reflection principle holds [8]

Theorem 2.1. *Let D_s and D_w be two domains with boundaries that include circular arcs a_s, a_w (they may be just linear segments). Suppose, there is a map $\omega\colon D_s \to D_w$ defined by $\omega = f(z)$ such that a_w corresponds to a_s. Let D_s^* be the domain obtained from D_s by inversion with respect to the circle C_s of which a_s forms a part. Choose $z \in D_s$ and $z^* \in D_s^*$ to be such points (inverse with respect to C_s, then the points $f(z)$ and $f(z^*)$ are inverse with respect to the circle C_w of which a_w forms a part.*

Corollary 2.2. *Let $w = f(z)$ be a function that maps an interior of a circle C onto the interior of another circle C^*. Then $f(z)$ is necessarily a linear transform.*

The last corollary comes obvious as far as we will show that the function $w = f(z)$ is regular at all points of a z-plane, that eventually comes to the fact that $w = f(z)$ is a rational function with only one pole, i.e a linear transform.

Consider D – a polygon in the w-plane with its interior angles to be $\pi\alpha_1,\ \pi\alpha_2,\ \dots,\ \pi\alpha_n$ and the corresponding vervices $a_1,\ \dots,\ a_n$ to be located in a z-plain. It is more convenient to introduce exterior angles $\pi\mu_1,\ \pi\mu_2,\ \dots,\ \pi\mu_n$ tied with the interior ones by relations $\pi\alpha_i + \pi\mu_i = \pi$. Now $w = f(z)$ is an analytical function that maps the upper half-plane $\mathrm{Im(z)} > 0$ onto the interior of D. Then the points $a_1,\ \dots,\ a_n$ are on real axis and are mapped by $w = f(z)$ onto a linear segment. Therefore, $f(z)$ is regular at all points on the real axis except at the points $a_1,\ \dots,\ a_n$ and can be continued analytically in each of the intervals bounded by the vertices. In other words, by the symmetry principle, we have the mirror image of the polygon D to some D' which turns out to be a conformal map of the half-plane $\mathrm{Im(z)} < 0$. Applying the symmetry principle once again, as a result, a point z returns to its original position, but we obtain another figure D'' that is congruent to D. As a result, $f(z)$ goes to $f_1(z) = Az + B$, with A, B to be some constants.

Then we have
$$\frac{f_1''(z)}{f_1'(z)} = \frac{f''(z)}{f'(z)}.$$

Thus, the function $g(z) = \frac{f'(z)}{f(z)}$ is single-valued in the whole z-plane, and has singularities at $a_1,\ \dots,\ a_n$, so we can consider the behaviour of $f(z)$ in the neighborhood of the point a_i. Assume that none of $a_1,\ \dots,\ a_n$ is ∞. Then one can show that

$$\frac{f''(z)}{f'(z)} = -\frac{\mu_i}{z - a_i} + k(z),$$

where $k(z)$ is some regular function in $z = a_i$. But then the function $g(z) + \frac{\mu_i}{z - a_i}$ is also regular at a_i. Perform of that procedure for all vertices allows to obtain a regular at $a_1,\ \dots,\ a_n$ function:

$$g_1(z) = g(z) + \sum_{i=1}^{n} \frac{\mu_i}{z - a_i}.$$

But then, taking into account that it is single-valued in the entire plane, we have that $g(z) = const$. Moreover, $g(z) = 0$ since we have a series for $f(z)$ due to its regularity at $z = \infty$:

$$f(z) = f(\infty) + c_1 z^{-1} + c_2 z^{-2} + \ \dots,$$

while $g_1(\infty) = 0$.

Hence we obtain the desired mapping function (Schwarz-Christoffel formula as follows

$$\frac{f''(z)}{f'(z)} = -\sum_{i=1}^{n} \frac{\mu_i}{z - a_i} \Rightarrow f(z) = \alpha \int_0^z \frac{\mathrm{dz}}{(z-a_1)^{\mu_1}(z-a_2)^{\mu_2}\dots(z-a_n)^{\mu_n}} + \beta, \tag{9}$$

where α, β are some integration constants related to parameters of the polygon.

For example, we can construct an analytic function $w = f(z)$ which maps the half-plane Im(z) > 0 onto the interior of the a triangle with its angles to be $\pi\alpha,\ \pi\beta,\ \pi\gamma$. Let the three vertices of the traingle correspond to the points $z = 0$, $z = 1$, $z = \infty$. The formula (9) can also be applied when the point a_i coincides with the point at ∞, it can be reached by means of a linear transformation, that takes a_i into the point at infinity, namely the linear substitution $z = a_n - \frac{1}{\xi}$. So in our case of the triangle (9) can be written as follows

$$f(z) = C_1 \int_0^z z^{\alpha-1}(1-z)^{\beta-1}\mathrm{dz} + \mathrm{C}_2.$$

Then one can find all sides of the triangle, for example the side c (which is opposite the angle $\pi\gamma$) – it can be given by the following beta function:

$$c = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \Rightarrow c = \frac{1}{\pi}\sin\pi\gamma\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma),$$

the last equality is valid in virtue by $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi x}$.

2.2 Circular Arcs

Now we have to find such $w = f(z)$ which maps the half-plane Re(z) > 0 onto the interior of a curvilinear polygon with interior angles $\pi\alpha_1,\ \dots,\ \pi\alpha_n$. In the previous case, we took an operator invariant under linear transform, now, when we have circular acrs, linear substitutions are replaced by general linear transforms, so we have to deal with the Schwarz derivative (1). This differential operator has precisely this property since the following property holds

Theorem 2.3.

$$\{W, z\} \equiv \{\frac{az+b}{cz+d}, z\} = \{w, z\}, \qquad ad - bc \neq 0, \tag{10}$$

Indeed, since

$$W' = \frac{ad-bc}{(cw+d)^2}W \Rightarrow \frac{W''}{W'} = \frac{w''}{w'} - \frac{2cw'}{cw+d},$$

one gets

$$\left(\frac{W''}{W'}\right)' - \frac{1}{2}\left(\frac{W''}{W'}\right)^2 = \left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2,$$

which completes a proof of (11).

At the points $a_1, \ldots, a_n$ the function $w = f(z)$ has singularities as well as its Schwarz derivative. Choose the vertex located at $z = a_i$, then the linear transform maps this vertex to the origin and the adjacent circles, which intersect at $z = a_i$, come to two straight lines, which meet at $\pi\alpha_i$ as well. $\{w, z\}$ is not affected by this linear transformation, so $\{w, z\}$ is singular at $z = a_i$. But then, we have

$$f(z) = (z - a_i)^\alpha f_1(z),$$

where $f_1(z)$ is regular at $z = a_i$, $f_1(a_i) \neq 0$. Then we will have an expression for the Schwarz derivative as follows

$$\{w, z\} = \frac{1}{2}\frac{1-\alpha_i^2}{(z-\alpha_i)^2} + \frac{\beta_i}{z - a_i} + f_2(z),$$

where parameters $\beta_i = \frac{1-\alpha_i^2}{\alpha_i}\frac{f_1'(a_i)}{f_1(a_i)}$ are real, and $f_2(z)$ is regular at $z = a_i$. We can do the same for all singular points a_i, in that way we obtain the following expression regular at $a_1, \ldots, a_n$.

$$\{w, z\} = \frac{1}{2}\sum_{i=1}^{n}\frac{1-\alpha_i^2}{(z-\alpha_i)^2} + \sum_{i=1}^{n}\frac{\beta_i}{z-\alpha_i} + \gamma. \tag{11}$$

Here, the constants $\gamma, \beta_1, \ldots, \beta_n$ depend on each other in a sense of the following relations obtained by inserting the decomposition of $f(z)$ near $z = \infty$ into the expression of $\{w, z\}$:

$$\begin{gathered}\gamma = 0, \qquad (12)\\ \sum_{i=1}^{n}\beta_i = 0,\\ \sum_{i=1}^{n}\left(2\alpha_i\beta_i - \alpha_i^2 + 1\right) = 0,\\ \sum_{i=1}^{n}\left(\alpha_i\left(1-\alpha_i^2\right) + \beta_i\alpha_i^2\right) = 0.\end{gathered}$$

In (11) there are $3n+1$ independent real parameters, and that number reduces to $3n-3$ due to relations (12), so it seems, that this number is more than the number of independent parameters implied by a poligon P_n, namely $3n-6$ ($3n$ real parameters for a circle on which linear transformations act, – that excludes extra 6 parameters), but this not the case due to a linear transformation of the upper half-plane onto itself (we have to eliminate another triple of constants). Thus, equations (12) exhaust all possible relations between the parameters α_i, β_i, γ. However, it is quite difficult to work with the equation (11) since it contains $n-3$ accessory parameters (apart from α_i, of course) which are very complicated to be found using only the geometry of P_n. The only case which is free of accessory parameters is that of a curvilinear triangle mentioned in the intro. If we take angles to be $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $\alpha_3 = \gamma$ the equation (11) reduces to

$$\{w,z\} = \frac{1}{(z-a)(z-b)(z-c)}\left[\frac{1-\alpha^2}{2}\cdot\frac{(a-b)(a-c)}{z-a} + \frac{1-\beta^2}{2}\cdot\frac{(b-a)(b-c)}{z-b} + \right.$$
$$\left. + \frac{1-\gamma^2}{2}\cdot\frac{(c-a)(c-b)}{z-c}.\right]$$

If we take the points $z_1 = 0,\ z_2 = \infty,\ z_3 = 1$ and let $b \to \infty$ we will obtain an expression for $\{w, z\}$ as in (5).

Let us take a closer look at equation (11) and find its particular solution. Here we can make progress thanks to

Theorem 2.4. *If u_1 and u_2 are linearly independent solutions of the linear ODE*

$$u''(z) + p(z)u = 0.$$

Then a function $w(z) = \frac{u_1(z)}{u_2(z)}$ is a solution of the equation

$$\{w, z\} = 2p(z)$$

But then (11), (12) imply the following

Corollary 2.5. *Let $w = f(z)$ maps $\mathrm{Im(z)} > 0$ onto a curvilinear polygon with n vertices on the real axis denoted by a_i and corresponding angles as $\pi\alpha_i$. Then $w = \frac{u_1(z)}{u_2(z)}$, where $u_1(z)$, $u_2(z)$ are two linearly independent solutions of the following linear ODE*

$$u''(z) + \left(\frac{1}{4}\sum_{i=1}^{n}\frac{1-\alpha_i^2}{(z-\alpha_i)^2} + \frac{1}{2}\sum_{i=1}^{n}\frac{\beta_i}{z-\alpha_i}\right)u(z) = 0, \tag{13}$$

where real β are constrained by these relations:

$$\sum_i \beta_i = 0,$$

$$\sum_{i=1}^{n}\left(-\alpha_i^2 + 2\alpha_i\beta_i + 1\right) = 0,$$

$$\sum_{i=1}^{n}\left(\alpha_i(1-\alpha_i^2) + \beta_i\alpha_i^2\right) = 0.$$

For the simplest case of a curvilinear polygon with two vertices, choose one of two points as $z = 0$ the theorem 2.4 with the corollary 2.5 together give:

$$2p(z) = \frac{1-\mu^2}{2z^2} + \frac{\beta}{z}.$$

But then the relations between α_i and β_i give that $\beta = 0$ and the Schwarz equation has such form

$$\{w, z\} = \frac{1-\mu^2}{2z^2},$$

with the corresponding 2-nd order ODE

$$u'' + \frac{1-\mu^2}{4z^2}u = 0.$$

Its solution can be written as follows

$$u = C_1 z^{\frac{1+\mu}{2}} + C_2 z^{\frac{1-\mu}{2}}.$$

Hence, we have the following conformal map from the polygon to the domain Im(z) > 0

$$w = \frac{A_1 z^\mu + A_2}{B_1 z^\mu + B_2}$$

In the case of a curvilinear triangle, the equation (13) takes a form of the hypergeometric equation (three vertices take its positions at points $z = \infty$, at $z = 0$ and $z = 1$:

$$z(1-z)y'' + (c - (a+b+1)z)\, y' - aby = 0, \tag{14}$$

where $a,\ b,\ c$ are defined using angles

$$a = \frac{1}{2}(1 + \beta - \alpha - \gamma),$$

$$b = \frac{1}{2}(1 - \beta - \alpha - \gamma),$$

$$c = 1 - \alpha.$$

The solution of (14) is given by the following definite integral for $b > 0,\ c > b$

$$F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\mathrm{dt}.$$

To apply the mapping theorem we need another solution of (14), let us obtain it by performing the substitution of $1 - z$ instead of z which gives another hypergeometric equation

$$z(1-z)y'' + (a + b - c + 1 - (a+b+1)z)\, y' - aby = 0.$$

It can be solved

$$y = \int_0^1 t^{b-1}(1-t)^{a-c}(1-zt)^{-a}\mathrm{dt}.$$

Replace z by $1 - z$, then the mapping function takes the following explicit form

Corollary 2.6.

$$w = f(z) = \frac{\int\limits_0^1 t^{-\frac{1}{2}(1+\alpha+\beta+\gamma)}(1-t)^{-\frac{1}{2}(1+\alpha-\beta-\gamma)}(1-zt)^{-\frac{1}{2}}(1+\beta-\alpha-\gamma)\mathrm{dt}}{\int\limits_0^1 t^{-\frac{1}{2}(1+\alpha+\beta+\gamma)}(1-t)^{-\frac{1}{2}(1+\gamma-\beta-\alpha)}(1-t+zt)^{-\frac{1}{2}}(1+\beta-\alpha-\gamma)\mathrm{dt}},$$

maps $\mathrm{Im(z)} > 0$ *onto a curvilinear triangle with the angles* $\pi\alpha$, $\pi\beta$, $\pi\gamma$ *sum of which is smaller than* π.

If $\alpha+\beta+\gamma > 1$ then these solutions should be replaced by other integral representations of hypergeometric functions, you can read more about them in [12].

3 Modular forms over $\mathrm{SL}(2, \mathbb{Z})$

3.1 Doubly-perioding functions

Firstly, let us construct a one-periodic complex function $f(z)$ (holomorphic or meromorphic), so there exists such $w \neq 0$ that $f(z)$ is invariant uner the action of a group $\mathbb{Z}w$

$$f(z+w) = f(z) \qquad \forall z \in \mathbb{C}$$

Put $w = 1$, so we have to find a $\mathbb{Z}$-periodic function on $\mathbb{C}$. Let us apply averaging for $f(z)$ which means that we have to consider the following periodic expression:

$$f(z) = \sum_{n\in\mathbb{Z}} \phi(z+n) \Rightarrow f(z) = \sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^2}.$$

We can guess that $f(z)$ is $1/(\sin(\pi z))^2$ which can be shown more explicitly. Consider a Lauren expansion for $f(z)$ nrear its poles $z = 0$:

$$f(z) = \frac{1}{z^2} + \sum_{n\neq 0} \frac{1}{(z+n)^2} = \frac{1}{z^2} + \text{ holomorphic near } z = 0.$$

But then we ahve for the inverse square of sine function

$$\frac{1}{\sin(\pi z)^2} = \frac{1}{(\pi z)^2} + \text{holomorphic at } 0.$$

Now, as $\mathrm{Im}(z) \to \infty$ we have that the following two expressions

$$f(z) \to 0$$
$$\pi^2/\sin(\pi z)^2$$

go to zero. Hence, by virtue of Liouville theorem

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^2} = \frac{\pi^2}{\sin^2(\pi z)}.$$

We can do the same for meromorphic functions f on $\mathbb{C}$ that are periodic with respect to a lattice Λ (they are called elliptic functions)

$$\Lambda = \mathbb{Z}w_1 + \mathbb{Z}w_2 \;\Leftrightarrow\; f(z+\lambda) = f(z) \qquad \forall z \in \mathbb{C},\; \lambda \in \Lambda,$$

where w_1 and w_2 are some linearly independent numbers. Now, the procedure of averaging which we used for the previous example can be applied for elliptic functions as well. Just as in the previous case, consider the following sums

$$\sum_{\lambda\in\Lambda} \frac{1}{(z+\lambda)^k}. \tag{15}$$

Choose $k = 2$ to define the Weierstrass $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{0\neq\lambda\in\Lambda} \Bigg(\frac{1}{(z+\lambda^2} - \frac{1}{\lambda^2} \Bigg), \tag{16}$$

the expression of which is invariant under the substitution $z \to z + \lambda$ in (16), since so is its derivative

$$\wp'(z) = -2\sum_{\lambda\in\Lambda} \frac{1}{(z+\lambda)^3} \Rightarrow \wp'(z+\lambda) = \wp'(z) \Rightarrow \wp(z+\lambda) = \wp(z) + A_\lambda.$$

where the constant A_λ can be found by choosing some specific z, for example take $z = -\frac{\lambda}{2}$, so A_λ turns to zero. In fact, we have the folllowing algebraic relation, which gives us a map between $(\wp, \wp')$ and its corresponding elliptical curve

Theorem 3.1. *Let Λ be a fixed lattice on which a Weierstrass function is defined. Then the following relation holds*

$$\wp'^2 = 4\wp^3 - 60g_2\wp - 140g_3, \tag{17}$$

where g_2 and g_3 are the following series defined for Λ:

$$g_2(\Lambda) = 60 \sum_{0\neq\lambda\in\Lambda} \frac{1}{\lambda^4},$$

$$g_3(\Lambda) = 140 \sum_{0\neq\lambda\in\Lambda} \frac{1}{\lambda^6}.$$

Hence we can explicitly write the uniformizing map between $(\wp, \wp')$ and the torus $\mathbb{C}/\Lambda$ which is given by (17)

$$\begin{tikzcd}
& \mathbb{C}/\Lambda \arrow[dr] & \\
\mathbb{C} \arrow[ur] \arrow[rr] & & \{y^2 = 4x^3 - 60g_2x - 140g_3\}
\end{tikzcd}$$

$$\begin{tikzcd} & z+\Lambda \arrow[dr] & \\ z \arrow[ur] \arrow[rr] & & (\wp(z), \wp'(z)) \end{tikzcd}$$

For a given lattice $\Lambda = \mathbb{Z}w_1 + \mathbb{Z}w_2$ we have an ordered basis given as w_1 and w_2, the ratio of which w_1/w_2 lies in the upper half-plane. Suppose, we want to choose another ordered basis, which does not change the lattice Λ:

$$\mathbb{C}/\left(\mathbb{Z}w_1 + \mathbb{Z}w_2\right) \cong \mathbb{C}/\left(\mathbb{Z}w_1' + \mathbb{Z}w_2'\right).$$

Actually, that change is implied by the action of the modular group $\mathrm{SL}_2(\mathbb{Z})$:

$$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Simply speaking we make transform the lattice basis in a linear way, so for a fixed $z \in \mathcal{H}$ – the upper half-plane, a basis $z,\ 1$ goes to $az+b, cz+d$, which is equivalent to $(az+b)/(cz+d), 1$. The first part here

$$\gamma(z) = \frac{az+b}{cz+d}$$

is called linear fractional transormations cause it sends $\mathcal{H}$ to itself. It is these $\gamma(z)$ that give the equivalence classes of moduli spaces

Theorem 3.2. *If for $z,\ z' \in \mathcal{H}$ there is such $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ that $\gamma z = z'$, then for the complex toris $\mathbb{C}/(\mathbb{Z}z + \mathbb{Z}) \cong \mathbb{C}/(\mathbb{Z}z' + \mathbb{Z})$.*

This statement gives rise to the notion of modular curve, namely the quotient $\Gamma/\mathcal{H}$, where $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

3.2 Elliptic modular forms

We have already introduced functions $g_2 = g_2(\Lambda)$ and $g_3 = g_3(\Lambda)$ through the elliptic curve map $(\wp, \wp') \to \mathbb{C}/\Lambda$. Certainly, these fucntions depend on the lattice (module), so in this sense we call them as modular forms. These functions have many amazing properties which give a lot of generalizations, but now we only need homogeneity:

$$g_2(\mu\Lambda) = \mu^{-4} g_2(\Lambda),$$
$$g_3(\mu\Lambda) = \mu^{-6} g_3(\Lambda).$$

Now, let us consider the general homogeneous functions F, which has a degree $-k$ on lattices, so

$$F(\mu\Lambda) = \mu^{-k} F(\Lambda).$$

One can normalize a basis of the lattice $w_1,\ w_2$ to make it $z = w_1/w_2,\ 1$, then the homogeneous property turns as

$$F(w_2^{-1}\Lambda) = w_2^k F(\Lambda).$$

As we have already mentioned a function of lattices don't depend on choice of basis. Now combine this basis transform along with normalization. Then we get for $z = w_2/w_2 \in \mathcal{H}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} \Rightarrow f(z) = F(\mathbb{Z} \cdot z + \mathbb{Z} \cdot 1) \Rightarrow$$
$$\Rightarrow F\left(\mathbb{Z} \cdot (az+b) + \mathbb{Z} \cdot (cz+d)\right) = (cz+d)^{-k} F\left(\mathbb{Z} \cdot \frac{az+b}{cz+d} + \mathbb{Z} \cdot 1\right).$$

Hence,

$$f(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

That gives us motivation to treat to modular forms as to elliptic modular forms of weight k, namely the holomorphic functions satisfying the automorphy condition, this is the definition we will use in the next subsection.

3.3 Fundemental domain

Here we will consider $\mathrm{SL}_2(\mathbb{Z})$ in more details. Let $\mathcal{H} \subset \mathbb{C}$ denote the upper half-plane, namely the set of $z \in \mathbb{C}$ such that $\mathrm{Im}(z) > 0$. Recall that $\mathrm{SL}_2(Z)$ is defined a s a set of two-by-two matrices with unit determinant:

$$SL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \mid ad - bc = 1 \right\}$$

with the matrix multiplication as the group operation. This is why we can regard $\mathrm{SL}_2(\mathbb{Z})$ as a discrete subgroup of the group of two-by-two matrices with the unit determinant $\mathrm{SL}_2(\mathbb{R})$.

A natural action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$ is given by the Möbius transformation:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \quad \gamma \cdot z := \frac{az+b}{cz+d},$$

where $z \in \mathcal{H} \setminus \{-d/c\}$. Hereafter, the dot in the definition of the action is usually omitted. Since $\mathrm{Im}(\gamma z) = \mathrm{Im}(z)/|cz+d|^2$, a point γz belongs to $\mathcal{H}$ and therefore the above is indeed an action.

Now we identify $\mathcal{H}$ with its image in the Riemann sphere. In this way, the action of $\mathrm{SL}_2(\mathbb{Z})$ can be extended to the points $z = -d/c$ and $z = \infty$:

$$\gamma \cdot \infty = \frac{a}{c}, \quad \gamma \cdot (-\frac{d}{c}) = \infty.$$

Before giving the definition of

We are ready to give a formal definition of a modular form.

Definition 3.3. *A modular form of weight k for the modular group $\mathrm{SL}_2(\mathbb{Z})$ is a function $f\colon \mathcal{H} \to \mathbb{C}$ satisfying the following three conditions:*

1. *f is a holomorphic function on $\mathcal{H}$.*

2. For any $z \in \mathcal{H}$ and any matrix in $\mathrm{SL}_2(\mathbb{Z})$ we have the following:

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z). \tag{18}$$

3. f is required to be bounded as $z \to i\infty$.

Using the holomorphy of f at ∞ we can use the Fourier decomposition for $f(z)$

$$f(z) = \sum_{n=0}^{\infty} a_n q^n \qquad q = e^{2\pi i z}. \tag{19}$$

If in addition $a(0) = 0$, then $f(z)$ is called a cusp form. The set of all modular forms for a given integer $k \in \mathbb{Z}$ is denoted as M_k.

Now we give a notion of a fundemental domain for $\mathrm{SL}_2(\mathbb{Z})$. The action of a modular group is given up to an equivalence of orbits. So, if we collect each one point of these orbits we get a fundamental set for $\mathrm{SL}_2(\mathbb{Z})$. Let us find the fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$. But firstly we want to prove that there are two particular elements in $\mathrm{SL}_2(\mathbb{Z})$, namely the following matrices: T and S which correspond to translation and rotation respectively:

Theorem 3.4. *$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ generate the entire $\mathrm{SL}_2(\mathbb{Z})$*

Take $z \in \mathcal{H}$ for which the lattice $\{mz + n \mid m, n \in \mathbb{Z}\}$ is defined. Suppose a point $cz + d$, where $c,\ d$ are relatively prime, is a point with minimal modulus. Then there exist integers $a,\ b$ such that

$$\gamma_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

By

$$\mathrm{Im}(\gamma z) = \frac{\mathrm{Im}(z)}{|cz+d|^2},$$

we get that

$$\mathrm{Im}(\gamma_1 z) \geq \mathrm{Im}(\gamma z) \qquad \forall \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

Now set the following point

$$z^* = T^n \gamma_1 z = \gamma_1 z + n \qquad \text{s.t} \qquad |\mathrm{Re}(z^*)| \leq \frac{1}{2},$$

We have the impossibility of $|z| < 1$ for then $\mathrm{Im}(-1/z^*) = \mathrm{Im}(z*)/|z*|^2 > \mathrm{Im}(z^*)$, which contradicts to the maximality off $\mathrm{Im}(z*)$. Therefor, $z^* \in \overline{F_1}$ and z is equivalent under $\mathrm{SL}_2(\mathbb{Z})$.

Suppose there are two equivalent points z_1 and $z_2 = \gamma z_1$ in F_1 and $\gamma \neq \pm 1$. Then we have that the coefficient $c \neq 0$ for γ. Since $\mathrm{Im}(z) > \sqrt{3}/2$ for all $z \in F_1$ we get

$$\frac{\sqrt{3}}{2} < \mathrm{Im}(z_2) = \frac{\mathrm{Im}(z_1)}{|cz_1+d|^2} \leq \frac{\mathrm{Im}(z_1)}{c^2 \mathrm{Im}(z_1)^2} < \frac{2}{c^2\sqrt{3}} \Rightarrow c = \pm 1$$

Assume that $\mathrm{Im} z_1 \leq \mathrm{Im} z_2$. But $|\pm z_1 + d| \geq |z_1| > 1$, so we get a contradiction, which leads to a proof of the theorem.

3.4 Eisenstein series

Consider the following function (known in literature as Eisenstein series) defined on $z \in \mathcal{H}$ with an even integer parameter $k \geq 4$:

$$G_k(z) = \sum_{m,n \in \mathbb{Z} \setminus \{0\}} (mz+n)^{-k}.$$

Our main goal is to show that $G_k(z)$ is a basic example of modular forms of weight k. But firstly, note that the limit

$$\begin{aligned}
\lim_{z \to i\infty} \sum_{m,n \in \mathbb{Z} \setminus \{0\}} (mz+n)^{-k} &= \sum_{n \in \mathbb{Z} \setminus \{0\}, m=0} n^{-k} + \lim_{z \to i\infty} \sum_{m,n \in \mathbb{Z} \setminus \{0\},\ m \neq 0} (mz+n)^{-k} \\
&= 2 \sum_{n=1}^{\infty} n^{-k} = 2\zeta(k),
\end{aligned}$$

equals to double zeta function $\zeta(k)$. So $G_k(z)$ has no nonnegative term in its Fourier expansion, and is holomorphic at $+i\infty$. Now check the invariance of the Eisenstein series under an action of $\mathrm{SL}_2(\mathbb{Z})$. Indeed,

$$G_k\Big|_k T(z) = \sum_{m,n \in \mathbb{Z}\ ; m,n \neq 0} (mz+m+n)^{-k} = G_k(z),$$

$$G_k\Big|_k S(z) = z^{-k} \sum_{m,n \in \mathbb{Z}\ ; m,n \neq 0} (nz-m)^{-k} = G_k(z).$$

Hence, we have established that $G_k(z)$ is a modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$. Now there is a turn of Fourier series for $G_k(z)$. We have already found the constant term, namely $2\zeta(k)$, for higher terms, let us apply the following relation:

Theorem 3.5.

$$\zeta(k) = -\frac{(2\pi i)^k}{2k!} B_k,$$

where B_k denotes the k-th Bernoulli number defined by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k.$$

It can be seen by comparing powers of the series given by logarithmic derivative with respect to z of the following expressions:

$$\pi z \cot(\pi z) = 1 + 2 \sum_{n=1}^{\infty} \frac{z^2}{z^2 + 4\pi^2 n^2} = 1 + 2 \sum_{n=1}^{\infty} \zeta(2n) \left(\frac{t}{2\pi} \right)^m,$$

$$\pi z \cot(\pi z) = i\pi z + \frac{2i\pi z}{e^{2i\pi z} - 1} \equiv \frac{t}{2} + \frac{t}{e^t - 1} = \frac{t}{2} + \sum_{i=0}^{\infty} \frac{B_i}{i!} t^i.$$

Comparing the k-th power of x we obtain the desired result. Now rewrite the last formula as follows

$$\pi\cot(\pi z) = i\pi\left(1-\frac{2}{1-e^{2\pi iz}}\right) = i\pi\left(1-2\sum_{n=0}^{\infty}e^{2\pi inz}\right)$$

Substituting $e^{2\pi iz} = q$, as it's appropriate for Fourier analysists, we have $\pi\cot(\pi z) = -i\pi\left(1+2n\sum_n q^n\right)$. But then

$$\frac{1}{z}+\sum_{n=-\infty,\ n\neq 0}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right) = -i\pi\left(1+2\sum_n q^n\right)$$

If we differentiate it $k-1$ times with respect to z, we obtain

$$(k-1)!\sum_{n=-\infty}^{\infty}(z+n)^{-k} = (2\pi i)^k\sum_{n=1}^{\infty}n^{k-1}q^n \tag{20}$$

By the definition of $G_k(z)$ one gets

$$G_k(z) = 2\zeta(k)+2\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}(mz+n)^{-k} = 2\zeta(k)+2\frac{(2\pi ik)^k}{(k-1)!}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}n^{k-1}q^{mn},$$

So by (20) and theorem 3.2:

$$G_k(z) = 2\zeta(k)\left(1-\frac{2k}{B_k}\sum_{n\geq 1}\sigma_{k-1}(n)q^n\right),$$

where $\sigma_r(n) = \sum_{d|n}d^r$ denotes the divisor sum function. One can "normalize" $G_k(z)$ by dividing to $2\zeta(k)$:

$$E_k(z) = 1-\frac{2k}{B_k}\sum_{n\geq 1}\sigma_{k-1}(n)q^n.$$

3.5 Weight formula

Let f be a meromorphic function defined on $\mathcal{H}$. Fix a point $p \in \mathcal{H}$, then let $v_p(f)$ be the order of zero of f at P, and $v_\infty(f)$ – the least integer n for which $a(n)$ is nonzero in $f(q = e^{2\pi inz}) = \sum\limits_{n=0}^{\infty}a(n)q^n$. Then one can prove the weight formula for f – non-zero modular function of weight k for $SL_2(\mathbb{Z})$:

Theorem 3.6. *Let* $p = -\frac{1}{2}+\frac{i\sqrt{3}}{2}$

$$\sum_{P\in\mathcal{H}\backslash \mathrm{SL}_2(\mathbb{Z}),\ \mathrm{p}\neq \mathrm{i,P}} v_P(f)+v_\infty(f)+\frac{1}{3}v_p(f)+\frac{1}{2}v_i(f) = \frac{k}{12}. \tag{21}$$

Using (21) we see that the $\mathbb{C}$-vector space M_k is one-dimensional for $4 \leq k \leq 10$ and M_k is generated by G_k. Therefore in that case we do not have a cusp form – it occurs the first time when $k = 12$. The following example represents a cusp form with of weight 12 for a group $\mathrm{SL}_2(\mathbb{Z})$:

$$\Delta(z) = \frac{1}{1728}\left(E_4^3(z) - E_6^2(z)\right).$$

In terms of Fourier expansions

$$E_4(z) = 1 + 240\sum_{n\leq 1}\sigma_3(n)q^n,$$

$$E_6(z) = 1 - 504\sum_{n\leq 1}\sigma_5(n)q^n,$$

it can be written as follows

$$\Delta(n) = \frac{1}{1728}\left(1728q - 41472q^2 + \ldots\right) = \sum_n \tau(n)q^n,$$

where $\tau(n)$ denotes the so-called Ramanujan's tau function.

Interesting fact: each space M_k of modular forms weight k is generated by only $E_4(z)$ and $E_6(z)$, that can be expressed by the following conjecture

Theorem 3.7.

$$f(z) = \sum_{a,b\in\mathbb{Z}_{\geq 0},\ 4a+6b=k} c_{ab}E_4(z)^a E_6(z)^b,$$

with some constants $c_{ab} \in \mathbb{C}$

It can be shown that the following decomposition takes place

Theorem 3.8.

$$M_k = \mathbb{C}E_k \oplus S_k,$$

Theorem 3.9. *The discriminant function* $\Delta(z)$ *gives an isomorphism between* M_k *and* M_{k-12}

Corollary 3.10. *Dimension formula:*

$$\dim_{\mathbb{C}} M_k = \begin{cases} \left\lfloor \frac{k}{12} \right\rfloor & \text{if } k \equiv 2 \pmod{12}, \\ 1 + \left\lfloor \frac{k}{12} \right\rfloor & \text{if } k \not\equiv 2 \pmod{12}. \end{cases}$$

4 Literature

References

[1] Bump D., *Automorphic forms and representations.* Cambridge Stud. Adv. Math., 55 Cambridge University Press, Cambridge, 1997.

[2] Masaaki Y., *Fuchsian Differential equation's. With special emphasis on the Gauss-Schwarz theory.* Aspects Math., E11 *Friedr. Vieweg and Sohn* , Braunschweig, 1987.

[3] Sasaki T. and Yoshida M., Schwarzian derivatives and uniformization. *The Kowalevski property* (Leeds, 2000), 271–286. CRM Proc. Lecture Notes, 32 American Mathematical Society, 2002.

[4] Sato H., Schwarzian derivatives of contact diffeomorphisms, *Lobachevskii J. Math.* **4** (1999), 89–98.

[5] J. S. Milne, *Milne, J. S. Introduction to Shimura varieties.*, (English summary), *Harmonic analysis, the trace formula and Schimura varieties*, 265–378. *American Mathematical Society, Providence, RI, 2005*

[6] Sebbar A., *Schwarzian equations and equivariant functions.*(English summary) *2017 MATRIX annals*, 449–460. MATRIX Book Ser., 2 *Springer*, 2019

[7] W.J. Harvey, *King's College London*, *Introductory lectures on* $\mathrm{SL}(2,\mathbb{Z})$ *and modular forms.*, November 2008.

[8] Nehari Z., *Conformal mapping.* Reprinting of the 1952 edition *Dover Publications, Inc.*, New York, 1975. vii+396 pp.

[9] Caratheodory C. (1954), *Theory of functions of a complex variable*, Vol. 2, translated by F. Steinhardt, Chelsea

[10] Ramakrishnan B. *A brief introduction to modular forms. Cyclotomic fields and related topics*, (Pune, 1999), 219–238. *Bhaskaracharya Pratishthana*, Pune, 2000

[11] Zagier D. *Introduction to modular forms. From number theory to physics*, (Les Houches, 1989). *Springer-Verlag*, Berlin, 1992

[12] Whittaker E. T.; Watson G. N. *A course of modern analysis—an introduction to the general theory of infinite processes and of analytic functions with an account of the principal transcendental functions.* Fifth edition. Edited by Victor H. Moll. With a foreword by S. J. Patterson. *Cambridge University Press*, Cambridge, 2021. lii+668 pp.

[13] Diamond F. and Shurman J., *A first course in modular forms.* Grad. Texts in Math., 228 *Springer-Verlag*, New York, 2005.

[14] Diamond F.; Im J. *Modular forms and modular curves.* Seminar on Fermat's Last Theorem (Toronto, ON, 1993–1994), 39–133. *CMS Conf. Proc.*, 17 Published by the American Mathematical Society, Providence, RI; for the, 1995

[15] Chan H. H. *Theta functions, elliptic functions and* π. With a foreword by Bruce Berndt De Gruyter Textbook *De Gruyter*, Berlin, 2020.

[16] Atorn K. *Covariant hyperelliptic functions of genus 2. Teoret. Mat. Fiz. 137 (2003)*, no. 1, 5–13; translation in *Theoret. and Math. Phys. 137 (2003)*, no. 1.

[17] Farkas, Hershel M.; Kra, Irwin *Theta constants, Riemann surfaces and the modular group.* (English summary) An introduction with applications to uniformization theorems, partition identities and combinatorial number theory *Grad. Stud. Math., 37 American Mathematical Society*, Providence, RI, 2001. xxiv+531 pp.

[18] Mumford D. *Tata lectures on theta. I.* With the assistance of C. Musili, M. Nori, E. Previato and M. Stillman Progr. Math., 28 *Birkhäuser Boston, Inc.*, Boston, MA, 1983. xiii+235 pp.

[19] Harder G.; Raghuram A. *Eisenstein cohomology for* GL_N *and the special values of Rankin-Selberg L-functions.* Ann. of Math. Stud., 203 *Princeton University Press*, Princeton, NJ, 2020. xi+220 pp.

[20] Roy B.; Vlasenko M. *Frobenius constants for families of elliptic curves. Q. J. Math.*, 74 (2023), no. 4.

[21] Ramakrishnan D.; Shahidi F. *Siegel modular forms of genus 2 attached to elliptic curves.* (English summary) *Math. Res. Lett. 14 (2007)*, no. 2.

[22] Hu J. *BGG category* $\mathcal{O}$ *and* $\mathbb{Z}$*-graded representation theory. East China Norm. Univ. Sci. Rep., 16* World Scientific Publishing Co. Pte. Ltd., Singapore, [2023],

[23] Milla L. *An efficient determination of the coefficients in the Chudnovskys' series for* $\frac{1}{\pi}$. *Ramanujan J. 57 (2022)*, no. 2.

[24] G-D. Andreea; Merca M. *Ramanujan's tau function as sums over partitions.* English summary. *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.*, 86 (2024), no. 3, 69–80.

[25] Schwarz, H.A.. "Ueber diejenigen Fälle, in welchen die Gaussische hypergeometrische Reihe eine algebraische Function ihres vierten Elementes darstellt.." Journal für die reine und angewandte Mathematik 75 (1873).