Gauge theories and integrable systems

Stariy Kolya
Scientific advisor: Nikolay Iorgov

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Introduction

Structures such as differential forms, bundles, and connections are now used in various areas of mathematics and physics. Perhaps their most well-known area of application is gauge theories, which describe the three interactions known to us. Since this area is rather abstract, let's try to start with the simplest example - the electromagnetic field, when considering which you can already get acquainted with the above concepts.

Electromagnetic field

The electromagnetic field is characterized by electric and magnetic intensity vectors

$$\overrightarrow{E} = (E_x, E_y, E_z), \overrightarrow{H} = (H_x, H_y, H_z)$$

.

The set of experimental data was generalized in Maxwell's equations

$$rot\,\overrightarrow{E} = -\frac{1}{c}\frac{\partial \overrightarrow{H}}{\partial t} \qquad div\,\overrightarrow{H} = 0$$

$$rot\,\overrightarrow{H} = \frac{1}{c}\frac{\partial \overrightarrow{E}}{\partial t} + \frac{4\pi}{c}\overrightarrow{j} \qquad div\,\overrightarrow{E} = 4\pi\rho$$

Tensors

How vector components change when changing coordinates.
Consider a smooth curve $\phi : \mathbb{R} \to \mathbb{R}^n$ defined in some coordinate system $(x^1, x^2, ..., x^n)$ by the equations $x^i = \phi^i(t)$. Let $\xi^i = \dot{\phi}_i(0)$ be the components of tangent vector.
Then in new coordinates we will obtain:

$$\tilde{\xi}^i = \frac{d\tilde{x}^i}{dt}(t=0) = \frac{df^i(\phi^1(t), ..., \phi^n(t))}{dt}(t=0) =$$

$$= \sum_{j=1}^{n} \frac{\partial f^i}{\partial x^j}(t=0)\xi^j$$

Tensors

Consider a function $\phi(\tilde{x}^1, ..., \tilde{x}^n)$. Then let's try to calculate the components of its gradient $grad\phi = (\frac{\partial\phi}{\partial\tilde{x}_1}, ..., \frac{\partial\phi}{\partial\tilde{x}_n}) = (\tilde{\eta}^1, ..., \tilde{\eta}^n)$ in the initial coordinates

$$\eta^i = \sum_{j=1}^{n} \frac{\partial\phi}{\partial\tilde{x}_j}\frac{\partial f^j}{\partial x_i} = \sum_{j=1}^{n} \tilde{\eta}^j \frac{\partial f^j}{\partial x_i}$$

Therefore, we see that when the coordinates are changed, the gradient components behave differently than the components of the tangent vector, they change according to the same law as the differential (tensor of another kind).

Scew symmetric differential forms

Modulus of vector fields over a ring of smooth functions $D(\mathbb{R}^n)$

$$\vec{v}(x^1, ..., x^n) = \sum_{j=1}^{n} v_j(x^1, ..., x^n) \frac{\partial}{\partial x_j}$$

Consider the dual space to $D(\mathbb{R}^n)$

$$\alpha : D(\mathbb{R}^n) \to F(\mathbb{R}) \qquad \alpha(f\,\vec{v}) = f\,\alpha(\vec{v}) \qquad \alpha(\vec{v_1}+\vec{v_2}) = \alpha(\vec{v_1})+\alpha(\vec{v_2})$$

The resulting module is called the space of differential forms $\Lambda^1(\mathbb{R}^n)$
A differential form as a linear combination of generators

$$\alpha = \sum_{i=1}^{n} \alpha_i(x^1, ..., x^n) dx^i$$

Differential forms and their examples

This definition can be extended to the case of two or more vector fields. In this way we can obtain 2-forms $\alpha : D(\mathbb{R}^n) \times D(\mathbb{R}^n) \to F(\mathbb{R}^n)$ Consider an antisymmetric 2-covector (external form). Suppose we have two vector fields $\vec{v}^1, \vec{v}^2$ in $\mathbb{R}^2$. Then we have the functional of the oriented area of the parallelogram built on these vectors $\tau(\vec{v}^1, \vec{v}^2)$ of $F(\mathbb{R}^2)$.

$$\det \begin{pmatrix} x^1 & y^1 \\ x^2 & y^2 \end{pmatrix} = \mathrm{x}^1 y^2 - x^2 y^1$$

In the case of an arbitrary number of vector fields

$$L(\vec{\xi}^1, ..., \vec{\xi}^n) = L(\xi_1^{i_1} e_{i_1}, ..., \xi_n^{i_n} e_{i_n}) = L(e_{i_1}, ..., e_{i_n}) \xi_1^{i_1} ... \xi_n^{i_n} =$$

$$= L(e_{i_1}, ..., e_{i_n}) \det \begin{pmatrix} \xi_1^{i_1} & ... & \xi_1^{i_n} \\ ... & ... & ... \\ \xi_n^{i_1} & ... & \xi_n^{i_n} \end{pmatrix}$$

Differential forms and their examples

The above example had the obvious property (scew symmetry) for arbitrary i,j: $\alpha(\vec{v}_1, ..., \vec{v}_i, ..., \vec{v}_j, ..., \vec{v}_n) = -\alpha(\vec{v}_1, ..., \vec{v}_j, ..., \vec{v}_i, ..., \vec{v}_n)$
Then such a func. α is called the outer differential form $\Lambda^k(\mathbb{R}^n)$
We write the forms in coordinates, for this we introduce the operation of the outer product
On basis vectors (1-forms) we have
$dx^i \wedge dx^j(\vec{v}_1, \vec{v}_2) = dx^i(\vec{v}_1)dx^j(\vec{v}_2) - dx^i(\vec{v}_2)dx^j(\vec{v}_1)$
Then the mapping ω is written as
$\omega = \sum_{1 \leqslant i_1 < i_2 \leqslant n} \omega_{i_1 i_2}(x^1, ..., x^n) dx^{i_1} \wedge dx^{i_2}$

Differential forms and their examples

One can extend this definition to forms of arbitrary degree (namely for p-,q-forms). The result of this operation will be (p+q)-form.

$$(\alpha \wedge \beta)(\vec{v}_1, ...\vec{v}_{p+q}) =$$

$$\frac{1}{p!q!} \sum \epsilon(s)\alpha(\vec{v}_{s^{-1}(1)}, ..., \vec{v}_{s^{-1}(p)})\beta(\vec{v}_{s^{-1}(p+1)}, ..., \vec{v}_{s^{-1}(p+q)})$$

here the summation is over all possible permutations of p+q elements.

It is possible to decompose the resulting form in terms of the basis

$$\alpha = \sum_{1 \leqslant i_1 < ... < i_k < ... \leqslant n} \alpha_{i_1 ... i_k}(x^1, ..., x^n) dx^{i_1} \wedge ... \wedge dx^{i_k}$$

From a property $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$ we have that $\alpha \wedge \alpha = 0$ if the degree of α is odd.

External differential

Let us define a product that can be used to construct forms of a higher degree $d : \Omega^k \to \Omega^{k+1}$

$$d\omega(x) = d(f_1 dx^1 + ... + f_n dx^n) = df_1 \wedge dx^1 + ... + df_n \wedge dx^n$$

$$d\alpha = \sum_{1 \leqslant i_1 < ... < i_k < ... \leqslant n} d\alpha_{i_1,...,i_k}(x^1, ..., x^n) \wedge dx^{i_1} \wedge ... \wedge dx^{i_k}$$

from this definition follows that $d^2\omega = 0$. But for spaces different from $\mathbb{R}^n$. k-forms obtained in this way may be non trivial, and the result of the composition can be considered as a de Rham complex.

$$F(\mathbb{R}^n) \xrightarrow{d} \Lambda^1(\mathbb{R}^n) \xrightarrow{d} \Lambda^2(\mathbb{R}^n) \xrightarrow{d} ... \xrightarrow{d} \Lambda^k(\mathbb{R}^n) \xrightarrow{d} ...$$

With the help of this complex, one can study the topology of space.

Inner product

Let $\vec{v}$ is a vector field and α is an external form of degree p. Then we can construct a form of degree p-1 using next definition

$$(i_v \alpha)(\vec{v}_1, ..., \vec{v}_{p-1}) = \alpha(\vec{v}, \vec{v}_1, ..., \vec{v}_{p-1})$$

Analysis of Maxwell equations

Let us rewrite the rotor and divergence operators in terms of differential forms. We consider a vector field on $\mathbb{R}^3$. Then we have a 3-form of volume $\tau = dx \wedge dy \wedge dz$. Let $\vec{v} = v_x\partial_x + v_y\partial_y + v_z\partial_z$

$$i_v\tau = v_x dy \wedge dz + v_y dz \wedge dx + v_z dx \wedge dy$$

$$d(i_v)\tau = (\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z})dx \wedge dy \wedge dz$$

When changing coordinates $\tilde{x}^i = f^i(x^1, x^2, x^3) \to \tilde{\tau} = J(f)\tau$, we obtain the divergence

$$\frac{i_v(\tilde{\tau})}{\tilde{\tau}} = \frac{dJ(f) \wedge i_v\tau}{J(f)\tau} + \frac{d(i_v\tau)}{\tau}$$

Since the field is arbitrary, the divergence changes only under the substitutions J(f)=const

Analysis of Maxwell equations, continued

Let's do the same for a rotor. Then we have a mapping

$$\sigma : D(\mathbb{R}^3) \to \Lambda^1(\mathbb{R}^3), d(\sigma(\vec{v})) = (\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y})dx \wedge dy +$$

$$+(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z})dy \wedge dz + (\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x})dz \wedge dx$$

Then we've

$$rot\vec{v} = (m \circ d \circ \sigma)(\vec{v}), wherem : \Lambda^2(\mathbb{R}^3) \to D(\mathbb{R}^3)$$

such that $i_{m(\alpha)}\tau = \alpha$
Hence we see that the map m is defined only on $\mathbb{R}^3$ cause it is associated with form $i_{m(\alpha)}\tau$ which must coincide with α.
From here we see that it is better to consider the rotor not as a vector field, but as an external differential of a covector field $v_x dx + v_y dy + v_z dz$

Analysis of Maxwell equations, continued

Let's try to find an invariant representation of Maxwell's equations, which does not depend on the metric in space or transformations of coordinates. Let's start with the 2-form Ω that corresponds to the electromagnetic field - the electromagnetic tensor

$$\Omega = cE_x dx \wedge dt + cE_y dy \wedge dt + cE_z dz \wedge dt +$$

$$H_x dy \wedge dz + H_y dz \wedge dx + H_z dx \wedge dy$$

Since we are in $\mathbb{R}^4 \to d\Omega = 0$.
Collecting coefficients at terms of the form $dx \wedge dy \wedge dz$ and $dx \wedge dy \wedge dt$ we obtain the corresponding Maxwell equations

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0, \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} + \frac{\partial H_z}{\partial t} = 0$$

Similarly, one can obtain the remaining components of the Maxwell equation. Hence, we have an invariant notation of Maxwell's equations.

$$d\Omega = 0$$

Analysis of Maxwell equations, continued

Because in $R^4 : d^2 = 0$ every closed form is exact ($\Omega = d\omega$,where ω is a 1-form)
This form is called a 4-potential of electromagnetic field

$$\omega = A_x dx + A_y dy + A_z dz + A_t dt$$

$$d\omega = d(A_i dx^i) = \sum_{\nu > \mu} (\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}) dx^\mu \wedge dx^\nu =$$

$$= F_{\mu\nu} dx^\mu \wedge dx^\nu (\mu < \nu)$$

ω is ambiguous cause $d\omega' = d(\omega + dS) = \Omega$ where dS is an exact 1-form. Therefore, the 4-potential A_μ is determined up to gauge transformations

$$A'_\mu = A_\mu + \frac{\partial S}{\partial x_\mu}$$

Vector Bundles

Consider an example that illustrates the concept of a bundle.
Let we have a func. $f : \mathbb{R}^4 \to \mathbb{R}^2$,namely it's a set of s=(x,f(x)).
Then there's a natural projection π from $\mathbb{R}^4 \times \mathbb{R}^2$ onto $\mathbb{R}^4 : (x, y) \to x, \pi \circ s = id$. Then a map s is called a section, and the construction itself is a trivial bundle.

Consider covering $\mathbb{R}^4$ by open sets U_i. Then we define functions (sticky cocycle) $g_{ij} : U_i \cap U_j \to GL(2, \mathbb{R})$ for each non-empty intersection, which have such properties

$$g_{ji}(x) = (g_{ij}(x))^{-1} \qquad g_{ij}(x)g_{jk}(x)g_{ki}(x) = I$$

Let's build a structure similar to a trivial bundle by gluing spaces $U_i \times \mathbb{R}^2$ over sets $(U_i \cap U_j) \times \mathbb{R}^2$. In fact we need to identify a point $(x, \vec{\xi})$ with $(x, g_{ij}\vec{\xi})$
As a result, we obtain a space with a natural projection onto $\mathbb{R}^4$, it's called a vector bundle with a structural group $GL(2, \mathbb{R})$

Vector bundles

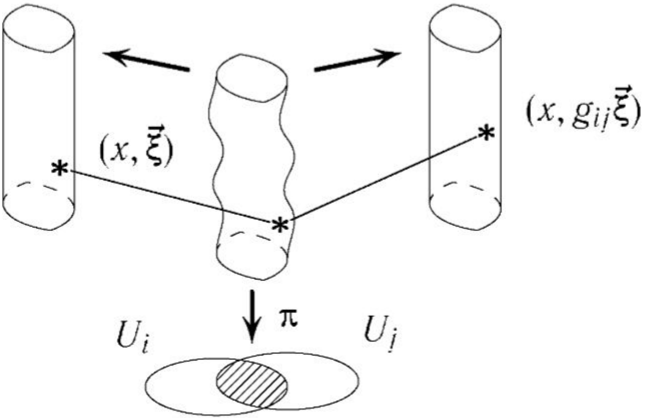

Figure: Bonding cylinders with twist.

Equivalent bundles

Let we have another gluing cocycle that corresponds to the vector bundle (P', π). If on each layer there exists a smooth mapping P such that $\pi' \circ F = \pi$, then bundles P,P' are equivalent

The sections of the bundle form a module over the ring of smooth functions on the base.

Connection on the bundle

Differentiation of a section of a direct bundle

$$(x, f(x)) \to (x, \partial_v f = df(\vec{v}))$$

Differentiation of a section of an arbitrary bundle

$$(x, f(x)) \to (x, \partial_v f_i = (\partial_v g_{ij}) f_j + g_{ij} \partial_v f_j)$$

Consider a new derivative along the vector field, at which the term linear in f_j disappears, and the new sections stick.

$$\nabla_v f_i = \partial_v f_i + A^i_v f_i, A^i_v is2 \times 2matrix$$

$$\nabla_v f_i = \partial_v (g_{ij} f_j) + A^i_v (g_{ij} f_j) = g_{ij} (g_{ij}^{-1} (\partial_v g_{ij}) f_j + \partial_v f_j + g_{ij}^{-1} A^i_v g_{ij} f_j) =$$

$$= g_{ij} \nabla_v f_j$$

Hence we've $A^i_v = -(\partial_v g_{ij}) g_{ij}^{-1} + g_{ij} A^j_v g_{ij}^{-1}$

Connection on the bundle

A connection is given in a vector bundle if for arbitrary neighborhoods and gluing cocycles a set of differential matrix 1-forms ω_i is given satisfying
$w_i = -dg_{ij}g_{ij}^{-1} + g_{ij}\omega_j g_{ij}^{-1}, A_v^i = \omega_i(\vec{v})$
Hence, we've $\nabla s = (x, df_i + \omega_i f_i)$
Invariant definition of a connection: The action of the connection operator transforms sections into sections with coefficients in 1-forms

$$\nabla : (sect.) \to (sect.) \otimes \Lambda^1(\mathbb{R}^4)$$

$$U_i \times \mathbb{R}^2 : \xi_1 = (x, e_1), \xi_2 = (x, e_2)$$

$$\nabla \xi_j = (x, \nabla e_j) = (x, \omega_i e_j) = (x, \omega_i^j) \to (\nabla \xi_1, \nabla \xi_2) = (\xi_1, \xi_2)\omega_i$$

The section is said to be horizontal with respect to the connection ∇ if $\nabla s = 0$

The connection curvature tensor and the Bianchi identity

We continue the diagram of differentiation of sections:

$$(sect.) \xrightarrow{\nabla} (sect.) \otimes \Lambda^1(\mathbb{R}^n) \xrightarrow{\tilde{\nabla}} (sect.) \otimes \Lambda^2(\mathbb{R}^n)$$

Let's consider the map (connection curvature tensor) $K = \tilde{\nabla} \circ \nabla$
Locally over U_i in basis ξ_1, ξ_2 we can write $K(\xi_1, \xi_2) = (\xi_1, \xi_2)\omega_i$
where ω_i is a matrix of differential 2-forms.

$$\tilde{\nabla}(\nabla(\xi_1, \xi_2)) = (\nabla\xi_1, \nabla\xi_2) \wedge \omega_i + (\xi_1, \xi_2) d\omega_i =$$

$$= (\xi_1, \xi_2)(\omega_i \wedge \omega_i) + (\xi_1, \xi_2) d\omega_i = (\xi_1, \xi_2)(\omega_i \wedge \omega_i + d\omega_i)$$

Comparing the coefficients at the basis terms, we have the Bianchi identity

$$\Omega_i = d\omega_i + \omega_i \wedge \omega_i \to d\Omega_i = \Omega_i \wedge \omega_i - \omega_i \wedge \Omega_i$$

Principal bundle

Consider as elementary cylinders the sets $U_i \times GL(2, \mathbb{R})$. Gluing will occur the same way as in the previous case:
$(x, G) \to (x, g_{ij}G)$. Then we get the principal bundle, in which the structural group and the layer are the same.
In the principal bundle, one can also introduce a connection, although it will no longer be a differentiation of sections, it will just be some subspace of horizontal vectors in the tangent subspace.

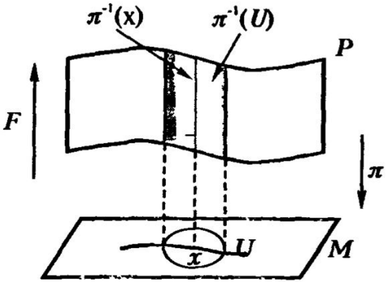

Figure: Global structure of a bundle.

Electromagnetic field

The electromagnetic field can be considered as a connection in the main bundle over $\mathbb{R}^4$ with the layer $U(1) in GL(1, \mathbb{C})$.
The gluing functions g_{ij} take on values in a group U(1). Therefore, the local forms of connection and curvature will have the form

$$\omega_i \wedge \omega_i = 0 \qquad \Omega_i \wedge \omega_i - \omega_i \wedge \Omega_i = 0$$

By changing the gauge, we can obtain another connection

$$\omega_i = -dg_{ij} g_{ij}^{-1} + g_{ij} \omega_j g_{ij}^{-1} = \omega_j + dS_{ij}$$

where $S_{ij} = dln(-g_{ij})$ or in terms of ω_i: $A^i_\mu = A^j_\mu + \frac{\partial S_{ij}}{\partial \mu}$
The components of the local connection form are transformed in the same way as the components of the 4-potential of the electromagnetic field.

Other examples

Yang Mills fields (describes the behavior of nucleons) are connections in the principal bundle with the structure group G=SU(2). Here, the connectivity and curvature forms will have a more complex form, since they are matrix.

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Gauge theories

Mathematically, gauge theories are connection theories in principal G-bundles. Here the connection is implied by some 1-form ω with values in the Lie algebra g=Lie G. In fact we've an affine variety of connections.

$$d + A_\alpha = g_{\alpha\beta}(d + A_\beta)g_{\alpha\beta}^{-1} \to A_\alpha = g_{\alpha\beta}A_\beta g_{\alpha\beta}^{-1} - dg_{\alpha\beta}g_{\alpha\beta}^{-1}$$

$$(A_\alpha) : A_\alpha^1 - A_\alpha^2 = F_\alpha = Ad_{g_{\alpha\beta}}(F_\beta) in \Gamma(Y \times P)$$

Measure on the manifold of connections

Quantum gauge theories=a measure on A invariant under the action of the gauge transformation group.

$$1) g = (g_\alpha), g_\alpha : U_\alpha \to G, g_\alpha = g_{\alpha\beta} g_\beta g_{\alpha\beta}^{-1}$$

$$2) A_\alpha \to A_\alpha^g : A_\alpha^g = g_\alpha A_\alpha g_\alpha^{-1} - dg_\alpha g \alpha^{-1}$$

Measure on the manifold of connections

Then we want to construct such a measure DA

$$\delta A^2 = \int_X Tr(\delta A \wedge *\delta A)$$

where Tr denotes a Killing form on a Lie algebra, * is a so called Hodge operation. Then we can get the following volume form

$$(\frac{\delta A}{Vol(g)})e^{-\frac{1}{g^2}S[A]}$$

As an action in $\mathbb{R}^4$ we can take either
$S_{YM}[A] = \int Tr(F_a \wedge *F_a), where F_A = (d + A)^2 = dA + A \wedge A$
It can be shown that our action is a holonomy along some loop
$= Pexp \int_C A = g_c$ which is taken from the expression
$\psi(1) = g_c\psi(0), where \qquad d\psi + A\psi = 0$
From a perturbation theory we can obtain g_c

$$g_c = 1 + \int_0^1 A_t dt + \int_{t_1<t_2} A_t(t_1)A_t(t_2)dt_1dt_2 + ...$$

About second fuctional

Let's consider the topological functional

$$S_{top}[A] = \int_X Tr(F_A \wedge F_A) = -8\pi^2 k$$

Here k is an integer number and it's exactly a 2-nd Chern class of a bundle P.
Hence we've such a measure

$$(\frac{DA}{Vol(g)})e^{-\frac{1}{g^2}S_{YM}[A]+\frac{i\theta}{8\pi^2}S_{top}[A]}, \theta in \mathbb{R}/2\pi\mathbb{Z}$$

For the coupling constant, we have $\frac{1}{g^2} = ln(\frac{\Lambda_{highenergy}}{\Lambda_{obs}})$

Observables

Local observables: $\int \mu O_1(x_1)...O_n(x_n)W_{R_1}(C_1)...W_{R_k}(C_k)$
here $O_i(x) = \frac{Tr(F_A \wedge *F_A)}{Vol(g)}$, $R_i inRep(G)$
There's a hypothesis that the product of O can be presented in the form $e^{-\Lambda_{QCD}(x_i - x_j)}$
We can also consider unlocal observables, namely $W_R[C] = Tr_R Rexp \int_C A$.
Next we make assumptions that systems energies are low, namely $x_i - x_j >> \Lambda_{high}^{-1} \quad or \quad C(loopsizes) >> \Lambda_{high}^{-1}$ Next, we have two approaches to regularize observables: 1)Associate X with a finite simplex complex, then A is 1-chain that takes values in a group. Further, from the corresponding curvature of the lattice, we can obtain the Haar measure on the edges.
2)We can also work with a discontinious theory (one introduces additional supersymmetric fields). Further, we can approach the measure μ_{YM} through the cohomologies of A. This reveals an interesting connection between 4D gauge theories and 2D conformal integrable systems.

Instantons

$$\Omega_x^2 = \Omega_x^{2+} \oplus \Omega_x^{2-} \to F_A = F_{A_+} + F_{A_-} \to$$

$\to S_{YM} = (F_A^+)^2 + (F_A^-)^2, S_{top} = (F_A^+)^2 + -(F_A^-)^2$ Let $F_A^+ = 0$, then $k > 0$.

Finite dimensional manifold $M = (A, F_A^+ = 0)/g$. Then if we take $\mu_{inst} = \delta_M inA/g$ we can obtain purely instanton contributions to the correlation function.

Corresponding partition function

$$Z(a', \epsilon', \Lambda) = \sum_{k=0}^{\infty} \Lambda^{2kN} \int_{M_{k,U(N)}(\mathbb{R}^4)} e^{-D\Lambda}$$

here M are connections in the bundle on euclidean space. cause we're interested in local properties of theory;
$\epsilon' = (\epsilon_1, \epsilon_2) in \mathbb{C}^2, a' = (a_1, ..., a_N) in C^N$.
A group $H = SO(4) \times G$ acts on $M_{k,U(N)}$

$$D\lambda = \partial_A((g_M)_{BC} V^C(a'*, \epsilon'*) dm^a \wedge dm^B + g_M(V(a', \epsilon'); V(a'*, \epsilon'*)))$$

$$e^{-D\lambda} = e^{-g_M(V(a',\epsilon');V(a'*,\epsilon'*))(1+\Omega+\frac{1}{2}\Omega\wedge\Omega+...+\frac{1}{(2kN)!}\Omega^{2kN})}$$

$$Z = \int \frac{\Omega^{2kN}}{(2kN)!} e^{-(V(a',\epsilon'))^2} = \sum_{\lambda^1,...,\lambda^N} \Lambda^{2N} \mu_{a',\epsilon'}(\lambda^1, ..., \lambda^N)$$

Here the sum is considered over N partitions
$abs(\lambda^1) + abs(\lambda^2) + ... + abs(\lambda^N) = k$

Connection with integrable systems

Let's consider Z in the limit of $\epsilon_2 \to 0, namely(a', h, \Lambda)$. Then $Z = e^{-\frac{1}{\epsilon_2} W(a', h, \Lambda)}$. In this case W can be considered as a free energy. This free energy also arises in another problem where it is necessary to find the spectrum in a quantum integrable system. Consider the periodic Toda chain:

$$H = -\frac{h^2}{2} \sum_{i=1}^{N} \partial_{x_i}^2 + \Lambda^2 \sum_{i=1}^{N} e^{x_i - x_{i+1}} (x_{N+1} = x_1)$$

Then the spectrum E will be related to the free energy as follows

$$E + \Lambda \frac{\partial}{\partial \Lambda} W, \partial W \partial a_i = 2\pi i n_i (n_i in \mathbb{Z})$$

Conclusions

1)We have clarified the connection between gauge theories and such mathematical concepts as differential forms, bundles and connections on them.

2)An attempt was made to approach an important problem, the connection between microscopic and macroscopic physics. So we got acquainted with a simplified model of such a problem and saw its connection with such integrable systems as spin chains.

Instead of epilogue

Thanks for your attention !!!