Research by Volodymyr V. Lyubashenko

Research Field

Quantum Groups, Hopf algebras, Tensor Categories and their applications to Low Dimensional Topology and Topological Field Theories, applications of Algebraic Geometry to Quantum Groups, Tensor Categories and Hopf Categories, A_∞-categories.

Published articles

Here is the description of the former research. Articles are numbered as in CV.

5. *Real and Imaginary Forms of Quantum Groups*, Proceedings of the first semester "Quantum Groups" of Euler IMI, Leningrad, 1990, Lect. Notes Math. **1510** (1992) 67–78.

Real forms of complex ribbon braided categories were defined via descent data following the approach of Deligne and Milne to symmetric categories. The definition is not equivalent to real forms of Hopf algebras (*-structure), but often includes them and gives new examples (imaginary structure).

10. *Quantum Function Algebra at Roots of 1*, Adv. Math. **108** (1994) n. 2, 205–262 (with C. De Concini).

We introduce a form of the quantum function algebra on a Drinfeld-Jimbo quantum group over the ring $Q[q, q^{-1}]$. Specializing q to a root of 1, we show that over the cyclotomic field this algebra is a projective module over its central subalgebra, which is the usual coordinate algebra of the group. We study induced Poisson-Lie structure of the group. A bundle of algebras on a complex simply connected Lie group with hamiltonian flows in the bundle is constructed. Some representations of the quantum function algebra in a root of 1 are constructed as an application.

12. *Modular Transformations for Tensor Categories*, J. Pure Appl. Algebra **98** (1995) n. 3, 279–327.

For an abelian braided tensor category we investigate a Hopf algebra F in it, the "braided function algebra". We show the existence of the object of integrals for any Hopf algebra in a rigid abelian braided category. If some assumptions of finiteness and non-degeneracy are satisfied, the Hopf algebra F has an integral and there are morphisms $S, T : F \to F$, called modular transformations. They yield a representation of the modular group. The properties of S are similar to those of the Fourier transform.

14. *Invariants of 3-manifolds and projective representations of mapping class groups via quantum groups at roots of unity*, Commun. Math. Phys. **172** (1995) 467–516.

Realization of the next article for modules over finite dimensional (factorizable and not) ribbon Hopf algebra with an example of the quotient $H = u_q(g)$ of the quantized universal enveloping algebra $U_q(g)$ at a root of unity q. The mapping class group $M_{g,1}$ of a surface of genus g with one hole projectively acts by automorphisms in the H-module $H^{*\otimes g}$, if H^* is endowed with the coadjoint H-module structure. There exists a projective representation of the mapping class group $M_{g,n}$ of a surface of genus g with n holes labelled by finite dimensional H-modules $X_1, \ldots, X_n$ in the vector space $\mathrm{Hom}_H(X_1 \otimes \ldots \otimes X_n, H^{*\otimes g})$. Also an invariant of closed oriented 3-manifolds is constructed, which for Hopf algebras reduces to Hennings invariant and for semisimple categories reduces to Reshetikhin–Turaev invariant.

15. *Ribbon Abelian Categories as Modular Categories*, J. Knot Theory Ramif. **5** (1996) n. 3, 311–403.

A category N of labeled ribbon graphs is extended by new generators called fusing, braiding, twist and switch with relations which can be called Moore–Seiberg relations. A functor to N is constructed from the category $Surf$ of oriented surfaces with labeled boundary and their homeomorphisms. Given an (eventually non-semisimple) k-linear abelian ribbon braided category C with some finiteness conditions we construct a functor from a central extension of N with the set of labels Ob C to k-vector spaces. Composing the functors we get a modular functor from a central extension of $Surf$ to k-vector spaces.

16. *Extensions and Contractions of the Lie Algebra of q-Pseudodifferential Symbols on the Circle*, J. Funct. Anal. **143** (1997) n. 1, 55–97 (with B. Khesin and C. Roger).

We construct cocycles on the Lie algebra of pseudo- and q-pseudodifferential symbols of one variable and on their close relatives: the sine-algebra and the Poisson algebra on two-torus. A "quantum" Godbillon-Vey cocycle on (pseudo)-differential operators appears in this construction as a natural generalization of the Gelfand-Fuchs 3-cocycle on periodic vector fields. A nontrivial embedding of the Virasoro algebra into (a completion of) q-pseudodifferential symbols is described. q-analogs of the KP and KdV-hierarchies admitting an infinite number of conserved charges are proposed.

17. *Squared Hopf algebras and reconstruction theorems*, Proc. of the Workshop "Quantum Groups and Quantum Spaces", Banach Center Publ. **40**, Inst. Math. Polish Acad. Sci., Warszawa 1997, 111–137.

I. *Squared Hopf algebras*, Mem. Amer. Math. Soc. **142** (1999), no. 677, 184 p.

Given an abelian k-linear rigid monoidal category V, where k is a perfect field, we define *squared coalgebras* as objects of cocompleted $V \otimes V$ (Deligne's tensor product of categories) equipped with the appropriate notion of comultiplication, which is a morphism in $V \otimes V \otimes V$. Based on this, (squared) bialgebras and Hopf algebras are defined without use of braiding. If V is the category of k-vector spaces, squared (co)algebras coincide with conventional ones. If V is braided, a braided Hopf algebra can be obtained from a squared one. The squared notions (coalgebras, bialgebras, Hopf algebras) are objects of the cocompleted tensor square of the initial category V, whence the terminology. The structure maps – comultiplication, multiplication etc. – are morphisms in tensor powers of V. The associativity and other properties mean equality of two composite morphisms in tensor powers of V.

For instance, a *squared bialgebra* is defined as an object of $V \otimes V$ having the structure of a squared coalgebra and of an algebra in the monoidal category $V \otimes V$ with compatibility axioms which require that the multiplication and the unit were homomorphisms of

coalgebras. (There are several monoidal structures in $V \otimes V$ and we choose a special one.) If V is braided, a squared Hopf algebra determines a braided Hopf algebra, but not vice versa. Furthermore, squared quasitriangular Hopf algebra is a solution to the problem of defining quantum groups in braided categories.
Any such braided abelian category V has a canonically associated squared bialgebra of that kind — the coend $\int^{X \in V} X \odot X^*$.
Reconstruction theorems give equivalence of squared co- (bi-, Hopf) algebras in V and corresponding fibre functors to V (which is not the case with the usual definitions). Philosophically, categories over the category V are fully encoded in terms of coalgebras living in $V \otimes V$ and vice versa.
The monoidal version of the reconstruction theorem also holds. Namely, the category of monoidal k -linear exact faithful functors $\omega : C \to V$ (C is essentially small) and the category of squared bialgebras in V are equivalent.

18. *Quantum supergroups of $GL(n|m)$ type: differential forms, Koszul complexes and Berezinians*, Duke Math. J. **90** (1997) n. 1, 1–62 (with A. Sudbery).

We introduce and study the Koszul complex for a Hecke R-matrix. Its cohomology, called the Berezinian, is used to define quantum superdeterminant for a Hecke R-matrix, generalizing the classical case [4]. Their behaviour with respect to Hecke sum of R-matrices is studied. Given a Hecke R-matrix in n-dimensional vector space, we construct a Hecke R-matrix in $2n$-dimensional vector space commuting with a differential. The notion of a quantum differential supergroup is derived. Its algebra of functions is a differential coquasitriangular Hopf algebra, having the usual algebra of differential forms as a quotient. Examples of superdeterminants related to these algebras are calculated. Several remarks about Woronowicz's theory are made.

22. *Integrals for braided Hopf algebras*, (with Yu. Bespalov, T. Kerler and V. Turaev), J. Pure Appl. Algebra, **148** (2000), no. 2, 113–164.

Let H be a Hopf algebra in a rigid braided monoidal category with split idempotents. We prove the existence of integrals on (in) H characterized by the universal property, employing results about Hopf modules, and show that their common target (source) object of integrals is invertible. This generalizes the result for rigid braided monoidal abelian categories [16]. The fully braided version of Radford's formula for the fourth power of the antipode is obtained. The results apply to topological Hopf algebras, e.g. a torus with a hole, which do not have additive structure.

II. *Non-Semisimple Topological Quantum Field Theories for 3-Manifolds with Corners*, Lect. Notes in Math., vol. 1765, Springer-Verlag, Heidelberg, 2001, vi+379 p. (with T. Kerler),

In this book we describe extended topological quantum field theories (TQFT's) as double functors between two naturally defined double categories: one of topological nature, made of 3-manifolds with corners, the other of algebraic nature, made of linear categories, functors, vector spaces and maps. The conventional notion of TQFT's of Atiyah's, as well as the notion of a modular functor from axiomatic conformal field theory are unified in this concept. We construct a large class of such extended TQFT's, assigning a double functor to every abelian modular category, which does not have to be semisimple.

27. *External tensor product of perverse sheaves*, Ukr. Math. J. **53** (2001), no. 3, 311–322.
30. *Tensor products of categories of equivariant perverse sheaves*, Cahiers Topologie Géom. Différentielle Catég. **XLIII-1** (2002), 49–79.

We prove that the Deligne tensor product of categories of equivariant constructible perverse sheaves is again such a category. Precisely, the product of categories on a complex algebraic G-variety X and an H-variety Y is the category corresponding to the $G\times H$-variety $X\times Y$ – product of constructible spaces.

Hopf categories

21. *Operations and isomorphisms in a triangulated Hopf category*, Methods of Func. Analysis and Topology, **5** (1999), no. 4, 37–53.

28. *Coherence isomorphisms for a Hopf category*, Noncommutative structures in mathematics and physics (September 24-27, 2000, Kyiv), (J. Wess and S. Duplij, eds.), NATO ARW Proc., Kluwer Acad. Publ., Dordrecht, 283–294.

Operations in some graded Hopf categories, Preprint MPI 2001 - 44, unpublished, `www.mpim-bonn.mpg.de/html/preprints/preprints.html`

31. *The triangulated Hopf category $n_+SL(2)$*, Applied Categorical Structures, **10** (2002), no. 4, 331–381.

32. *A model of the 2-category of equivariant derived categories*, Proc. of the First Ukrainian Math. Congress (Kyiv), August 2001, Inst. of mathematics NASU, 2002, 307–322.

A definition of a *triangulated Hopf category* is proposed. It arises from the Lusztig's theory of quantum groups which establishes a bijection between canonical basis and some set of simple perverse sheaves. With this theory is associated a hypothetical example of a triangulated Hopf category.

A new feature of the proposed definition is the use of the whole equivariant derived category instead of only semi-simple complexes. A triangulated Hopf category consists of a family of equivariant derived categories D_V depending on a vector space V equipped with

1. operad-like operations – triangulated functors J^X between equivariant derived categories, generalizing the operation

$$x_1\otimes\ldots\otimes x_n\mapsto\Delta^m(x_1\ldots x_n)=(\Delta\otimes 1)\ldots\Delta(x_1\ldots x_n)$$

for ordinary Hopf algebras, which depend on a parameter set X taken from the set of unions of strata of some stratified space;

2. (associativity, coassociativity as particular cases of) coherence isomorphisms of functors;

3. functorial distinguished triangles $J^U \to J^X \to J^F \to$ are given for any closed embedding of parameter sets $F \subset X$ with $U = X - F$,

such that

i) a quadratic equation for coherence isomorphisms holds;

ii) diagram made with given distinguished triangles for any pair of closed embeddings $F \subset Z \subset W$ is an octahedron.

The above triangles $J^U \to J^X \to J^F \to$ reflect the specifics of triangulated categories. Working with semisimple abelian categories we would ask for the relation $J^X = J^U \oplus J^F$.

The coherence isomorphism imposes the use of parameter sets X and they are not necessarily maximal possible = unions of all strata.

In the example of Lusztig's category the parameterizing sets are unions of Bruhat cells.

The rôle of braiding is played by the shift functor $L \mapsto L[Q(V)]$, where $Q(V)$ is some integer-valued quadratic form depending on the data.

At the moment I have not proved i), but only it's particular cases:

– associativity-associativity equation;

– coassociativity-coassociativity equation;

– coassociativity-coherence equation.

The full equation i) should follow from the above and one remaining case:

– associativity-coherence equation.

35. *Special PROPs and homotopy bialgebras*, Math. bulletin of the Shevchenko Sci. Soc., **1** (2004), 59–76, in Ukrainian.

For a braided category $\mathcal{C}$ we construct a special PROP $\overline{\mathcal{C}}$ such that functors of special PROPs Bialg $\to \overline{\mathcal{C}}$ are in bijection with braided bialgebras in $\mathcal{C}$.

Research on A_∞-categories

33. *Category of A_∞-categories*, Homology, Homotopy and Applications **5** (2003), no. 1, 1–48.

We define natural A_∞-transformations and construct A_∞-category of A_∞-functors. The notion of non-strict units in an A_∞-category is introduced. The 2-category of (unital) A_∞-categories, (unital) functors and transformations is described.

36. *Free A_∞-categories*, Theory and Applications of Categories **16** (2006), no. 9, 174–205 (with O. Manzyuk).

For a differential graded k-quiver $\mathcal{Q}$ we define the free A_∞-category $\mathcal{F}\mathcal{Q}$ generated by $\mathcal{Q}$. The main result is that the restriction A_∞-functor $A_\infty(\mathcal{F}\mathcal{Q}, \mathcal{A}) \to A_1(\mathcal{Q}, \mathcal{A})$ is an equivalence, where objects of the last A_∞-category are morphisms of differential graded k-quivers $\mathcal{Q} \to \mathcal{A}$.

38. *A construction of quotient A_∞-categories*, Homology, Homotopy and Applications **8** (2006), no. 2, 157–203 (with S. Ovsienko)

We construct an A_∞-category $D(C|B)$ from a given A_∞-category C and its full subcategory B. The construction is similar to a particular case of Drinfeld's quotient of differential graded categories. We use $D(C|B)$ to construct an A_∞-functor of K-injective resolutions of a complex. The conventional derived category is obtained as the 0-th cohomology of the quotient of differential graded category of complexes over acyclic complexes.

39. *Unital A_∞-categories*, Problems of topology and related questions (V. V. Sharko, ed.), Proc. of Inst. of Mathematics NASU, vol. 3, no. 3, Inst. of Mathematics, Nat. Acad. Sci. Ukraine, Kyiv, 2006, 235–268 (with O. Manzyuk).

We prove that three definitions of unitality for A_∞-categories suggested by the first author, by Kontsevich and Soibelman, and by Fukaya are equivalent.

41. *A_∞-bimodules and Serre A_∞-functors*, Geometry and Dynamics of Groups and Spaces (M. M. Kapranov, S. Kolyada, Yu. I. Manin, P. Moree, and L. Potyagailo, eds.), Progress in Mathematics, vol. 265, Birkhäuser Verlag, Basel, 2008, 565–645 (with O. Manzyuk).

We define A_∞-bimodules and show that this notion is equivalent to an A_∞-functor with two arguments which takes values in the differential graded category of complexes of k-modules, where k is a ground commutative ring. Serre A_∞-functors are defined via A_∞-bimodules likewise Kontsevich and Soibelman. We prove that a unital closed under shifts A_∞-category $\mathcal{A}$ over a field k admits a Serre A_∞-functor if and only if its homotopy category $H^0\mathcal{A}$ admits a Serre k-linear functor. The proof uses categories enriched in $\mathcal{K}$, the homotopy category of complexes of k-modules, and Serre $\mathcal{K}$-functors. Also we use a new A_∞-version of the Yoneda Lemma generalizing the previously obtained result.

42. *Quotients of unital A_∞-categories*, Theory Appl. Categ. 20 (2008), no. 13, 405–496 (with O. Manzyuk).

Assuming that $\mathcal{B}$ is a full A_∞-subcategory of a unital A_∞-category $\mathcal{C}$ we construct the quotient unital A_∞-category $\mathcal{D}$ =‘$\mathcal{C}/\mathcal{B}$’. It represents the A^u_∞-2-functor $\mathcal{A} \mapsto A^u_\infty(\mathcal{C},\mathcal{A})_{\text{mod}\mathcal{B}}$, which associates with a given unital A_∞-category $\mathcal{A}$ the A_∞-category of unital A_∞-functors $\mathcal{C} \to \mathcal{A}$, whose restriction to $\mathcal{B}$ is contractible. Namely, there is a unital A_∞-functor $e : \mathcal{C} \to \mathcal{D}$ such that the composition $\mathcal{B} \hookrightarrow \mathcal{C} \to \mathcal{D}$ is contractible, and for an arbitrary unital A_∞-category $\mathcal{A}$ the restriction A_∞-functor $A^u_\infty(e,\mathcal{A}) : A^u_\infty(\mathcal{D},\mathcal{A}) \to A^u_\infty(\mathcal{C},\mathcal{A})_{\text{mod}\mathcal{B}}$ is an equivalence.
Let $\mathbf{C}_k$ be the differential graded category of differential graded k-modules. We prove that the Yoneda A_∞-functor $Y : \mathcal{A}^{\text{op}} \to A_\infty(\mathcal{A},\mathbf{C}_k)$ is a full embedding for an arbitrary unital A_∞-category $\mathcal{A}$. Therefore, any unital A_∞-category is equivalent to a differential graded category with the same set of objects.

III. *Pretriangulated A_∞-categories*, Proceedings of the Institute of Mathematics of NAS of Ukraine, vol. 76, Institute of Mathematics of NAS of Ukraine, Kyiv, 2008, 599 p. (with Yu. Bespalov and O. Manzyuk),

The framework of differential graded categories and functors is too narrow for many problems, and it is preferable to consider wider class of A_∞-functors even dealing with differential graded categories. We have noticed that many features of A_∞-categories and A_∞-functors come from the fact that they form a symmetric closed multicategory. In the first part of this book the theory of multicategories is presented including its new parts: closed multicategories and multicategories enriched in symmetric multicategories. In the second part we apply this theory to (differential) graded k-linear quivers. In this setting we start to construct two ingredients of pretriangulated categories: the monad of shifts and the Maurer–Cartan functor. We finish the construction in the third part, where various properties of A_∞-categories and A_∞-functors are discussed. In particular we obtain the monad of pretriangulated A_∞-categories and prove that zeroth homology of a pretriangulated A_∞-category is a triangulated category. In appendices we do some algebra in 2-categories necessary for taking tensor products of A_∞-categories with differential graded categories.

44. *Homotopy unital A_∞-algebras*, J. Algebra **329** (2011), no. 1, 190–212, Special Issue Celebrating the 60th Birthday of Corrado De Concini.

In this article we find a cofibrant replacement A^{hu}_∞ of the **dg**-operad Ass of associative differential graded algebras with units. As a graded operad it is freely generated by a nullary homotopy unit i and operations $m_{n_1,n_2,\dots,n_k}$, $k \ge 1$, $n_1, \dots, n_k \in \mathbf{Z}_{\ge 0}$, of arity $n = \sum_{q=1}^k n_q$, $n + k \ge 3$, and of degree $4 - n - 2k$. The projection morphism $A^{hu}_\infty \to$ Ass is a homotopy isomorphism. It turns out that A^{hu}_∞-algebras are precisely homotopy unital A_∞-algebras in the sense of Fukaya.
We construct also a cofibrant replacement for the regular Ass-bimodule Ass describing morphisms of associative **dg**-algebras with units. This A^{hu}_∞-bimodule F^{hu}_1 is freely generated by elements $f_{n_1,n_2,\dots,n_k}$, $k \ge 1$, $n_1, \dots, n_k \in \mathbf{Z}_{\ge 0}$, of arity $n = \sum_{q=1}^k n_q$, $n+k \ge 2$, and degree $3-n-2k$. Instead of $f_{0,0}$ a special notation v is used. The projection map $F^{hu}_1 \to$ Ass is a homotopy isomorphism. It turns out that $(A^{hu}_\infty, F^{hu}_1)$-algebras can be identified with

homotopy unital A_∞-morphisms, defined entirely in terms of Fukaya's homotopy unital A_∞-data.

Research on curved algebras and coalgebras

45. *Bar and cobar constructions for curved algebras and coalgebras*, Matematychni Studii **40** (2013), no. 2, 115–131.

We provide bar and cobar constructions as functors between some categories of curved algebras and curved augmented coalgebras over a graded commutative ring. These functors are adjoint to each other.