Symmetries of Kolmogorov equations and multi-layer quasi-geostrophic problem

by

© **Serhii D. Koval**

A thesis submitted to the School of Graduate Studies in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Department of Mathematics and Statistics

Memorial University

October 2026

St. John's, Newfoundland and Labrador, Canada

Abstract

The purpose of this thesis is to explore symmetry properties and construct invariant solutions of three classes of differential equations using advanced Lie group methods.

Within the class of (1+2)-dimensional ultraparabolic linear equations, we distinguish a fine Kolmogorov backward equation with a quadratic diffusivity. Modulo the point equivalence, it is a unique equation within this class whose essential Lie invariance algebra is five-dimensional and nonsolvable. We compute the point symmetry pseudogroup of this equation and analyze its structure. We then exhaustively classify its Lie invariant solutions, discuss the generation of these solutions using point and linear generalized symmetries, and carry out some peculiar generalized reductions.

We solve the group classification problem for two peculiar classes of Kolmogorov backward equations with power diffusivity. Within these classes, the remarkable Fokker–Planck and the fine Kolmogorov backward equations are singled out by their exceptional symmetry properties. We extend the known results on these two equations to their counterparts under a nontrivial discrete equivalence transformation. Additionally, we carry out symmetry analysis of the generic equations from the above classes and study the solution generation via actions by recursion operators.

We carry out an extended symmetry analysis of the multi-layer quasi-geostrophic problem. This model is given by a system of an arbitrary number of coupled barotropic vorticity equations. Using original methods, we compute its maximal Lie invariance algebra and its complete point-symmetry pseudogroup. After classifying one- and two-dimensional subalgebras of the Lie invariance algebra, we exhaustively study codimension-one, -two and -three Lie reductions. We integrate the obtained reduced systems and construct wide families of exact solutions of the model, including rediscovered representations of stationary and travelling baroclinic Rossby waves, coherent baroclinic eddies, hetons and localized dipolar vortices.

Acknowledgments

First and foremost, I would like to express my deepest gratitude to my supervisor, Professor Alex Bihlo, for his guidance and support throughout my graduate studies. I am particularly grateful to him for suggesting research problems that were as captivating as they were challenging, and for providing me with the opportunities to pursue my own research interests and collaborate with other mathematicians.

I cannot thank my co-supervisor, Professor Roman Popovych, enough for his profound impact on my professional growth. I deeply appreciate the countless hours of discussions we shared, his brilliant ideas, our fruitful collaboration and his care about my life. This thesis would not have been written without his advice and mentorship.

I am sincerely grateful to Professor Mikhail Kotchetov for his significant impact on my education and for our collaboration [65]. I also thank him for inviting me to various conferences and workshops, where I had the pleasure of meeting Professor Mitja Mastnak. I am very grateful to Professors Mastnak and Kotchetov for supporting my AARMS postdoctoral fellowship, and I look forward to our future collaboration.

I extend my thanks to the examining committee, Professors Stephen Anco, Graham Cox, and Marco Merkli, for reviewing this thesis, for interesting discussions and for providing valuable feedback.

I am very grateful to Professor Yuri Bahturin for his support and for his constant care about my life and well-being. My gratitude also extends to Afework Solomon for his mentorship in teaching, his moral support, and for being an excellent office neighbor.

Throughout my graduate studies, I have had the privilege of collaborating with many remarkable mathematicians. I am grateful to all my co-authors for enriching my research and influencing me. In particular, I would like to thank Yevhenii Chapovskyi,

Dmytro Popovych, Elsa Dos Santos Cardoso-Bihlo, Maryna Nesterenko, Vyacheslav Boyko and Oleksandra Vinnichenko.

I am also thankful to my colleagues in the department for our daily discussions. I would like to thank Sebastiano Argenti, Daniela Martinez Correa, Christopher Lang and Cameron Ruether for the stimulating seminars we shared.

I acknowledge the financial support provided by Prof. Alex Bihlo's NSERC grant, the School of Graduate Studies and the AARMS graduate scholarship. I am also thankful for the teaching assistant fellowship offered by the Department of Mathematics and Statistics.

Finally, I am deeply grateful to my family for their support and encouragement throughout this journey. I also wish to sincerely thank my friend, Serhii Tytov, for his unwavering support and for our enduring friendship.

Statement of contribution

This thesis is based on three peer-reviewed publications [71, 72, 75].

In Chapter 1, we establish the necessary mathematical framework. The exposition primarily follows the approaches to the symmetry analysis of differential equations presented in the books by Olver [103, 105], Bluman, Cheviakov and Anco [20], Hydon [51] and Vinogradov [149]. To support the theoretical discussion, we include several illustrative examples adapted from my Master's research [70, 74].

The Chapters 2 and 3 are based on [75] and [72], respectively. These chapters extend the research initiated in my Master's thesis [70]. While the research problems were proposed by Prof. Bihlo and Prof. Popovych, I was responsible for all primary computations.

Specific sections, namely Sections 2.6 and 2.7, were written in collaboration with Prof. Popovych, who also provided guidance on the final group classification results in Chapter 3 and suggested investigating the generation of solutions via the action of generalized symmetries. All computational results were independently verified by Prof. Dos Santos Cardoso-Bihlo.

Chapter 4 is based on the paper [71]. The problem was suggested by Prof. Bihlo and the idea of embedding the system into classes of differential equations was contributed by Prof. Popovych. I proposed the matrix interpretation of the model. The algebra was computed in collaboration with Prof. Popovych and Prof. Bihlo, while all subsequent computations remain my own contribution. Additionally, Prof. Bihlo provided the physical interpretation and illustrations of the exact solutions obtained, and Prof. Popovych suggested representation of bounded solutions in terms of Herglotz wave functions.

Contents

Abstract ii

Acknowledgments iii

Statement of contribution v

Contents vi

List of tables x

List of figures xi

Glossary xii

Introduction 1

1 Methods of extended symmetry analysis 4

1.1 Geometry of partial differential equations 4

1.2 Lie symmetries 8

1.2.1 Basic notions 8

1.2.2 Lie symmetries of linear heat equation 11

1.3 Pseudogroups, groupoids and classes of differential equations 13

1.3.1 Pseudogroups 14

1.3.2 Basics on classes of differential equations and their equivalence groupoids . . . 17
1.4 Methods for computing point symmetry pseudogroups . . . 22
1.4.1 Direct method . . . 22
1.4.2 Point symmetry pseudogroup of the heat equation . . . 24
1.4.3 Algebraic methods . . . 25
1.5 Optimized procedure for Lie reductions . . . 28
1.6 Generalized symmetries and generalized reductions . . . 31
1.6.1 Basics on generalized symmetries . . . 31
1.6.2 Generalized symmetries of the heat equation . . . 34
1.6.3 Generalized reductions and solution generation . . . 35

2 Extended symmetry analysis of (1+2)-dimensional fine Kolmogorov backward equation 37

2.1 Introduction . . . 37
2.2 Lie invariance algebra . . . 40
2.3 Point symmetry pseudogroup . . . 41
2.4 Classification of subalgebras . . . 48
2.5 Lie reductions . . . 54
2.6 Generating new solutions from known ones . . . 59
2.7 Generalized reductions . . . 64
2.8 Conclusion . . . 68
Appendix 2.A Two-by-two matrix groups as semidirect products . . . 70

3 Surprising symmetry properties and exact solutions of Kolmogorov backward equations with power diffusivity 73

3.1 Introduction . . . 73
3.2 Equivalence groupoids . . . 79

3.3 Equivalence group and equivalence algebra of the original class 82
3.4 Equivalence group and equivalence algebra of the gauged class 85
3.5 Classification of Lie symmetries 88
3.6 Remarkable Fokker–Planck equation and its counterpart 92
3.7 Fine Kolmogorov backward equation and its counterpart 98
3.8 Generic case 101
3.8.1 Point symmetry pseudogroup 102
3.8.2 Classification of subalgebras 104
3.8.3 Lie reductions of codimension one 106
3.8.4 Lie reductions of codimensions two and three 110
3.8.5 Generating solutions 112
3.9 Conclusion 117
Appendix 3.A Algebraic method for finding equivalence groups 122
3.A.1 The original class 123
3.A.2 The gauged class 127
Appendix 3.B Darboux transformations of linear heat equations with potentials 131

4 Symmetry analysis of multi-layer quasi-geostrophic problem 134
4.1 Introduction 134
4.2 Multi-layer quasi-geostrophic problem 139
4.2.1 Mathematical model 139
4.2.2 Vector representation of the model 142
4.2.3 Properties of the vertical coupling matrix 142
4.2.4 Barotropic and baroclinic modes 146
4.2.5 Numerical example 147
4.3 Conservation laws and Hamiltonian structure 148

4.4 Lie invariance algebra and Hamiltonian symmetries . . . 152
4.5 Point symmetry pseudogroup . . . 161
4.6 Equivalence groupoid, pseudogroup and algebra . . . 166
4.7 Classification of subalgebras . . . 170
4.8 Codimension-one Lie reductions . . . 175
4.8.1 Subalgebra family 1.1 . . . 176
4.8.2 Subalgebra family 1.2 . . . 193
4.8.3 Subalgebra family 1.3 . . . 210
4.9 Codimension-two Lie reductions . . . 218
4.9.1 Subalgebra family 2.1 . . . 219
4.9.2 Subalgebra family 2.2 . . . 221
4.9.3 Subalgebra family 2.4 . . . 223
4.9.4 Subalgebra family 2.6 . . . 225
4.10 Conclusion . . . 226
Appendix 4.A Eigenvectors of the matrix of vertical coupling via orthogonal polynomials . . . 231

Bibliography **236**

List of tables

2.1 G^{ess}-inequivalent codimension-one Lie reductions of the equation (2.2) . 55

List of figures

4.1 Three-layer model 141

4.2 Baroclinic simple plane waves (4.34). 184

4.3 Coherent baroclinic eddy (4.35). 185

4.4 Coherent baroclinic heton (4.36). 186

4.5 Superposition of two baroclinic Rossby waves (4.37). 186

4.6 An example of linear superposition of various solution patterns. 187

4.7 Larichev–Reznik baroclinic modon (4.41) in the case of the same eigenbasis in the inner and outer domains for two values of the parameter tuples (r_0, ϱ). 191

4.8 Larichev–Reznik barotropic modon (4.41) for two values of the parameter tuples $(\tilde{\varrho}, \hat{\varrho}, r_0)$. 192

4.9 Traveling Rossby wave (4.57) in three layer model at the time snapshots $t = 0$ s, $t = 3600$ s and $t = 7200$ s from the last row to the first row. . . 207

Glossary

$\mathcal{L}$	a system or a class of differential equations
θ	a tuple of arbitrary elements
$\mathcal{L}_\theta$	system of differential equations in the class $\mathcal{L}$
$\mathcal{G}^\sim$	equivalence groupoid of a class
$G^\sim$	(usual) equivalence group of a class
$\mathfrak{g}^\sim$	equivalence algebra of a class
G	point-symmetry pseudogroup
$\mathfrak{g}$	Lie-invariance algebra of a system of differential equations
$\mathfrak{s}$	subalgebra of a Lie invariance algebra
$\mathfrak{m}$	megaideal of a Lie algebra
J^r	r-th jet space
Q	vector field
$Q^{(r)}$	rth prolongation of Q
D_x	Operator of total derivative with respect to x
$\bar{\mathcal{F}}$	class of the linear (1+2)-dimensional second-order ultraparabolic equations of the form (3.3)
$\mathcal{F}$	class of ultraparabolic (1+2)-dimensional Kolmogorov backward equations of the form (3.1)
$\mathcal{F}'$	class of ultraparabolic (1+2)-dimensional Kolmogorov backward equations of the form (3.2)
R, Q	Operators in total derivatives
Υ	Associative algebra of operators in total derivatives or associative algebra of recursion operators
$\Upsilon^{(-)}$	Lie algebra associated with the associative algebra Υ
F	m-by-m matrix of vertical coupling of (4.1)
A^+	the Moore–Penrose inverse of a matrix A
$\bar{1}$	the all-ones m-column

Introduction

The invention of the concept of the *(finite) continuous local transformation group* by Sophus Lie, and its further development into the theory of transformation groups in collaboration with Engel in [87], paved the way for one of the most significant breakthroughs in mathematics. It gave rise to new branches of geometry and algebra, including the theories of topological groups, Lie groups and Lie algebras, algebraic groups, and the research areas stemming from them. A comprehensive investigation of the history of Lie's theory can be found in the remarkable book by Borel [23]. Lie also pioneered the study of geometry of differential equations, making fundamental contributions to this field. In particular, Lie solved the group classification problems for three classes of differential equations: second-order ordinary differential equations [87], second-order linear partial differential equations in two independent variables over $\mathbb{C}$ in [86] and nonlinear Klein–Gordon equations of the form $d^2z/dx\,dy = F(z)$ [85]. The modern development of the theory of group classification began with the works of Ovsyannikov [112, 113], and this has been the field of intensive research ever since, see [2, 53, 79, 80, 146] and references therein.

The methods developed within the theory of group analysis of differential equations allow one to construct particular solutions of a system of differential equations, provided it admits a nontrivial symmetry (pseudo)group. This is why, since Lie's time, the study of point symmetries of differential equations has gained great popularity in both applied and theoretical sciences.

The present thesis is devoted to symmetry analysis of certain particular (classes of) systems of partial differential equations. It is structured as follows.

Chapter 1 introduces the reader to the fundamental notions and concepts of the symmetry analysis of differential equations. Our presentation of these basics is intentionally concise and rather survey-like, since there are already many excellent

monographs in the literature on the subject, for example [19–21, 51, 103, 105]. In contrast, more attention is paid to comprehensively describing the variety of methods and approaches used in extended symmetry analysis of differential equations. More specifically, we will focus on methods of constructing point-symmetry pseudogroups of differential equations, and carrying out Lie and generalized reductions.

Continuing research initiated in my Master Theses [69, 70], we carry out extended symmetry analysis of a fine Kolmogorov backward equation with a quadratic diffusivity in Chapter 2. Modulo the point equivalence, it is a unique equation within the class of (1+2)-dimensional ultraparabolic second-order linear partial differential equations whose essential Lie invariance algebra is five-dimensional and nonsolvable. Using the direct method, we compute the point symmetry pseudogroup of this equation and analyze its structure. In particular, we single out its essential subgroup and classify its discrete elements. We exhaustively classify all subalgebras of the corresponding essential Lie invariance algebra up to inner automorphisms and up to the action of the essential point-symmetry group. This allowed us to classify Lie reductions and Lie invariant solutions of the equation under consideration. We also discuss the generation of its solutions using point and linear generalized symmetries and carry out its peculiar generalized reductions. As a result, we construct wide families of its solutions parameterized by an arbitrary finite number of arbitrary solutions of the (1+1)-dimensional linear heat equation or one or two arbitrary solutions of (1+1)-dimensional linear heat equations with inverse square potentials.

In Chapter 3, using the original advanced version of the direct method, we efficiently compute the equivalence groupoids and the equivalence groups of two peculiar classes of Kolmogorov backward equations with power diffusivity and solve the problems of their complete group classifications. The results on the equivalence groups are double-checked with the algebraic method. Within these classes, the remarkable Fokker–Planck and the fine Kolmogorov backward equations are distinguished by their exceptional symmetry properties. We extend the known results on these two equations to their counterparts with respect to a nontrivial discrete equivalence transformation. Additionally, we carry out Lie reductions of the equations under consideration up to the point equivalence, exhaustively study their hidden Lie symmetries and generate wider families of their new exact solutions via acting by their recursion operators on constructed Lie-invariant solutions. This analysis reveals eight powers of the space variable with exponents -1, 0, 1, 2, 3, 4, 5 and 6 as values of the diffusion coefficient

that are prominent due to symmetry properties of the corresponding equations.

In Chapter 4, we carry out an extended symmetry analysis of the multi-layer quasi-geostrophic problem. This model is given by a system of a non-fixed number of coupled barotropic vorticity equations. Conservation laws and a Hamiltonian structure for the general case of the model are correctly described for the first time. Using original methods, we compute the maximal Lie invariance algebra and the complete pseudogroup of point symmetries of the model. After classifying one- and two-dimensional subalgebras of the Lie invariance algebra, we exhaustively study codimension-one, -two and -three Lie reductions. Notably, among invariant submodels of the original nonlinear model, we obtain uncoupled systems of well-known linear equations, including the Helmholtz, modified Helmholtz, Laplace, Klein–Gordon, Whittaker, Bessel and linearized Benjamin–Bona–Mahony equations. Integration of these systems significantly depends on spectral properties of the model's vertical coupling matrix, which we also revisit in detail. As a result, we construct wide families of exact solutions, including rediscovered representations of stationary and travelling baroclinic Rossby waves, coherent baroclinic eddies and localized dipolar vortices. We illustrate relevant physical solutions using real-world geophysical data for a three-layer ocean model.

Chapter 1

Methods of extended symmetry analysis

1.1 Geometry of partial differential equations

The most suitable framework for studying symmetry-like objects for differential equations is algebro-geometric, where differential equations are considered in terms of the varieties they define in the (infinite) jet space. This is why we start with recalling the basics of the geometry of jets. For this purpose, we will follow Chapters 3 and 4 of [22] and [149, Chapter 0] in the exposition of this material.

Denote by $X = \mathbb{R}^n$ and $U = \mathbb{R}^m$ the spaces of independent and dependent variables coordinatized by tuples $(x_1, \dots, x_n)$ and $(u^1, \dots, u^m)$, respectively, and endowed with the standard Euclidean topology. The product space $E := X \times U$ has a natural structure of a trivial fiber bundle (E, X, π, U) with the base space X, fiber U and the canonical projection $\pi\colon E \to X$. For $p \in X$, denote by $\Gamma(p)$ the set of local sections of π whose domain contains p.

Notation. In this chapter, the indices i and α run through the sets $\{1, \dots, n\}$ and $\{1, \dots, m\}$, respectively, unless another range is specified, and the summation over the repeated indices whenever they appear in the different multipliers is understood unless the other is specified.

For any $r \in \mathbb{N}_0$, we let $I = (i_1, \dots, i_r)$ be an unordered r-tuple (or r-multiset) with

$1 \leqslant i_l \leqslant n$ for all $l \in \{1, \ldots, r\}$, $r \in \mathbb{N}$. The order of this tuple is defined as $|I| = r$, and the corresponding partial derivative operator is

$$\partial_I := \frac{\partial^r}{\partial x_{i_1} \cdots \partial x_{i_r}}.$$

The local sections $\sigma, \eta \in \Gamma(p)$ are said *to have the same r-jet at p* if

$$\partial_I \sigma^\alpha|_p = \partial_I \eta^\alpha|_p \quad \text{for all} \quad 0 \leqslant |I| \leqslant r \text{ and } \alpha \in \{1, \ldots, m\},$$

where $(\sigma^1, \ldots, \sigma^m)$ denotes the push-forward $(\mathrm{pr}_U)_*\sigma$ of a local section $\sigma \in \Gamma(p)$ by the natural projection $\mathrm{pr}_U \colon E \to U$. In what follows, we will identify $\sigma \colon X \to E$ with $(\mathrm{pr}_U)_*\sigma \colon X \to U$ when the context is clear. The relation that two local sections have the same r-jet defines an equivalence relation on $\Gamma(p)$. The equivalence class of r-jet at p, as well as a representative of this equivalence class, is denoted by $j^r_p\sigma$. Here, r is called the order, p is the source and $\sigma(p)$ is the target of the jet.

The rth jet space of the bundle (E, X, π, U) is given by the set

$$J^r := \{j^r_p\sigma \mid p \in X, \sigma \in \Gamma(p)\}.$$

Note that to indicate the dependence on the bundle $\pi \colon E \to X$, it is usually common to denote rth jet space as $J^r(\pi)$. But we will omit this dependence for the sake of simplicity of the presentation.

The manifold structure on J^r arises from that structure on $E = X \times U$. More specifically, the coordinate tuple for J^r is $(x, u_{(r)})$, where $x^i(j^r_p\sigma) := x^i(p)$, and the $m_{(r)} := m\binom{n+r}{r}$ derivatives coordinates:

$$u^\alpha_I \colon J^r \to \mathbb{R}, \quad u^\alpha_I(j^r_p\sigma) := \partial_I \sigma^\alpha|_p.$$

with $u^\alpha_0(j^r_p\sigma) := u^\alpha\big(\sigma(p)\big)$. In other words, $u_{(r)}$ is the tuple of all partial derivatives of order ranging from 0 to r.

The projection $\pi \colon E \to X$ naturally extends to surjective submersions $\pi_r \colon J^r \to X$ and $\pi_{r,k} \colon J^r \to J^k$, where $0 \leqslant k \leqslant r$ and $J^0 := E$:

$$\pi_r(j^r_p\sigma) := p, \quad \pi_{r,k}(j^r_p\sigma) := j^k_p\sigma$$

under the identification $j^0_p\sigma = \sigma(p)$. It is clear that for any $0 \leqslant s \leqslant k$ we have $\pi_{r,s} = \pi_{k,s} \circ \pi_{r,k}$, hence we obtain an inverse (or projective) system $\big((J^r)_{r\in\mathbb{N}_0}, (\pi_{r,k})_{k\leqslant r}\big)$. The inverse limit of this system $J^\infty := \varprojlim J^r$ is called the jet space of the fiber bundle (E, X, π, U). The inverse system $\big((J^r)_{r\in\mathbb{N}_0}, (\pi_{r,k})_{k\leqslant r}\big)$ naturally gives rise to the direct system

$$\big((C^\infty(J^r))_{r\in\mathbb{N}_0}, (\pi^*_{r,k})_{k\leqslant r}\big)$$

of algebras of smooth local functions on jet spaces, where $\pi^*_{r,k}\colon C^\infty(J^k) \to C^\infty(J^r)$ denotes the pullback of $\pi_{r,k}$. The direct limit

$$C^\infty(J^\infty) := \varinjlim C^\infty(J^r)$$

is defined to be the coordinate algebra of the jet space J^∞ and it is usually called the *algebra of differential functions*. It is more convenient to denote the algebra $C^\infty(J^\infty)$ by $\mathcal{A}$. The elements of this algebra are smooth local functions that depend on x, u and derivatives of u up to some finite order. They are denoted by $f[u]$ and are called *differential functions*. The maximal order of the derivative of u included in the arguments of $f[u]$ is called *the order of* $f[u]$ and denoted by $\operatorname{ord} f[u]$. If $f[u]$ does not depend on the derivatives of u, it is convenient to define its order as $\operatorname{ord} f[u] := -\infty$.

It is also important to emphasize that the jet space J^r can be endowed with the structure of a fiber bundle over X, where the fibers are spaces $U^r = \mathbb{R}^{m_{(r)}}$ with coordinates given by tuples $u_{(r)}$ and the bundle projection is exactly the submersion $\pi_r\colon J^r \to X$. Using this bundle structure, we define the rth prolongation of a local function $f\colon X \to U$ to be the projection of the jet of the local section $p \mapsto (p, f(p))$, $p \in \operatorname{dom} f$, onto the fiber U^r, that is, $\operatorname{pr}^{(r)} f\colon X \to U^r$ such that

$$\operatorname{pr}^{(r)} f(x) := (\partial_I f(x) \mid 0 \leqslant |I| \leqslant r).$$

The jet space J^r carries a natural contact structure defined by *contact forms*. A differential one-form on J^r is called a contact form if its pullback along every prolongation is zero, that is, $\theta \in \wedge^1 J^r$ is a contact form provided that $(j^r\sigma)^*\theta = 0$ for any local section σ of π. In the local coordinates $(x, u_{(r)})$ of J^r, the space of contact forms is spanned by $\omega^\alpha_I := \mathrm{d}u^\alpha_I - u^\alpha_{I,i}\mathrm{d}x_i$, where I runs through the set of unordered tuples of $\{1, \dots, n\}$ with $0 \leqslant |I| \leqslant r$.

Another concept useful in the theory of symmetries of differential equations, which comes from the jet geometry, is the notion of prolongation of a (local) vector field $Q \in \Gamma(TE)$. Such a vector field Q defines the flow $\phi_t \colon E \to E$, which can be viewed as a local one-parameter transformation group. The flow ϕ_t can be naturally extended to the space J^r:

$$(J^r\phi)_t(j^r_p\sigma) := j^r_{\phi_t(p)}(\phi_t \cdot \sigma),$$

where the action $\phi_t \cdot \sigma$ is defined pointwise, $(\phi_t \cdot \sigma)(p) := \big(\phi_t(p), \phi_t(\sigma(p))\big)$. The rth prolongation of Q, denoted by $Q^{(r)}$ or $\mathrm{pr}^{(r)}Q$, is defined to be the generator of the flow $(J^r\phi)_t$, that is,

$$Q^{(r)} := \frac{\mathrm{d}}{\mathrm{d}t}(J^r\phi)_t\Big|_{t=0}.$$

Representing the vector field Q via the local coordinates of TE, $Q = \xi^i\partial_i + \eta^\alpha\partial_{u^\alpha}$, where $\xi^i, \eta^\alpha \in C^\infty(E)$, the explicit formula for rth prolongation can be written as follows:

$$Q^{(r)} := Q + \sum_{I\colon 1\leqslant|I|\leqslant r} \eta^{\alpha,I}\partial_{u^\alpha_I}, \quad \text{where} \quad \eta^{\alpha,I} := \mathrm{D}_I(\eta^\alpha - \xi^i u^\alpha_{x^i}) + \xi^i u^\alpha_{i,I}$$

and $\mathrm{D}_I := \mathrm{D}_{i_1}\cdots\mathrm{D}_{i_k}$ for a k-tuple $I = (i_1, \dots, i_k)$. Here, D_i denotes the operator of total derivative with respect to x_i,

$$\mathrm{D}_i := \partial_{x_i} + \sum_I u^\alpha_{i,I}\frac{\partial}{\partial u_I}.$$

Remark 1.1. Typically, differential equations are defined over some open domain $M \subset E = X \times U$ rather than over the entire space E. In this case, the rth jet space of M can be defined as $J^r(M) := \pi^{-1}_{r,0}M$, where $\pi_{r,0}\colon J^r \to E$.

We are now prepared to define the notion of a system of differential equations in this geometric framework. Consider $l \in \mathbb{N}$ differential functions of order up to r, $L^\mu(x, u_{(r)})$, $\mu = 1, \dots, l$. The system of differential equations $\mathcal{L}$ consists of l equations of the form

$$\mathcal{L}\colon \quad L^\mu(x, u_{(r)}) = 0, \quad \mu = 1, \dots, l.$$

In other words, a system of differential equations $\mathcal{L}$ is the vanishing condition for the functions $L^\mu(x, u_{(r)})$. To translate this into geometric language, it would seem intuitive to associate with the system $\mathcal{L}$ a variety of points in the jet bundle J^∞ satisfying this system. But differential algebra requires us to also account for all differential consequences of the system. Therefore, we associate a system of differential equations $\mathcal{L}$ with a level set

$$\mathcal{L}\colon \quad \{(x, u_{(r)}) \mid L^\mu(x, u_{(r)}) = 0,\ \mathrm{D}_I L^\mu(x, u_{(r)}) = 0 \text{ for all } \mu \text{ and all tuples } I\} \subset J^\infty,$$

which is usually called a *diffiety* and, abusing notation, is also denoted by $\mathcal{L}$.[1] A local function $u = f(x)$ is called a *(local) solution* of $\mathcal{L}$ if the graph of its infinite prolongation $\mathrm{pr}^{(\infty)} f(x)$ lies within the diffiety $\mathcal{L}$.

1.2 Lie symmetries

1.2.1 Basic notions

We emphasize that point transformations in this thesis are considered to be local.

Let $\mathcal{L}$ be a system of differential equations of order r defined on a domain $M \subset X \times U$.

Definition 1.2. A local diffeomorphism Φ of M is called a *point symmetry* of the system $\mathcal{L}$ if it preserves the solution set of $\mathcal{L}$, that is, if a local function $f\colon X \to U$ is a solution of $\mathcal{L}$ then the pullback $\Phi^* f$ is also a solution of $\mathcal{L}$. Geometrically, it means that the transformation Φ preserves the diffiety $\mathcal{L}$.

Most of the research in the field of symmetry analysis of differential equations is concentrated on computing and studying Lie symmetries. These symmetries arise as flows of some (local) vector fields and can be identified systematically. For this, we should require that the differential equations satisfy certain nondegeneracy conditions, for example, to be of maximal rank. This means that the Jacobian of $L^1, \dots, L^l$

[1] Considering a system of differential equations along with all its prolongations in the infinite jet space J^∞ is necessary to properly account for the underlying differential structure. Additionally, while J^∞ carries a natural topology of a Fréchet space by being a projective limit of finite-dimensional manifolds J^k, we treat J^∞ purely algebraically.

with respect to x and $u_{(r)}$ is of rank l on the solution set of $\mathcal{L}$, see more details in [103, Section 2.6]. Geometrically, this condition can be interpreted as the regularity requirement for the diffiety $\mathcal{L}$.

Theorem 1.3 (Theorem 2.31 in [103])**.** *Let $\mathcal{L}$ be a system of differential equations of maximal rank defined on M, Q be a local vector field on M and ϕ_t be the flow of Q. Then ϕ_t is a symmetry of the system $\mathcal{L}$ if and only if*

$$Q^{(r)}[L^\mu(x,u_{(r)})]=0,\quad \mu=1,\dots,l,\quad \textit{whenever}\quad (x,u_{(r)})\in\mathcal{L}. \tag{1.1}$$

The equation (1.1) is called the *infinitesimal invariance criterion*. When it holds, the vector field Q and its flow ϕ_t are called the *Lie-symmetry vector field* and the *one-parameter local group of Lie symmetries*, respectively. In fact, both Q and ϕ_t are often referred to as Lie symmetries in the literature.

Remark 1.4. In classical textbooks such as [20, 103], one-parameter groups of symmetry transformations are treated infinitesimally via their generators, as permitted by the infinitesimal criterion in Theorem 1.3. Determining these generators is a straightforward algorithmic procedure that in many important cases can be carried out by hand. One of the primary focuses of this thesis is finding complete sets of point symmetries of systems of partial differential equations in the sense of Definition 1.2 and analyzing their underlying structures. Finding the entire set of such symmetries is a very complicated and cumbersome problem, which usually requires solving overdetermined systems of nonlinear partial differential equations. We will discuss the methods for addressing this problem in Section 1.4.

It is obvious that a linear combination of Lie-symmetry vector fields, if defined, also satisfies the infinitesimal invariance criterion (1.1) and thus is a Lie-symmetry vector field. It follows from Hadamard's Lemma, see Lemma 1.12 below, that the Lie bracket of two such symmetries, when defined, is again a Lie symmetry. Therefore, the set of Lie-symmetry vector fields is closed under the Lie bracket.

These closure properties say that the set of Lie-symmetry vector fields of a system of differential equations $\mathcal{L}$ constitutes a *Lie pseudoalgebra*. (Note that the Jacobi identity is inherited from the standard properties of the Lie bracket of vector fields.) The latter object can be treated as a "sheaf version" of a Lie algebra of global vector fields on M. The Lie pseudoalgebra of Lie symmetry vector fields of a system of

differential equations is usually called *Lie symmetry pseudoalgebra* or *Lie invariance pseudoalgebra*. To highlight that the set of Lie symmetries is exhaustive, we use the terms *maximal Lie invariance pseudoalgebra* or *maximal Lie-symmetry pseudoalgebra*. This emphasis is necessary because many authors identify specific subalgebras as "Lie invariance (pseudo)algebras", whereas our terminology explicitly denotes the complete set of symmetries.

Let us rigorously define the notion of Lie pseudoalgebra of vector fields. Let $\mathfrak{g}$ be a set of local vector fields on a domain (or, more generally, a manifold) M. In the spirit of the theory of partial functions, see [82], we can endow $\mathfrak{g}$ with the addition operation and the Lie bracket via restricting the corresponding global versions of these operations to the intersection of the domains of the involved terms. That is, for local vector fields $Q_1, Q_2 \in \mathfrak{g}$ with domains $U_1, U_2 \subset M$, define $Q_1 + Q_2 := Q_1|_{U_1\cap U_2} + Q_2|_{U_1\cap U_2}$ and $[Q_1, Q_2] := [Q_1|_{U_1\cap U_2}, Q_2|_{U_1\cap U_2}]$. The operations of addition and Lie bracket on the right-hand sides of these definitions are globally defined for the vector fields with the domain $U_1 \cap U_2$.

Definition 1.5. A set $\mathfrak{g}$ of local vector fields on a domain M is called a *Lie pseudoalgebra* if it is closed with respect to the partial operations of linear combination and taking the Lie bracket, and additionally satisfies the following conditions:

1. the global vector field $0 \in TM$ belongs to $\mathfrak{g}$;
2. if $Q \in \mathfrak{g}$, then the restriction $Q|_U$ of Q to any open subset U of its domain also belongs to $\mathfrak{g}$;
3. given any collection $\{U_\alpha\}$ of open sets of M, a local vector field Q with the domain $\bigcup_\alpha U_\alpha$ belongs to $\mathfrak{g}$ if and only if for every α, its restriction to U_α belongs to $\mathfrak{g}$.

While it is a common practice to treat the sets of Lie-symmetry vector fields as just Lie algebras, merely by forgetting the locality of the vector fields, we emphasize that Lie pseudoalgebras constitute the most appropriate algebraic structure. This treatment appeared in the literature in very recent works [26, 148].

However, to streamline the presentation of the results, in this thesis, we primarily treat the set of Lie-symmetry vector fields as Lie algebras. While this approach sacrifices the degree of formality, it is well justified. Firstly, in practice, Lie-symmetry pseudoalgebras usually contain essential Lie subalgebras that capture information

about all "nontrivial" and interesting symmetries of systems of differential equations. Secondly, one of the aims of symmetry analysis is to find the complete set of point symmetry transformations of differential equations. It turns out that it can be achieved effectively since only algebraic substructures of pseudoalgebras are essential in the course of carrying out computations.

Remark 1.6. While in this thesis we assume that M is a domain in the Euclidean space $X \times U$, the underlying framework remains conceptually unchanged when M is a manifold. However, explicit computations may become more involved due to nontrivial parameterizations or local coordinate representations of M. Additionally, the geometry of M or imposed initial and boundary conditions can restrict which local symmetries extend to global symmetries of the problem. For a concrete illustration, see the symmetry analysis of the barotropic vorticity equation on a rotating sphere [37]. At the same time, while M can itself lack symmetries like isometries or shifts, a differential equation can still possess non-trivial symmetries. For example, any linear homogeneous partial differential equation (such as the heat equation on a manifold) admits linear superposition of solutions, providing an infinite-dimensional symmetry algebra that is independent of the symmetries of the domain itself.

1.2.2 Lie symmetries of linear heat equation

To show examples of all symmetry-related structures introduced in this chapter, we carry out symmetry analysis of the linear heat equation following [74].

The (1+1)-dimensional (linear) heat equation

$$u_t = u_{xx} \tag{1.2}$$

is one of the simplest but most fundamental equations of mathematical physics. Studying symmetries and related objects of the equation (1.2) was initiated by Sophus Lie himself [86] in the course of group classification of second-order linear partial differential equations in two independent variables. In particular, he computed its Lie invariance (pseudo)algebra and showed that it gives a unique (modulo the point equivalence) maximal Lie-symmetry extension in the class of linear (1+1)-dimensional second-order evolution equations.

The maximal Lie invariance (pseudo)algebra $\mathfrak{g}$ of the equation (1.2) is spanned by

the vector fields

$$\mathcal{P}^t=\partial_t,\quad \mathcal{D}=2t\partial_t+x\partial_x-\tfrac{1}{2}u\partial_u,\quad \mathcal{K}=t^2\partial_t+tx\partial_x-\tfrac{1}{4}(x^2+2t)u\partial_u,$$
$$\mathcal{G}^x=t\partial_x-\tfrac{1}{2}xu\partial_u,\quad \mathcal{P}^x=\partial_x,\quad \mathcal{I}=u\partial_u,\quad \mathcal{Z}(f)=f(t,x)\partial_u,$$

where the parameter function f depends on (t,x) and runs through the solution set of the equation (1.2). Since functions $f(t,x)$ are, in general, defined locally, the set $\mathfrak{g}$ indeed constitutes a Lie pseudoalgebra. At the same time, we can redefine the commutator and linear combination of vector fields from $\mathfrak{g}$ by formally globalizing the relevant operations and thus turn $\mathfrak{g}$ into an infinite-dimensional Lie algebra. This approach makes sense since in the particular case of the heat equation, the Lie-symmetry vector fields $\mathcal{Z}(f)$ are in some sense trivial and not interesting for the study. Moreover, their finding is a no-go problem, so from now on $\mathfrak{g}$ is a Lie algebra.

The vector fields $\mathcal{Z}(f)$ constitute the infinite-dimensional abelian ideal $\mathfrak{g}^{\rm lin}$ of $\mathfrak{g}$ associated with the linear superposition of solutions of (1.2), $\mathfrak{g}^{\rm lin}:=\{\mathcal{Z}(f)\}$. Thus, the algebra $\mathfrak{g}$ can be represented as a semidirect sum, $\mathfrak{g}=\mathfrak{g}^{\rm ess}\mathbin{\in}\mathfrak{g}^{\rm lin}$, where

$$\mathfrak{g}^{\rm ess}=\langle\mathcal{P}^t,\mathcal{D},\mathcal{K},\mathcal{G}^x,\mathcal{P}^x,\mathcal{I}\rangle \tag{1.3}$$

is a subalgebra of $\mathfrak{g}$, called the *essential Lie invariance algebra* of (1.2).

Up to the antisymmetry of the Lie bracket of vector fields, the nonzero commutation relations between the basis elements of $\mathfrak{g}^{\rm ess}$ are exhausted by

$$[\mathcal{D},\mathcal{P}^t]=-2\mathcal{P}^t,\quad [\mathcal{D},\mathcal{K}]=2\mathcal{K},\quad [\mathcal{P}^t,\mathcal{K}]=\mathcal{D},$$
$$[\mathcal{P}^t,\mathcal{G}^x]=\mathcal{P}^x,\quad [\mathcal{D},\mathcal{G}^x]=\mathcal{G}^x,\quad [\mathcal{D},\mathcal{P}^x]=-\mathcal{P}^x,\quad [\mathcal{K},\mathcal{P}^x]=-\mathcal{G}^x,$$
$$[\mathcal{G}^x,\mathcal{P}^x]=\tfrac{1}{2}\mathcal{I}.$$

The algebra $\mathfrak{g}^{\rm ess}$ is nonsolvable. Its Levi decomposition is given by $\mathfrak{g}^{\rm ess}=\mathfrak{f}\mathbin{\in}\mathfrak{r}$, where the radical $\mathfrak{r}$ of $\mathfrak{g}^{\rm ess}$ coincides with the nilradical of $\mathfrak{g}^{\rm ess}$ and is spanned by the vector fields $\mathcal{G}^x$, $\mathcal{P}^x$ and $\mathcal{I}$. The Levi factor $\mathfrak{f}=\langle\mathcal{P}^t,\mathcal{D},\mathcal{K}\rangle$ of $\mathfrak{g}^{\rm ess}$ is isomorphic to $\mathrm{sl}(2,\mathbb{R})$, the radical $\mathfrak{r}$ of $\mathfrak{g}^{\rm ess}$ is isomorphic to the (real) rank-one Heisenberg algebra $\mathrm{h}(1,\mathbb{R})$, and the real representation of the Levi factor $\mathfrak{f}$ on the radical $\mathfrak{r}$ coincides, in the basis $(\mathcal{G}^x,\mathcal{P}^x,\mathcal{I})$, with the representation $\rho_1\oplus\rho_0$ of $\mathrm{sl}(2,\mathbb{R})$. Here ρ_n is the

standard irreducible representation of $\mathrm{sl}(2,\mathbb{R})$ on $\mathbb{R}^{n+1}$. More specifically,

$$\rho_n(\mathcal{P}^t)_{ij}=(n-j)\delta_{i,j+1},\quad \rho_n(\mathcal{D})_{ij}=(n-2j)\delta_{ij},\quad \rho_n(-\mathcal{K})_{ij}=j\delta_{i+1,j},$$

where $i,j\in\{0,\dots,n\}$, $n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$, and δ_{kl} is the Kronecker delta, i.e., $\delta_{kl}=1$ if $k=l$ and $\delta_{kl}=0$ otherwise, $k,l\in\mathbb{N}_0$. Thus, the entire algebra $\mathfrak{g}^{\rm ess}$ is isomorphic to the so-called special Galilei algebra $\mathrm{sl}(2,\mathbb{R})\in_{\rho_1\oplus\rho_0}\mathrm{h}(1,\mathbb{R})$, which is denoted by $L_{6,2}$ in the classification of indecomposable Lie algebras of dimensions up to eight with nontrivial Levi decompositions from [144].

The radical $\mathfrak{r}$ and its derived algebra $\mathfrak{r}':=[\mathfrak{r},\mathfrak{r}]=\langle\mathcal{I}\rangle$, which coincides with the center $\mathfrak{z}(\mathfrak{g}^{\rm ess})$ of $\mathfrak{g}^{\rm ess}$, are the only proper megaideals and, moreover, the only proper ideals of $\mathfrak{g}^{\rm ess}$. Another basis of $\mathfrak{g}^{\rm ess}$, which stems from the Iwasawa decomposition of $\mathrm{SL}(2,\mathbb{R})$ and is thus more convenient in many aspects, is $(\mathcal{Q}^+,\mathcal{D},\mathcal{P}^t,\mathcal{G}^x,\mathcal{P}^x,\mathcal{I})$, where $\mathcal{Q}^\pm:=\mathcal{P}^t\pm\mathcal{K}$.

1.3 Pseudogroups, groupoids and classes of differential equations

The study of symmetries of differential equations is local by its nature. A very nice structure that captures this feature and arises in many other applications is given by local Lie transformation groups [103, Definition 1.20] (also called Lie groupuscule). Each such object comes along with a smooth local action on a manifold. However, this notion does not suffice to cover all the cases, including those when the corresponding set of transformations cannot be parameterized by a finite number of parameters, which is common, e.g., for the point symmetry groups of all linear and many nonlinear systems of partial differential equations. Moreover, a local Lie transformation group is not a usual group (e.g., as the collection of Möbius transformations [83]) but rather a groupoid (in the algebraic sense), and the concept of discrete point symmetries fails beyond the framework of such groups. This is why it is more convenient for our purpose to use the more general notion of pseudogroup.

While connected finite-dimensional local Lie groups suffice for studying point symmetries in many classical examples, they cannot account for discrete symmetries, symmetries arising from linear superposition of solutions, and generalized Galilean shifts.

For these purposes, the framework of transformation pseudogroups provides a more natural and unified setting. Pseudogroups are, by definition, the broadest possible objects in this setting, containing local Lie symmetry groups as subsets. Furthermore, the transformation composition within a pseudogroup is a well-defined ("global", in the sense of Definition 1.8) algebraic operation, whereas the transformation composition within a local Lie group may require restriction of transformation domains. Throughout this thesis, all point-symmetry objects are treated from the pseudogroup perspective.

In this section, we discuss all the necessary concepts from pseudogroup theory based on [75, Appendix A]. To make the presentation of the results of the thesis self-contained, we also recall the basic theory of classes of differential equations following [146, Section 2].

1.3.1 Pseudogroups

We first present an (enhanced) definition of a pseudogroup of transformations of a topological space, which is often used in the literature [129, Section 1.1] and can be further specified for topological spaces with additional structures (i.e., *localé*) such as manifolds, and then modify it. A *pseudogroup of transformations* on a topological space X is a set G of homeomorphisms between nonempty open subsets of X with the operation of map composition that satisfy the following properties:

1. The identity map of X belongs to G.
2. If $\Phi \in G$, then its restriction to any open nonempty subset of its domain belongs to G.
3. If $\Phi \in G$, then $\Phi^{-1} \in G$.
4. If $\Phi_i \in G$, $\Phi_i \colon U_i \to V_i$, $i = 1, 2$, with $V_1 \cap U_2 \neq \varnothing$, then $\Phi_2 \circ \Phi_1 \colon \Phi_1^{-1}(V_1 \cap U_2) \to \Phi_2(V_1 \cap U_2)$ belongs to G.
5. Given any collection $\{U_\alpha\}$ of open sets of X, a homeomorphism from $\bigcup_\alpha U_\alpha$ onto an open set of X belongs to G if and only if for every α, its restriction to U_α belongs to G.

In view of property 4, the operation in G is partially defined, i.e., not all pairs

of elements of G can be composed. This is why the (algebraic) concept of groupoid comes in handy.

Definition 1.7. An *(algebraic) groupoid* is a nonempty set G together with a unary operation (inverse) $^{-1}\colon G \to G$ and a partial binary operation[2] (multiplication) $*\colon G \times G \rightharpoonup G$ satisfying the following properties:

Associativity: If $a * b$ and $b * c$ are defined, then $(a * b) * c$ and $a * (b * c)$ are defined and equal to each other. Moreover, $a * b$ and $(a * b) * c$ are defined if and only if $b * c$ and $a * (b * c)$ are defined as well.

Inverse: For every $a \in G$, $a * a^{-1}$ and $a^{-1} * a$ are defined.

Identity: If $a * b$ is defined, then $a * b * b^{-1} = a$ and $a^{-1} * a * b = b$.

As defined above, any pseudogroup of transformations on a topological space is a groupoid, where the map composition is considered as the multiplication. The partially defined operation makes the study of the corresponding structure involved and challenging. However, by modifying the above definition of a pseudogroup, one can endow a pseudogroup with the structure of an *inverse monoid* [82], which makes analyzing it more pleasant. The modification consists in treating homeomorphisms between open subsets of a topological space X as partially defined functions from X to X, allowing the empty function to be an element of a pseudogroup and setting the composition of partial functions as the pseudogroup operation.

Definition 1.8. Given a topological space X, a *pseudogroup G of transformations* on X is a collection of partial homeomorphisms on X, whose domains[3] and images are open subsets of X and that satisfy the following properties:

1. The identity function of X is in G.
2. The restriction of any element $\Phi \in G$ to any open set contained in its domain is also an element of G.
3. The composition $\Phi_1 \circ \Phi_2$ of any pair of elements of G is also in G.

[2] "$*$" is not a usual (total) binary operation since it is not necessarily defined for all pairs of elements of G.

[3] The (natural) domain of a partial function $\Phi\colon X \rightharpoonup Y$ is the domain of Φ viewed as a function. A *partial homeomorphism* between topological spaces is a homeomorphism between their subspaces viewed as a partial function.

4. The inverse of an element Φ is in G.
5. Given any collection $\{U_\alpha\}$ of open sets of X, a partial homeomorphism of X with the domain $\bigcup_\alpha U_\alpha$ onto an open set of X belongs to G if and only if for every α, its restriction to U_α belongs to G.

In what follows, we treat pseudogroups in the sense of Definition 1.8.

It is clear that Definition 1.8 can be generalized to any localé.

Definition 1.9. ([82, p. 6]) An *inverse semigroup* is a nonempty set S with an associative binary operation $*\colon S \times S \to S$ such that for every $a \in S$, there exists a unique $a^{-1} \in S$, called the *inverse* of a, satisfying $a^{-1}*a*a^{-1} = a^{-1}$ and $a*a^{-1}*a = a$. An *inverse monoid* M is an inverse semigroup with *neutral element* $e \in M$, $e * a = a * e = a$ for all $a \in M$.

See more [82, Chapter 1] for an excellent presentation of the basic theory of inverse semigroups and inverse monoids.

Properties 1, 3 and 4 of Definition 1.8 ensure that any pseudogroup is an inverse monoid. Throughout the paper, when referring to pseudogroups, we mean them in the sense of Definition 1.8.

An element a of a set with a binary operation $*$ is called an idempotent if $a*a = a$. One can completely characterize the set of idempotents in any pseudogroup.

Lemma 1.10. *An element of a pseudogroup is an idempotent if and only if it is a restriction of the identity map.*

Proof. Given a pseudogroup G of transformations on X, every id_U, where U is an open subset of X, is an idempotent. Conversely, consider an idempotent $\Phi \in G$, $\Phi^2 = \Phi$. Therefore, $\Phi^2 = \Phi \circ \mathrm{id}_{\mathrm{dom}\,\Phi}$, and thus $\Phi = \mathrm{id}_{\mathrm{dom}\,\Phi}$. □

The required notions are introduced by analogy with their counterparts for groups with modification caused by the fact that any pseudogroup is rather an inverse monoid.

A subset of a pseudogroup G that itself is a pseudogroup of transformations between open sets of a topological space X is called a *pseudosubgroup* of G. Given any pseudogroup G, the set of its idempotents, $G^{\mathrm{id}} := \{\mathrm{id}_U \mid U \text{ is an open subset of } X\}$, and the entire G itself are pseudosubgroups of G called the *improper pseudosubgroups*

of G. They are the minimal and the maximal pseudosubgroups of G with respect to the set inclusion, and the other pseudosubgroups of G are called *proper*. If $G = G^{\rm id}$, then there are no proper pseudosubgroups in G.

Given a subgroup H of G, we call a pseudosubgroup N of G *invariant with respect to* H if $\Phi N\Phi^{-1} \subseteq N$ for any $\Phi \in H$, i.e., the pseudosubgroup N is stable under the conjugation by any element Φ of H. It is obvious that then $\Phi N\Phi^{-1} = N$ or, equivalently, $\Phi N = N\Phi$ for any $\Phi \in H$.

If H is a subgroup of G, N is a pseudosubgroup of G that is invariant with respect to H $G = NH$ and $N \cap H = \{{\rm id}_X\}$, then we say that G is the *semidirect product* of H acting on N, writing $G = H \ltimes N$ or $G = N \rtimes H$, or that G *splits over* N, or even that G is the semidirect product of N and H.

In a similar way, one can try to introduce the more general notions of normal pseudosubgroup and splitting a pseudogroup over such a pseudosubgroup. We can suggest the following definitions, although their naturality requires a thorough study. A *normal pseudosubgroup* N of G is such a pseudosubgroup of G that $\Phi N = N\Phi$ for any $\Phi \in G$, i.e, for any $\Phi \in G$ and for any $\Psi \in N$ we have $\Phi \circ \Psi = \hat\Psi \circ \Phi$ and $\Psi \circ \Phi = \Phi \circ \check\Psi$ for some $\hat\Psi, \check\Psi \in N$. If H and N are respectively a pseudosubgroup and a normal pseudosubgroup of G, $G = NH$ and $N \cap H = G^{\rm id}$, then $G = HN$ as well and we say that G is the *semidirect product* of H acting on N, writing $G = N \rtimes H$ or $G = H \ltimes N$, or that G *splits over* N, or even that G is the semidirect product of N and H. We can weaken these notions even more. A pseudosubgroup N of G is called a *right normal pseudosubgroup* of G if $\Phi N \subseteq N\Phi$ for any $\Phi \in G$, i.e, for any $\Phi \in G$ and for any $\Psi \in N$ we have $\Phi \circ \Psi = \hat\Psi \circ \Phi$ for some $\hat\Psi \in N$. Given a pseudosubgroup H and a right normal pseudosubgroup of G with $G = NH$ and $N \cap H = G^{\rm id}$, we say that G is the *right semidirect product* of H acting on N, writing $G = N \rtimes H$, or that G *right splits over* N, or even that G is the right semidirect product of N and H. Left normal pseudosubgroups and left semidirect product are defined analogously.

1.3.2 Basics on classes of differential equations and their equivalence groupoids

The presentation of the theoretical background on classes of differential equations in this section will serve two purposes. First, it will allow us to rigorously describe

the direct method for computing point-symmetry pseudogroups. Second, Chapter 3 of this thesis is devoted to group classification of a certain class of Fokker–Planck equations. The theory of classes of differential equations is an indispensable tool in the course of addressing this problem.

We assume notation from Section 1.1. Consider a system of differential equations

$$\mathcal{L}_\theta\colon L^\mu(x,u_{(r)},\theta_{(q)}(x,u_{(r)}))=0,\quad \mu=1,\dots,l$$

parameterized by a tuple of arbitrary elements

$$\theta(x,u_{(r)}):=(\theta^1(x,u_{(r)}),\dots,\theta^k(x,u_{(r)})).$$

Here $\theta_{(q)}$ denotes the tuple of derivatives of the tuple θ with respect to x and $u_{(r)}$ up to order q. The tuple θ belongs to the solution set $\mathcal{S}$ of an auxiliary system of differential equations and differential inequalities in θ, $S(x,u_{(r)},\theta_{(q')}(x,u_{(r)}))=0$ and $\Sigma(x,u_{(r)},\theta_{(q')}(x,u_{(r)}))\neq 0$, where x and $u_{(r)}$ play the role of independent variables, and S and Σ are tuples of smooth functions on x, $u_{(r)}$ and $\theta_{(q')}$. By $\mathcal{L}|_{\mathcal{S}}$ we denote the class of systems $\mathcal{L}_\theta$ with the arbitrary-element tuple θ running through the set $\mathcal{S}$, $\mathcal{L}|_{\mathcal{S}}:=\{\mathcal{L}_\theta\mid\theta\in\mathcal{S}\}$. Up to the gauge equivalence of systems from $\mathcal{L}|_{\mathcal{S}}$, which is usually trivial [126], the correspondence $\theta\mapsto\mathcal{L}_\theta$ between $\mathcal{S}$ and $\mathcal{L}|_{\mathcal{S}}$ is bijective.

Example 1.11. Consider the most general form of (1+2)-dimensional ultraparabolic[4] Fokker–Planck equations,

$$u_t+B(t,x,y)u_y=A^2(t,x,y)u_{xx}+A^1(t,x,y)u_x+A^0(t,x,y)u+C(t,x,y)$$
$$\text{with}\quad A^2\neq 0,\quad B_x\neq 0.$$

We denote the entire class of these equations by $\bar{\mathcal{F}}$. Thus, the arbitrary-element tuple $\bar\theta:=(B,A^2,A^1,A^0,C)$ of the class $\bar{\mathcal{F}}$ runs through the solution set of the system of the inequalities $A^2\neq 0$ and $B_x\neq 0$ with no restrictions on A^0, A^1 and C.

A *(point) equivalence groupoid* $\mathcal{G}^\sim$ of the class $\mathcal{L}|_{\mathcal{S}}$ is the small category with $\mathcal{L}|_{\mathcal{S}}$ or, equivalently, with $\mathcal{S}$ as the set of objects and with the set of point transformations of (x,u), that is, local diffeomorphisms in the space with the coordinates (x,u), between

[4] A linear second-order partial differential equation in n independent variables is called *parabolic*, *normally parabolic* or *ultraparabolic* if the rank of the symmetric matrix constituted by the coefficients of second-order partial derivatives is less than n, is equal to $n-1$ or less than $n-1$, respectively.

pairs of systems from $\mathcal{L}|_{\mathcal{S}}$ as the set of morphisms,

$$\mathcal{G}^{\sim}=\{\mathcal{T}=(\theta,\Phi,\tilde{\theta})\mid \theta,\tilde{\theta}\in\mathcal{S},\Phi\in \mathrm{Diff}^{\mathrm{loc}}_{(x,u)}\colon \Phi_*\mathcal{L}_{\theta}=\mathcal{L}_{\tilde{\theta}}\}.$$

The triples $\mathcal{T}$ are called *admissible (point) transformations* within the class $\mathcal{L}|_{\mathcal{S}}$.

The definitions of all notions in this paragraph arise from the general theory of groupoids. The *source* and *target* maps $\mathrm{s},\mathrm{t}\colon \mathcal{G}^{\sim}\to\mathcal{S}$ are defined by $\mathrm{s}(\mathcal{T})=\theta$ and $\mathrm{t}(\mathcal{T})=\tilde{\theta}$ for any $\mathcal{T}=(\theta,\Phi,\tilde{\theta})\in\mathcal{G}^{\sim}$. This inspires the groupoid notation $\mathcal{G}^{\sim}\rightrightarrows\mathcal{S}$, where the pair of arrows "$\rightrightarrows$" denotes the pair of the source and target maps. Admissible transformations $\mathcal{T}=(\theta,\Phi,\tilde{\theta})$ and $\mathcal{T}'=(\theta',\Phi',\tilde{\theta}')$ are called *composable* if $\tilde{\theta}=\theta'$, and then their *composition* is $\mathcal{T}\star\mathcal{T}':=(\theta,\Phi'\circ\Phi,\tilde{\theta}')$, which defines a partial multiplication on the set $\mathcal{G}^{\sim}$. For any object $\theta\in\mathcal{S}$, the *unit at* θ is the admissible transformation $\mathrm{id}_{\theta}:=(\theta,\mathrm{id},\theta)$, where id is the identity transformation of the space (x,u). This gives rise to the *object inclusion map* $\mathcal{S}\ni\theta\mapsto\mathrm{id}_{\theta}\in\mathcal{G}^{\sim}$, that is, the object set $\mathcal{S}$ can be regarded as the object set for the base groupoid $\mathcal{S}\rightrightarrows\mathcal{S}:=\{\mathrm{id}_{\theta}\mid\theta\in\mathcal{S}\}$. The inverse of $\mathcal{T}$ is $\mathcal{T}^{-1}:=(\tilde{\theta},\Phi^{-1},\theta)$, where Φ^{-1} is the inverse of Φ. All required groupoid properties like associativity of the partial multiplication, its consistency with the source and target maps, natural properties of units and inverses are obviously satisfied. The s-*fibre over* $\theta\in\mathcal{S}$, $\mathrm{s}^{-1}(\theta)\subset\mathcal{G}^{\sim}$, is the set of admissible transformations in $\mathcal{G}^{\sim}$ with the source θ, $\mathrm{s}^{-1}(\theta):=\{\mathcal{T}\mid s(\mathcal{T}):=\theta\}$. Similarly defined the notion of t-*fibre over* $\theta\in\mathcal{S}$. The subset $\mathcal{G}(\theta,\tilde{\theta})$ is the set of admissible transformations from the system $\mathcal{L}_{\theta}$ to the system $\mathcal{L}_{\tilde{\theta}}$, or, equivalently, morphisms from the object θ to the object $\tilde{\theta}$, in other words, $\mathcal{G}(\theta,\tilde{\theta}):=\mathrm{s}^{-1}(\theta)\cap\mathrm{t}^{-1}(\tilde{\theta})$.

The *vertex group* $\mathcal{G}_{\theta}:=\mathcal{G}(\theta,\theta)=\mathrm{s}^{-1}(\theta)\cap\mathrm{t}^{-1}(\theta)$ is associated with the point symmetry pseudogroup G_{θ} of the system $\mathcal{L}_{\theta}$, $G_{\theta}:=\{\Phi\in\mathrm{Diff}^{\mathrm{loc}}_{(x,u)}\mid(\theta,\Phi,\theta)\in\mathcal{G}_{\theta}\}$. The *fundamental groupoid* $\mathcal{G}^{\mathrm{f}}$ of the class $\mathcal{L}|_{\mathcal{S}}$ is the disjoint union of vertex groups of objects $\theta\in\mathcal{S}$,

$$\mathcal{G}^{\mathrm{f}}:=\sqcup_{\theta\in\mathcal{S}}\mathcal{G}_{\theta}:=\{(\theta,\Phi,\theta)\mid\theta\in\mathcal{S},\Phi\in G_{\theta}\}.$$

In other words, the groupoid $\mathcal{G}^{\mathrm{f}}$ is constituted by admissible transformations generated by point symmetries of systems from $\mathcal{L}|_{\mathcal{S}}$. Since it has the same object set $\mathcal{S}$ and the same vertex groups as $\mathcal{G}$ and $\mathcal{T}^{-1}\mathcal{G}_{\tilde{\theta}}\mathcal{T}=\mathcal{G}_{\theta}$ for any $\mathcal{T}\in\mathcal{G}(\theta,\tilde{\theta})$, $\mathcal{G}^{\mathrm{f}}$ is a normal subgroupoid of the equivalence groupoid $\mathcal{G}^{\sim}$, which is also called the *fundamental*

subgroupoid of $\mathcal{G}^{\sim}$.

Denote by ϖ^k, $k \in \{0,\dots,r\}$, the projection from the space with coordinates $(x, u_{(r)}, \theta)$ to the spaces with coordinates $(x, u_{(k)})$. To simplify the presentation, let us denote ϖ^0 by ϖ.

The *usual equivalence (pseudo)group* $G^{\sim}$ of the class $\mathcal{L}|_{\mathcal{S}}$ is the (pseudo)group of point transformations $\mathcal{T}$ in the space with coordinates $(x, u_{(r)}, \theta)$ that

1. are projectable to the k-jets for any $k \in \{0,\dots,r\}$,
2. are consistent with contact structure of the rth jet space that is, $\varpi^r_*\mathcal{T}$ is the prolongation of $\varpi^k_*\mathcal{T}$ to rth order jets $(x, u_{(r)})$, and
3. preserve the class $\mathcal{L}|_{\mathcal{S}}$, that is, $\mathcal{T}_*\mathcal{L}_\theta \in \mathcal{L}|_{\mathcal{S}}$ for any $\theta \in \mathcal{S}$.

The elements of $G^{\sim}$ are called *equivalence transformations* of the class $\mathcal{L}|_{\mathcal{S}}$. The (pseudo)group $G^{\sim}$ naturally acts on the space with coordinates $(x, u_{(r')}, \theta)$, where $r' < r$, if the arbitrary-element tuple θ depends only on $(x, u_{(r')})$.

Two restrictions on equivalence transformations, their projectability and locality with respect to arbitrary elements, can be weakened to generalize the notion of usual equivalence (pseudo)group in several ways. This leads to notions of generalized equivalence (pseudo)group and extended equivalence (pseudo)group, respectively, or extended generalized equivalence (pseudo)group if both restrictions are weakened simultaneously, see [56, 92, 107, 111, 122, 126].

The *action groupoid* $\mathcal{G}^H$ of a (pseudo)subgroup H of the usual equivalence (pseudo)group $G^{\sim}$ of the class $\mathcal{L}|_{\mathcal{S}}$,

$$\mathcal{G}^H := \{(\theta, \varpi_*\mathcal{T}, \mathcal{T}_*\theta) \mid \theta \in \mathcal{S}, \mathcal{T} \in G^{\sim}\},$$

is a wide subgroupoid of the equivalence groupoid $\mathcal{G}^{\sim}$ of this class, $\mathcal{G}^H \subseteq \mathcal{G}^{\sim}$ and $s(\mathcal{G}^H) = \mathcal{S}$. Recall that a subgroupoid of a groupoid is called wide if their sets of objects coincide. We say that an admissible transformation $\mathcal{T}$ in the class $\mathcal{L}|_{\mathcal{S}}$ is generated by an equivalence transformation from H if $\mathcal{T} \in \mathcal{G}^H$.

The *kernel point symmetry (pseudo)group* $G^{\cap} := \cap_{\theta\in\mathcal{S}} G_\theta$ of systems from the class $\mathcal{L}_{\mathcal{S}}$ consists of common point symmetries of these systems. This (pseudo)group can be associated with the normal (pseudo)subgroup $\tilde{G}^{\cap}$ of $G^{\sim}$ whose elements are obtained

from elements of $G^{\cap}$ by the standard prolongation to r'th order jets $(x, u_{(r')})$ and the trivial prolongation to the arbitrary-element tuple θ, $G^{\cap} = \varpi_* \tilde{G}^{\cap}$. Thus, $\tilde{G}^{\cap}$ is the unfaithful (pseudo)subgroup of $G^{\sim}$ under the action of $\mathcal{L}_{\mathcal{S}}$.

To study a Lie group it is often useful to consider the Lie algebra of vector fields, which is constituted by infinitesimal generators of one-parameter subgroups of this group. An analogous approach is useful for pseudogroups, that is, the infinitesimal counterpart of the *point symmetry (pseudo)group* G_θ of a system $\mathcal{L}_\theta$ is *the maximal Lie invariance (pseudo)algebra* $\mathfrak{g}_\theta$ of the system $\mathcal{L}_\theta$, see Section 1.2. The infinitesimal counterparts of the *equivalence (pseudo)group* $G^{\sim}$ and the *kernel point symmetry (pseudo)group* $G^{\cap}$ of the class $\mathcal{L}|_{\mathcal{S}}$ are called the *equivalence (pseudo)algebra* $\mathfrak{g}^{\sim}$ and *the kernel Lie invariance (pseudo)algebra* $\mathfrak{g}^{\cap}$ of the class $\mathcal{L}|_{\mathcal{S}}$, respectively, $\mathfrak{g}^{\cap} = \cap_{\theta\in\mathcal{S}}\mathfrak{g}_\theta$.

The most convenient classes of differential equations to study within the framework of group analysis are normalized classes. A class $\mathcal{L}|_{\mathcal{S}}$ is called *normalized* if its equivalence groupoid coincides with the action groupoid of the equivalence (pseudo)group $G^{\sim}$ of this class, $\mathcal{G}^{\sim} = \mathcal{G}^{G^{\sim}}$. In other words, all the information about the admissible transformations of the normalized class $\mathcal{L}|_{\mathcal{S}}$ can be deduced from its equivalence group $G^{\sim}$.

A class $\mathcal{L}|_{\mathcal{S}}$ is called *seminormalized* if its equivalence groupoid coincides with the Frobenius product of the fundamental subgroupoid $\mathcal{G}^{\rm f}$ and the action groupoid of the equivalence group $G^{\sim}$ of the class $\mathcal{L}|_{\mathcal{S}}$, that is, $\mathcal{G}^{\rm f} \star \mathcal{G}^{G^{\sim}} = \mathcal{G}^{\sim}$. Note that for any class $\mathcal{L}|_{\mathcal{S}}$, the Frobenius product of subgroupoids $\mathcal{G}^{\rm f} \star \mathcal{G}^{G^{\sim}} := \{\mathcal{T} \star \mathcal{T}' \mid \mathcal{T} \in \mathcal{G}^{\rm f}, \mathcal{T}' \in \mathcal{G}^{G^{\sim}}, {\rm s}(\mathcal{T}) = {\rm t}(\mathcal{T}')\}$ is a well-defined subgroupoid of $\mathcal{G}^{\sim}$ since the fundamental subgroupoid $\mathcal{G}^{\rm f}$ is a normal subgroupoid in $\mathcal{G}^{\sim}$.

Replacing the notion of equivalence (pseudo)group in the above definitions of normalization with extended, generalized or extended generalized equivalence pseudogroup leads to a generalization of the notions of normalization. In the literature, this is reflected by specifying in which sense the normalization is considered: in the usual, generalized, extended or extended generalized sense. In this thesis, we consider only normalization and seminormalization in the usual sense. This is justified by our description of equivalence groups and groupoids in Chapters 3 and 4. Roughly speaking, the point transformations for the equations under study exhibit no nonlocalities or intertwining of independent and dependent variables with arbitrary elements.

1.4 Methods for computing point symmetry pseudogroups

The infinitesimal invariance criterion (Theorem 1.3) allows us to find the (pseudo)algebras of Lie-symmetry vector fields in a systematic and algorithmic way. Nevertheless, for the cases considered in this thesis, the required computations are highly non-trivial, even for finding infinitesimal generators, see, e.g., Chapter 4.

By computing the flows of Lie-symmetry vector fields, we can obtain a pseudogroup of continuous point symmetries of a system of differential equations. However, much like the exponentiation of a Lie algebra does not account for connected components of a Lie group, exponentiating Lie-symmetry pseudoalgebras, we may miss many discrete point symmetries.

Discrete symmetries are very important when studying reductions of the system of differential equations and constructing its group-invariant solutions. In many works on the analysis of the symmetry of differential equations, discrete symmetries were tried to be found by guesswork, but usually incorrectly, see, e.g., [1,55,95,101]. Over the last two decades, however, two systematic approaches for finding *complete* pseudogroups of point symmetries of differential equations have been developed: the *direct method* [15, 16, 107, 127] and the *algebraic method* [13, 48, 50, 51]. The exposition in this section follows [26, 69].

1.4.1 Direct method

The name of this approach is quite descriptive, hinting at its systematic and computationally complicated nature. The direct method is based on the definition of a point symmetry Φ of a system of differential equations $\mathcal{L}$, that is, it is a point transformation in the space with coordinates (x, u) that preserves the solution set of $\mathcal{L}$, see Definition 1.2. In view of Hadamard's Lemma, see Lemma 1.12, Φ preserves the diffiety $\mathcal{L}$ if and only if it preserves the form of the system $\mathcal{L}$, $\Phi_*\mathcal{L} = \mathcal{L}$.

Lemma 1.12 (Hadamard's Lemma, see [78])**.** *Let M be a manifold and $F\colon M^m \to \mathbb{R}^l$ $(l \leqslant m)$ be of maximal rank on the regular level set $S_F = \{x \in M \mid F(x) = 0\}$, $F = (F^1, \dots, F^l)$. Then a smooth real-valued function $f\colon M \to \mathbb{R}$ vanishes on S_F if*

and only if there exists $Q_1,\dots,Q_l\in C^\infty(M)$ *such that*

$$\forall x\in M\colon f(x)=Q_1(x)F^1(x)+\dots+Q_l(x)F^l(x).$$

More specifically, consider the most general point transformation Φ in the space with coordinates (x,u),

$$\Phi\colon\quad \tilde x=X(x,u),\ \tilde u=U(x,u),$$

with a nonzero Jacobian $|\partial(X,U)/\partial(x,u)|\neq 0$ and assume that $\Phi_*\mathcal{L}=\mathcal{L}$. To implement the latter condition, we express all the required derivatives of $\tilde u$ with respect to $\tilde x$ in terms of the derivatives of u with respect to x using chain rule and substitute the derived expressions into system $\mathcal{L}$, yielding $\Phi_*^{-1}\mathcal{L}$, which should be identically satisfied on the solution set of the system $\mathcal{L}$. To take into account the last condition, we fix the ranking of the derivatives of u consistent with the structure of $\mathcal{L}$. We then substitute the expressions for leading derivatives in view of $\mathcal{L}$ and its differential consequences into $\Phi_*^{-1}\mathcal{L}$ and split the resulting system with respect to parametric derivatives of u. These determining equations are often difficult to derive and solve, even for simple models such as the linear (1+1)-dimensional heat equation, see [74].

The most efficient trick to handle the above problems is to embed the system $\mathcal{L}$ as an element into a class of differential equations $\mathcal{L}|_{\mathcal{S}}:=\{\mathcal{L}_\theta\mid\theta\in\mathcal{S}\}$, e.g., $\mathcal{L}=\mathcal{L}_{\theta^0}$ for some $\theta^0\in\mathcal{S}$. If this class is normalized in the usual sense, the point symmetry (pseudo)group G_θ of any equation $\mathcal{L}_\theta$ from $\mathcal{L}|_{\mathcal{S}}$ is contained in the projection of the equivalence (pseudo)group $G^\sim$ of $\mathcal{L}|_{\mathcal{S}}$ onto the space of independent and dependent variables. Therefore, to find the pseudogroup of point symmetries G using the point equivalence pseudogroup $G^\sim$, it suffices to specify the values of the arbitrary-element tuple, e.g., $\theta=\theta^0$ [15]. Such an approach is justified, since, as a rule, computation of common constraints on admissible transformations of the entire normalized class is simpler than a straightforward computation of the point-symmetry (pseudo)group of a fixed system of equations, although it still requires solving nonlinear, highly-coupled partial differential equations.

Naturally, the best-case scenario is when the system of differential equations $\mathcal{L}$ can be embedded in the normalized class $\mathcal{L}|_{\mathcal{S}}$, which is very well studied in the literature.

1.4.2 Point symmetry pseudogroup of the heat equation

Let us now return to the example with the linear $(1+1)$-dimensional heat equation (1.2) from Section 1.2. This equation belongs to the class $\mathcal{E}$ of linear (1+1)-dimensional second-order evolution equations of the general form

$$u_t = A(t,x)u_{xx} + B(t,x)u_x + C(t,x)u + D(t,x) \quad \text{with} \quad A \neq 0. \tag{1.4}$$

Here the tuple of arbitrary elements of $\mathcal{E}$ is $\theta := (A,B,C,D) \in \mathcal{S}_{\mathcal{E}}$, the parameter-functions A, B, C and D are smooth functions of (t,x) and $\mathcal{S}_{\mathcal{E}}$ is the solution set of the auxiliary system consisting of the single inequality $A \neq 0$ [5] and the equations meaning that the arbitrary elements depend at most on (t,x),

$$A_u = A_{u_x} = A_{u_t} = A_{u_{tt}} = A_{u_{tx}} = A_{u_{xx}} = 0$$

and similar equations for B, C and D.

To find the point symmetry pseudogroup G of the equation (1.2), we start by considering the equivalence groupoid of the class $\mathcal{E}$, which in turn is a natural choice for a (normalized) superclass for the equation (1.2). We use the papers [111, 127] as reference points for many known results on admissible transformations of the class $\mathcal{E}$.

Proposition 1.13 ([127]). *The class $\mathcal{E}$ is normalized in the usual sense. Its usual equivalence pseudogroup $G^{\sim}_{\mathcal{E}}$ consists of the transformations of the form*

$$\tilde t = T(t), \quad \tilde x = X(t,x), \quad \tilde u = U^1(t,x)u + U^0(t,x), \tag{1.5a}$$

$$\tilde A = \frac{X_x^2}{T_t}A, \quad \tilde B = \frac{X_x}{T_t}\left(B - 2\frac{U^1_x}{U^1}A\right) - \frac{X_t - X_{xx}A}{T_t}, \quad \tilde C = -\frac{U^1}{T_t}\mathrm{E}\frac{1}{U^1}, \tag{1.5b}$$

$$\tilde D = \frac{U^1}{T_t}\left(D + \mathrm{E}\frac{U^0}{U^1}\right), \tag{1.5c}$$

where T, X, U^0 and U^1 are arbitrary smooth functions of their arguments with $T_t X_x U^1 \neq 0$, and $\mathrm{E} := \partial_t - A\partial_{xx} - B\partial_x - C$.

The normalization of the class $\mathcal{E}$ means that its equivalence groupoid coincides with the action groupoid of the pseudogroup $G^{\sim}_{\mathcal{E}}$.

[5] In local Lie symmetry analysis, the distinction between $A > 0$ and $A < 0$ is not restrictive, hence for local equivalence classification both signs of A are intentionally included.

Theorem 1.14. *The point symmetry pseudogroup G of the (1+1)-dimensional linear heat equation (1.2) is constituted by the point transformations of the form*

$$\begin{aligned}&\tilde t=\frac{\alpha t+\beta}{\gamma t+\delta},\quad \tilde x=\frac{x+\lambda_1 t+\lambda_0}{\gamma t+\delta},\\&\tilde u=\sigma\sqrt{|\gamma t+\delta|}\,\exp\left(\frac{\gamma(x+\lambda_1 t+\lambda_0)^2}{4(\gamma t+\delta)}-\frac{\lambda_1}{2}x-\frac{\lambda_1^2}{4}t\right)\big(u+h(t,x)\big),\end{aligned}\tag{1.6}$$

where α, β, γ, δ, λ_1, λ_0 and σ are arbitrary constants with $\alpha\delta-\beta\gamma=1$ and $\sigma\neq0$, and h is an arbitrary solution of (1.2).

Proof. The linear heat equation (1.2) corresponds to the value $(1,0,0,0)=:\theta^0$ of the arbitrary-element tuple $\theta=(A,B,C,D)$ of class $\mathcal{E}$. Its vertex group $\mathcal{G}_{\theta^0}:=\mathcal{G}^{\sim}_{\mathcal{E}}(\theta^0,\theta^0)$ is the set of admissible transformations of the class $\mathcal{E}$ with θ^0 as both their source and target, $\mathcal{G}_{\theta^0}=\{(\theta^0,\Phi,\theta^0)\mid\Phi\in G\}$. This argument allows us to use Proposition 1.13 in the course of computing the pseudogroup G.

The rest of the proof is integrating the equations (1.5), where both the source value θ of the arbitrary-element tuple and its target value $\tilde\theta$ coincide with θ^0, with respect to the parameter functions T, X, U^1 and U^0. It can be found in [74]. □

The explicit presentation of the point-symmetry pseudogroup G of the heat equation (1.2) allows us to systematically recover its complete symmetry structure, including discrete transformations. For example, in [74] this approach enabled us to show that the spatial reflection involution, $(t,x,u)\mapsto(t,-x,u)$, is actually a continuous symmetry belonging to the one-parameter subgroup generated by $\mathcal{P}^t+\mathcal{K}$, that is, it belongs to the exponent $\exp(\mathfrak{g})$ of the Lie invariance algebra $\mathfrak{g}$ of (1.2). Consequently, correctly interpreting the action of G on $\mathfrak{g}$ led to the first correct classification of subalgebras of the essential Lie invariance algebra $\mathfrak{g}^{\rm ess}$ of the equation (1.2) and completed the arrangement of its Lie reductions.

1.4.3 Algebraic methods

The basis of algebraic methods of finding the complete point symmetry pseudogroup G of differential equations $\mathcal{L}$ is constituted by works of Hydon [48, 50, 51], where he suggested the idea that the pushfoward Φ_* of any element Φ of G is an automorphism

of the Lie-symmetry algebra $\mathfrak{g}$, $\Phi_*\mathfrak{g} = \mathfrak{g}$. In his works, Hydon focused his attention on the case when the Lie symmetry algebra $\mathfrak{g}$ is finite-dimensional and developed what is now called *the automorphism-based version of the algebraic method*. More specifically, when the algebra $\mathfrak{g}$ is finite-dimensional, choosing its basis $(Q^1, \dots, Q^d)$, where $d := \dim \mathfrak{g}$, the condition $\Phi_*\mathfrak{g} = \mathfrak{g}$ can be expressed as

$$\Phi_* Q^i = \sum_{j=1}^{d} a_{ji} Q^j, \quad i = 1, \dots, d,$$

where $(a_{ji})_{i,j=1,\dots,d}$ is the matrix of an automorphism of the algebra $\mathfrak{g}$ in that basis. Substituting the general form of the matrix of an automorphism of $\mathfrak{g}$ into the above condition and splitting it with respect to basis elements Q^i yields a system of determining differential equations on the components of an arbitrary point symmetry transformation Φ of $\mathcal{L}$. The obtained system of differential equations is linear and, if $d > 1$, is overdetermined, however, it is usually not sufficient for finding the general form of elements of G. This is why, after integrating this system, one should continue computations using the direct method and the derived constraints on Φ to complete the system of constraints on Φ. The latter step usually does not create computational difficulties.

The most expensive part of the automorphism-based version of the algebraic method is computing the automorphism group of the algebra $\mathfrak{g}$. This complication also prevents the method from being applied to systems of differential equations whose symmetry algebras are infinite-dimensional. To account these difficulties, Hydon's approach was extended in [16] via replacing the condition $\Phi_*\mathfrak{g} = \mathfrak{g}$ with $\Phi_*\mathfrak{m} \subseteq \mathfrak{m}$ for any megaideal $\mathfrak{m} \subseteq \mathfrak{g}$. This approach is called the *megaideal-based version of the algebraic method*.

Recall that a *megaideal* $\mathfrak{m}$ of a Lie algebra $\mathfrak{g}$ is a subspace of $\mathfrak{g}$ that is invariant under any element $\mathfrak{T}$ of the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of $\mathfrak{g}$, $\mathfrak{T}\mathfrak{m} \subseteq \mathfrak{m}$, [16,125]. The latter in particular implies $\mathfrak{m} \subseteq \mathfrak{T}\mathfrak{m}$. Another name for megaideal is *fully characteristic ideal*, see [47, Exrecise 14.1.1], which stems from the property that any megaideal is, in particular, a characteristic ideal of $\mathfrak{g}$.

Of course, to determine all megaideals of a given Lie algebra $\mathfrak{g}$, it is necessary to construct the complete group of automorphisms of $\mathfrak{g}$. But to find the point symmetries of a system of differential equations it is enough to construct sufficiently many megaideals of the algebra of Lie symmetries of that system, and the latter can be

done algebraically, without any knowledge of the automorphism group.

The following assertions are obvious.

1. The trivial subalgebra $\{0\}$ and the entire algebra $\mathfrak{g}$ itself are megaideals of $\mathfrak{g}$.
2. If $\mathfrak{i}_1$ and $\mathfrak{i}_2$ are megaideals of $\mathfrak{g}$, then so are $\mathfrak{i}_1+\mathfrak{i}_2$, $\mathfrak{i}_1\cap\mathfrak{i}_2$ and $[\mathfrak{i}_1,\mathfrak{i}_2]$, i.e., sums, intersections and Lie products of megaideals are again megaideals.
3. If $\mathfrak{i}_2$ is a megaideal of $\mathfrak{i}_1$ and $\mathfrak{i}_1$ is a megaideal of $\mathfrak{g}$ then $\mathfrak{i}_2$ is a megaideal of $\mathfrak{g}$, i.e., megaideals of megaideals are again megaideals.

These properties imply that all elements of the derived series, upper and lower central series of a Lie algebra are its megaideals. In particular, the center and the derivative of a Lie algebra are its megaideals. We also have that the radical $\mathfrak{r}$ and the nilradical $\mathfrak{n}$ (i.e., the maximal solvable and nilpotent ideals, respectively) of $\mathfrak{g}$, as well as different Lie products, sums and intersections involving $\mathfrak{g}$, $\mathfrak{r}$ and $\mathfrak{n}$, are megaideals of $\mathfrak{g}$.

To apply the megaideal-based version of the algebraic method for finding point symmetries of a system of differential equations, it is necessary to find sufficiently many useful megaideals. Indeed, some of the megaideals may be useless in the course of finding a general form of a point symmetry Φ, for example, the sum $\mathfrak{i}_1+\mathfrak{i}_2$ of proper megaideals $\mathfrak{i}_1$ and $\mathfrak{i}_2$ will not give more constraints on components of Φ than $\mathfrak{i}_1$ and $\mathfrak{i}_2$ separately. We call megaideals of $\mathfrak{g}$ that are not sums of other megaideals *essential*. Moreover, within this megaideal-based approach, it is guaranteed that we will not find more constraints on components of Φ than those arising from $\Phi_*\mathfrak{g}\subseteq\mathfrak{g}$; the megaideals only simplify the latter condition, allowing us to efficiently perform the computations. This is why it is instructive to follow the strategy:

1. Construct as many essential megaideals $\mathfrak{m}$ of the algebra $\mathfrak{g}$ as possible.
2. For a point transformation Φ of the space of independent and dependent variables, applying the condition $\Phi_*\mathfrak{m}\subseteq\mathfrak{m}$, find the general constraints imposed on Φ using all the megaideals constructed in the previous step.
3. If under the derived constraints the condition $\Phi_*\mathfrak{g}\subseteq\mathfrak{g}$ is satisfied, then no more constraints on Φ can be found within the algebraic method. If not, return to step 1 and find more megaideals or use the condition $\Phi_*\mathfrak{g}\subseteq\mathfrak{g}$ itself if possible.

Megaideals can also be constructed using the following assertion, which was first proven in [37].

Proposition 1.15. *If* $\mathfrak{i}_0$, $\mathfrak{i}_1$ *and* $\mathfrak{i}_2$ *are megaideals of* $\mathfrak{g}$ *then the set* $\mathfrak{s}$ *of elements of* $\mathfrak{i}_0$ *whose commutators with arbitrary elements from* $\mathfrak{i}_1$ *belong to* $\mathfrak{i}_2$ *is also a megaideal of* $\mathfrak{g}$.

Proof. It is clear that $\mathfrak{s}$ is a linear subspace of $\mathfrak{g}$. Let z_0 be any element of $\mathfrak{s}$. Since $\mathfrak{i}_0$ and $\mathfrak{i}_1$ are megaideals, for any element $z_1 \in \mathfrak{i}_1$ and any automorphism $\mathfrak{T} \in \mathrm{Aut}(\mathfrak{g})$ we have $\mathfrak{T}z_0 \in \mathfrak{i}_0$ and $[z_0, \mathfrak{T}^{-1}z_1] \in \mathfrak{i}_2$. Applying $\mathfrak{T}$ to the latter conditions gives us $\mathfrak{T}[z_0, \mathfrak{T}^{-1}z_1] = [\mathfrak{T}z_0, \mathfrak{T}\mathfrak{T}^{-1}z_1] \in \mathfrak{T}\mathfrak{i}_2 = \mathfrak{i}_2$, therefore $\mathfrak{T}z_0 \in \mathfrak{s}$. □

For example, applying Proposition 1.15 to $\mathfrak{i}_0 = \mathfrak{g}$ and $\mathfrak{i}_2 = \{0\}$ gives us the centralizer of $\mathfrak{i}_1$, and $\mathfrak{i}_0 = \mathfrak{g}$ and $\mathfrak{i}_1 = \mathfrak{i}_2$ gives the normalizer of $\mathfrak{i}_1$. Nevertheless, not all megaideals constructed using this assertion can be related to structural objects of the Lie algebra $\mathfrak{g}$.

Once all possible constraints on a point transformation Φ have been found within the framework of the algebraic method, we should proceed to apply the direct method to the transformation Φ and find the remaining constraints on its components. We will apply the megaideal-based version of the algebraic method to construct the point symmetry pseudogroup of the multi-layer quasi-geostrophic problem in Chapter 4.

Analogs of both versions of the algebraic methods were developed for computing equivalence pseudogroups of classes of differential equations. In Chapter 3, we will compute the equivalence groups of two special classes of Kolmogorov equations using the automorphism-based version of the algebraic method.

1.5 Optimized procedure for Lie reductions

One of the most valuable applications of the Lie theory of symmetries of differential equations is the systematic derivation of particular exact invariant solutions. When a system of differential equations admits a nontrivial symmetry (pseudo)group, it becomes possible to decrease the order or decrease the number of dependent variables in the system using the method of *Lie reductions*.

Since the point symmetry pseudogroup acts naturally on the Lie-invariant solutions, we arrive at an important classification problem: construct and exhaustively classify all possible Lie reductions without redundancies. This problem turns out to

be difficult, and it is frequently addressed incorrectly in the existing literature. Deriving the complete classification involves cumbersome computations and requires careful analysis of many inequivalent cases. This is why a key step to successfully perform the above procedure is optimization.

Let $\mathcal{L}$ be a system of differential equations with three independent variables (t, x, y) and m dependent variables $u = (u^1, \dots, u^m)$. The optimized procedure of Lie reduction for such systems was instructively described in [148] in the course of studying Lie reductions of the dispersionless Nizhnik equation. To efficiently carry out Lie reductions of $\mathcal{L}$, one should follow the next steps.

1. Compute the maximal Lie invariance (pseudo)algebra $\mathfrak{g}$ and the complete point symmetry (pseudo)group G of $\mathcal{L}$.
2. Find complete and irredundant lists of G-inequivalent subalgebras of $\mathfrak{g}$ of dimension one and two. Among them, select subalgebras appropriate for Lie reductions.

It is common to carry out classification of subalgebras of $\mathfrak{g}$ modulo the action of the inner automorphism group $\mathrm{Inn}(\mathfrak{g})$ of $\mathfrak{g}$, which is equivalent to the action of the identity component G_{id} of the pseudogroup G. But this approach ignores the action of discrete symmetries of $\mathcal{L}$, thus potentially leading to superfluous cases to consider in the next steps.

3. *Codimension-one Lie reductions.* For a suitable one-dimensional subalgebra $\mathfrak{s}_1 := \langle Q_1 \rangle$ from the list obtained in the previous step, find a suitable $\mathfrak{s}_1$-invariant ansatz. The starting point for finding such an ansatz is to consider the differential constraint $Q_1[u] = 0$, where $Q[u]$ denotes the characteristic of a vector field Q (see Section 1.6). This gives rise to an ansatz $\rho_1 \colon \mathbb{R}^3 \times \mathbb{R}^m \to \mathbb{R}^2 \times \mathbb{R}^m$, $\rho_1 \colon (t, x, y, u) \to (z_1, z_2, v)$ where the tuple (z_1, z_2, v) denotes the tuple of invariants of the flow of Q_1. The reduced system $\rho_{1*}\mathcal{L} = \hat{\mathcal{L}}_1$ is $\operatorname{ord}\mathcal{L}$ system of differential equations in m dependent variables $v^1, \dots, v^m$ and two dependent variables (z_1, z_2). Note that it is usually useful to choose the ansatz ρ_1 with the purpose of deriving the reduced system $\hat{\mathcal{L}}$ of the simplest form possible. If this system can be completely integrated or its solution can be expressed as a general solution of systems that were comprehensively studied in the framework

of Lie symmetry analysis, then further consideration of the reduced system $\hat{\mathcal{L}}_1$ is needless. In such cases, the consideration of Lie reductions of $\mathcal{L}$ with respect to subalgebras G-equivalent to those containing $\mathfrak{s}_1$ is also needless. Pull back each obtained solution $\hat{f}$ of the reduced system $\hat{\mathcal{L}}$ to the solutions of the original system, that is, compute $\rho_1^*\hat{f}$.

4. If the system cannot be integrated or has not been studied previously, compute the normalizer $\mathrm{N}_{\mathfrak{g}}(\mathfrak{s}_1)$ of $\mathfrak{s}_1$ in $\mathfrak{g}$, the stabilizer $\mathrm{St}_G(\mathfrak{s}_1)$ of $\mathfrak{s}_1$ in G, the maximal Lie invariance (pseudo)algebra $\hat{\mathfrak{g}}_1$ and the point symmetry (pseudo)group $\hat{G}_1$ of $\hat{\mathcal{L}}_1$. Compute the subalgebra $\tilde{\mathfrak{g}}_1 := \rho_{1*}\mathrm{N}_{\mathfrak{g}}(\mathfrak{s}_1)$ of $\hat{\mathfrak{g}}$ and the subgroup $\tilde{G}_1 := \rho_{1*}\mathrm{St}_G(\mathfrak{s}_1)$ of $\hat{G}_1$ of induced symmetries. Carry out the Lie reduction procedure for $\hat{\mathcal{L}}_1$ only if $\tilde{\mathfrak{g}}_1 \neq \hat{\mathfrak{g}}_1$ or at least $\tilde{G}_1 \neq \hat{G}_1$. The elements of the complement $\hat{\mathfrak{g}}_1 \setminus \tilde{\mathfrak{g}}_1$ are called *hidden symmetries* of the original system $\mathcal{L}$. In many cases, these symmetries are the only essential ones for carrying out Lie reductions of $\hat{\mathcal{L}}$ and constructing hiddenly invariant solutions, see [70, Apprndix B].

5. *Codimension-two Lie reductions.* For each two-dimensional subalgebra of $\mathfrak{g}$ that has passed the selection in steps 2 and 3, say $\mathfrak{s}_2 := \langle Q_1, Q_2\rangle$, construct a suitable ansatz

$$\rho_2\colon \mathbb{R}^3 \times \mathbb{R}^m \to \mathbb{R} \times \mathbb{R}^m, \quad \rho_2\colon (t, x, y, u) \to (\omega, \varphi),$$

where the tuple (ω, φ) is a tuple of integrals of the system $Q_1[u] = 0$ and $Q_2[u] = 0$. Similarly to Step 3, we can choose ρ_2 in a way that makes the reduced system $\rho_{2*}\mathcal{L} = \hat{\mathcal{L}}_2$ of ordinary differential equations the simplest possible.

6. Similarly to step 4, compute the normalizer $\mathrm{N}_{\mathfrak{g}}(\mathfrak{s}_2)$ and the stabilizer $\mathrm{St}_G(\mathfrak{s}_2)$, and the induced symmetries

$$\tilde{\mathfrak{g}}_2 := \rho_{2*}\mathrm{N}_{\mathfrak{g}}(\mathfrak{s}_2) \quad \text{and} \quad \tilde{G}_2 := \rho_{1*}\mathrm{St}_G(\mathfrak{s}_2).$$

7. If possible, find the general solution of the reduced system $\hat{\mathcal{L}}_2$ or at least some of its particular solutions. Use the transformations from $\tilde{G}_2$ to gauge the constants of integration. Pull back the obtained solutions under ρ_2.

8. *Codimension-three Lie reductions.* It may be useful to reduce $\mathcal{L}$ to algebraic equations and analyze whether they can lead to essentially new solutions that were not constructed in the previous steps.

We will apply the above procedure for constructing invariant solutions of the fine Kolmogorov backward equation and the multi-layer quasi-geostrophic problem in Chapters 2 and 4, respectively.

1.6 Generalized symmetries and generalized reductions

1.6.1 Basics on generalized symmetries

Thus far, our discussion has been focused only on the point transformations acting on the spaces of independent and dependent variables. Emmy Noether was the first to extend this concept within the infinitesimal framework by including the derivatives of the dependent variables, see [102], and thereby introducing the concept of generalized symmetries. Since then, these symmetries have proven their importance in the modern study of integrable systems. See [103, pp. 374–379] for an excellent exposition on the history and development of the theory of generalized symmetries and their applications, as well as other monographs on the subject [20–22, 52, 77].

More specifically, consider vector fields on TJ^∞ of the form $Q = \xi^i[u]\partial_{x_i} + \eta^\alpha[u]\partial_{u^\alpha}$, where the components $\xi^i[u]$ and $\eta^\alpha[u]$ are differential functions. These vector fields are called *generalized vector fields*. Let $\mathcal{L}$ be a system of differential equations given by $L^\mu(x, u_{(r)}) = 0$, $\mu = 1, \dots, l$ and assume that it satisfies the nondegeneracy conditions from Section 1.2.

Definition 1.16. A generalized vector field $Q = \xi^i[u]\partial_{x_i} + \eta^\alpha[u]\partial_{u^\alpha}$ is called a *generalized symmetry* of a system of differential equations $\mathcal{L}$ if and only if

$$Q^{(\infty)}[L^\mu(x, u_{(r)})] = 0, \quad \mu = 1, \dots, l, \text{ whenever } (x, u_{(r)}) \in \mathcal{L}. \tag{1.7}$$

This definition is a direct analogue of the Lie invariance criterion, Theorem 1.3. However, we are now determining an object which resides in the infinite jet space, which significantly complicates the derivation and solving of determining equations, if it is at all possible. Because of this inherent to infinite jet space computational complexity, the exhaustive descriptions of generalized symmetry algebras with complete proofs have only been presented for a small number of specific systems of differential

equations. Notably, the generalized symmetry algebras even of such fundamental and simple models of mathematical physics as the linear (1+1)-dimensional heat equation [74], the Burgers equation [119], the linear Korteweg–de Vries equation [126] and the (1+1)-dimensional Klein–Gordon equation [110] were fully described only recently. See also [110] for a review of advances in this field. We also refer to works of Anco and Pohjanpelto on generalized symmetries of some (1+3)-dimensional equations. More specifically, the Maxwell equations in vacuum [6], the neutrino and graviton equations [7], and, more generally, the massless spin field equations [117].

Recall that $\mathcal{A}$ denotes the coordinate algebra $C^\infty(J^\infty)$ of the infinite jet space J^∞.

Definition 1.17. Let $Q[u] := (Q_1[u], \dots, Q_m[u]) \in \mathcal{A}^m$ be an m-tuple of differential functions. The generalized vector field

$$\mathbf{v}_Q := Q_\alpha[u]\partial_{u^\alpha}$$

is called an *evolutionary vector field* and $Q[u]$ is called its *characteristic*.

The prolongation formula for any evolutionary vector field takes a very simple form:

$$\mathbf{v}_Q^{(\infty)} := \sum_I \mathrm{D}_I Q_\alpha[u]\partial_{u^\alpha}.$$

Any generalized vector field Q has an associated *evolutionary representative* $\mathbf{v}_Q$ in which the characteristic $Q[u]$ has components

$$Q_\alpha[u] := \eta^\alpha - \xi^i u_i^\alpha.$$

The following assertion tells us that these two generalized vector fields determine essentially the same symmetry.

Proposition 1.18 (Proposition 5.5 in [103])**.** *A generalized vector field Q is a symmetry of a system of differential equations if and only if its evolutionary representative $\mathbf{v}_Q$ is.*

If the characteristic of an evolutionary vector field $\mathbf{v}_Q$ vanishes on the solutions of a system of differential equations, then obviously any of its prolongations vanishes on solutions as well. Therefore $\mathbf{v}_Q$ is automatically a generalized symmetry, which is

called *trivial*. This notion gives rise to the equivalence relation on the space of generalized symmetries: two generalized symmetries are called equivalent if their difference is a trivial generalized symmetry.

Definition 1.19. Let $\mathcal{L}$ be a system of differential equations. A *recursion operator* for $\mathcal{L}$ is a linear (pseudo)differential operator $\mathrm{R}\colon \mathcal{A}^m \to \mathcal{A}^m$ in the space of m-tuples of differential functions with the property whenever $\mathbf{v}_{\mathrm{Q}}$ is an evolutionary symmetry of $\mathcal{L}$, so $\mathbf{v}_{\tilde{Q}}$ is with characteristic $\tilde{Q}[u] = \mathrm{R}Q[u]$.

Recursion operators play an important role in the study of generalized symmetries. First of all, if we are fortunate enough to find such an operator, we can generate an infinite hierarchy of generalized symmetries starting with $\mathbf{v}_Q$ and successively applying R to the characteristic Q, that is, $Q^k[u] := \mathrm{R}^k Q[u]$ is a characteristic of a generalized symmetry. It turns out that such equations that admit a recursion operator are integrable, in the sense that they possess infinite hierarchies of conserved quantities. A basic example of an integrable system admitting a recursion operator is the famous Korteweg–de Vries equation, see [103, Section 5.2].

In this thesis, we are focused on studying generalized symmetries of linear systems of differential equations. In this case, recursion operators can generate the entire algebra of (essential) generalized symmetries, and they may be further used for generalized reduction and generating new solutions from known ones.

Let $\Delta[u] = 0$ be a system of linear differential equations associated with a linear differential operator Δ. Denote by Σ the quotient algebra of evolutionary generalized symmetries with respect to the equivalence of generalized symmetries; we denote the canonical representatives of equivalence classes by $\eta^\alpha[u]\partial_{u^\alpha}$. This algebra Σ filtration by the order $\operatorname{ord}(\eta^\alpha[u]\partial_{u^\alpha})$ of evolutionary symmetries,

$$\Sigma = \bigcup_{k\in\mathbb{N}_0\cup\{-\infty\}} \Sigma^k, \quad \Sigma^k := \{\eta^\alpha[u]\partial_{u^\alpha} \mid \operatorname{ord}(\eta^\alpha[u]\partial_{u^\alpha}) \leqslant k\},$$

where $\operatorname{ord}(\eta^\alpha[u]\partial_{u^\alpha}) := \max(\operatorname{ord}\eta^\alpha[u])$. The algebra Σ contains an important subalgebra Λ of *linear generalized symmetries*. A generalized symmetry $\eta^\alpha[u]\partial_u^\alpha$ is called *linear* if there exist tuples I_α with $|I_\alpha| \leqslant \operatorname{ord}(\eta^\alpha[u]\partial_u^\alpha)$ and functions $\eta^{\beta,I_\alpha}(x)$ such that $\eta^\alpha[u] := \sum_{\beta=1}^m \sum_{I_\alpha} \eta^{\beta,I_\alpha}(x) u^\beta_{I_\alpha}$. In other words, the components of the characteristic of a linear generalized symmetry are linear functions with respect to the derivatives of u. The algebra Λ inherits the filtration from Σ: $\Lambda = \bigcup_{k\in\mathbb{N}_0} \Lambda^k$ with $\Lambda^k := \Lambda \cap \Sigma^k$.

Recursion operators for systems of linear differential equations can be characterized as follows.

Proposition 1.20 (Proposition 5.22 in [103]). *Let $\mathcal{L}$ be a system of linear differential equations. A linear differential operator $\mathrm{R}\colon \mathcal{A}^m \to \mathcal{A}^m$ not depending on u or its derivatives is a recursion operator for $\mathcal{L}$ if and only if $Q = \mathrm{R}[u]$ is a characteristic of a linear generalized symmetry of the system.*

In other words, the elements of the Lie algebra of linear generalized symmetries Λ determine the elements of the associative algebra of recursion operators, which is usually denoted Υ, of the system $\mathcal{L}$, and the other way around.

1.6.2 Generalized symmetries of the heat equation

It is surprising that the explicit description of generalized symmetries of the linear heat equation (1.2) is not well known and was presented only in [2, Section 10.1] without proof. We have not only remedied this drawback in [74], but also studied the structure of this algebra in detail.

Theorem 1.21. *The algebra of generalized symmetries of the (1+1)-dimensional linear heat equation (1.2) is $\Sigma = \Lambda \in \Sigma^{-\infty}$, where*

$$\Lambda := \langle \mathfrak{Q}^{kl},\ k,l \in \mathbb{N}_0 \rangle, \quad \Sigma^{-\infty} := \{\mathfrak{Z}(h)\}$$

with $\mathfrak{Q}^{kl} := (\mathrm{G}^k \mathrm{D}_x^l u)\partial_u$, $\mathrm{G} := t\mathrm{D}_x + \frac{1}{2}x$, $\mathfrak{Z}(h) := h(t,x)\partial_u$, and the parameter function h runs through the solution set of (1.2).

Thus, the algebra Λ of the linear generalized symmetries of the equation (1.2) is generated by the two recursion operators D_x and G from the simplest linear generalized symmetry $u\partial_u$, and both the recursion operators and the seed symmetry are related to Lie symmetries. In particular, on the solution set of the equation (1.2), the generalized symmetries associated with the basis elements $\mathcal{P}^t$, $\mathcal{D}$, $\mathcal{K}$, $\mathcal{G}^x$, $\mathcal{P}^x$ and $\mathcal{I}$ of $\mathfrak{g}^{\rm ess}$ are, up to sign, $\mathfrak{Q}^{02}$, $2\mathfrak{Q}^{11} + \frac{1}{2}\mathfrak{Q}^{00}$, $\mathfrak{Q}^{20}$, $\mathfrak{Q}^{10}$, $\mathfrak{Q}^{01}$ and $\mathfrak{Q}^{00}$, respectively, see [103, Example 5.21]. By analogy with $\mathfrak{g}^{\rm ess}$, we can call Λ the essential algebra of generalized symmetries of the equation (1.2).

Lemma 1.22. *In the sense of unital algebras, the algebra* $\Upsilon_{\mathfrak{r}}$ *is isomorphic to the quotient algebra of the universal enveloping algebra* $\mathfrak{U}(\mathfrak{r})$ *of* $\mathfrak{r}$ *by the two-sided ideal* $(\iota(\mathcal{I})+1)$ *generated by* $\iota(\mathcal{I})+1$, $\Upsilon_{\mathfrak{r}} \simeq \mathfrak{U}(\mathfrak{r})/(\iota(\mathcal{I})+1)$, *where* $\iota\colon \mathfrak{r} \hookrightarrow \mathfrak{U}(\mathfrak{r})$ *is the canonical embedding of the Lie algebra* $\mathfrak{r}$ *in its universal enveloping algebra* $\mathfrak{U}(\mathfrak{r})$. *Moreover, this gives rise to an isomorphism between the associated Lie algebras* $\Upsilon_{\mathfrak{r}}^{(-)}$ *and* $\big(\mathfrak{U}(\mathfrak{r})/(\iota(\mathcal{I})+1)\big)^{(-)}$.

Consider the associative algebra $\Upsilon_{\mathfrak{r}}$ generated by the differential operators D_x and G. This algebra admits the following presentation by generators and relations:

$$\Upsilon_{\mathfrak{r}} = \big\langle \mathrm{D}_x, \mathrm{G} \mid [\mathrm{D}_x, \mathrm{G}] = \tfrac{1}{2} \big\rangle. \tag{1.8}$$

The application of Bergman's diamond lemma [11] to the algebra $\Upsilon_{\mathfrak{r}}$ with the presentation (1.8) leads to a description of the basis of $\Upsilon_{\mathfrak{r}}$.

Lemma 1.23. *Fixed any ordering* $(\mathrm{Q}^1, \mathrm{Q}^2)$ *of* $\{\mathrm{D}_x, \mathrm{G}\}$, $\mathrm{Q}^1 < \mathrm{Q}^2$, *the monomials of the form* $\mathbf{Q}^\alpha := (\mathrm{Q}^1)^{\alpha_1}(\mathrm{Q}^2)^{\alpha_2}$ *with* $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ *constitute a basis of the algebra* $\Upsilon_{\mathfrak{r}}$.

In view of the definition $\mathrm{W}(1,\mathbb{R}) := \mathfrak{U}\big(\mathrm{h}(1,\mathbb{R})\big)/(c-1)$ of the first Weyl algebra $\mathrm{W}(1,\mathbb{R})$ and the isomorphisms $\mathfrak{r} \simeq \mathrm{h}(1,\mathbb{R})$, $\mathfrak{U}(\mathfrak{r}) \simeq \mathfrak{U}(\mathrm{h}(1,\mathbb{R}))$ and $\mathrm{W}(n,\mathbb{R})^{\mathrm{op}} \simeq \mathfrak{U}\big(\mathrm{h}(n,\mathbb{R})\big)/(c+1)$, Lemma 1.22 gives the following.

Corollary 1.24. *The algebra* Λ *of the linear generalized symmetries of the equation* (1.2) *is isomorphic to the Lie algebra* $\mathrm{W}(1,\mathbb{R})^{(-)}$ *associated with the first Weyl algebra* $\mathrm{W}(1,\mathbb{R})$.

1.6.3 Generalized reductions and solution generation

Generalized symmetries can provide a framework for finding exact solutions of systems of differential equations, extending the classical methods based on Lie symmetries. This approach was first described in [120, Section 5]. However, the computations involving generalized symmetries are much more complicated than their counterparts involving Lie symmetries. This is why such usage of generalized symmetries is not quite common in the literature.

In analogy to constructing Lie reductions, consider a (finite-dimensional) subalgebra Θ of Σ with a basis $(\mathfrak{Q}_1, \dots, \mathfrak{Q}_r)$. The system $\mathcal{S}$ of differential constraints

$\mathfrak{Q}_1[u] = 0, \dots, \mathfrak{Q}_r[u] = 0$ is formally compatible with $\mathcal{L}$. However, unlike standard Lie reductions, there is no unified way to integrate the extended system $\mathcal{L} \cup \mathcal{S}$, and the deep analysis of the structure of the algebra Σ becomes essential in this context. Furthermore, since the point symmetry (pseudo)group G acts on generalized symmetries as well, only G-inequivalent subalgebras of Σ should be used for constructing solutions that are invariant with respect to generalized symmetries.

For linear systems $\mathcal{L}$, Proposition 1.20 shows us that $\mathrm{Q} \in \Upsilon$ if and only if $(\mathrm{Q}u)\partial_u \in \Lambda$. This implies that the function $\mathrm{Q}h$ is a solution of $\mathcal{L}$ whenever the function h is. This provides one more way for generating solutions of linear systems of differential equations, in addition to the standard solution generation via the action of the symmetry (pseudo)group.

To efficiently use this approach, one should definitely select only G-inequivalent elements of Υ, but even this selection does not guarantee nontrivial results. In addition, it is necessary to carefully analyze how particular elements of Υ act on known families of solutions of $\mathcal{L}$. For brevity, we refer to $\langle(\mathrm{Q}u)\partial_u\rangle$-invariant solutions of $\mathcal{L}$ as Q-*invariant solutions* of $\mathcal{L}$. Thus, a solution h of $\mathcal{L}$ is Q-invariant if and only if $\mathrm{Q}h = 0$.

It is clear that given an element Q of Υ, the action by any element from the principal left ideal $\Upsilon\mathrm{Q}$ on any Q-invariant solution of $\mathcal{L}$ results in the zero solution of $\mathcal{L}$. More generally, any element R of Υ with $\mathrm{QR} \in \Upsilon\mathrm{Q}$ maps the space of Q-invariant solutions of $\mathcal{L}$ to itself. In particular, this is the case for any element of $\Upsilon\mathrm{Q} + \mathrm{C}_\Upsilon(\mathrm{Q})$, i.e., for the sum of any elements of the principal left ideal $\Upsilon\mathrm{Q}$ and of the centralizer $\mathrm{C}_\Upsilon(\mathrm{Q})$ of Q in Υ. This is why the study of the algebraic structure of Υ is relevant in the context of generating solutions of $\mathcal{L}$ via acting by elements of Υ.

We will demonstrate the capabilities of this method in Chapter 2 by applying it to the fine Kolmogorov backward equation.

Chapter 2

Extended symmetry analysis of (1+2)-dimensional fine Kolmogorov backward equation

2.1 Introduction

Fokker–Planck and Kolmogorov equations are widely used to model various real-world phenomena in the broad range of sciences, including various branches of physics, biology and economics [42, 130]. For instance, Kolmogorov [68] suggested to describe the Brownian motion with ultraparabolic Fokker–Planck equations of the general form

$$u_t + xu_y = (f(t,x,y)u)_{xx} + (g(t,x,y)u)_x,$$

where f and g are arbitrary smooth functions depending on (t,x,y). He also constructed (up to the correcting multiplier $\frac{1}{4}$ [70]) the fundamental solutions of such equations with constant parameter functions f and g. The equations of the above form with $f = \text{const}$ and $g = 0$ appeared later for modeling processes of evolution of cell populations [131, Eq. 30] with constant diffusivity. The Fokker–Planck equations are also appreciated in modeling lasers and other nonlinear systems far from thermal equilibrium [130, Chapter 12]. Allowing the diffusivity to be a power function, one

derives a slight generalization,

$$u_t + xu_y = |x|^\beta u_{xx}, \tag{2.1}$$

where β is a real parameter. At the same time, these equations constitute a family of interesting objects for the study within the framework of group analysis of differential equations. We turn the collection of equations (2.1) into a class $\mathcal{F}$ by declaring β to be its only arbitrary element, i.e., the tuple of arbitrary elements of $\mathcal{F}$ is $\theta = (\beta)$, where β runs through the solution set of the system of equations $\beta_t = \beta_x = \beta_y = 0$. See [12, 17, 126, 146] and references therein for the definition and transformational properties of classes of differential equations. Symmetry properties of the class $\mathcal{F}$ were partially considered in [151]. Each equation of the form (2.1) belongs to the general class $\bar{\mathcal{F}}$ of $(1+2)$-dimensional ultraparabolic linear partial differential equations

$$u_t + B(t,x,y)u_y = A^2(t,x,y)u_{xx} + A^1(t,x,y)u_x + A^0(t,x,y)u + C(t,x,y)$$
$$\text{with} \quad A^2 \neq 0, \quad B_x \neq 0.$$

The tuple of arbitrary elements of the class $\bar{\mathcal{F}}$ is $\bar{\theta} := (B, A^2, A^1, A^0, C)$, where the components of $\bar{\theta}$ range through the solution set of the system of inequalities $A^2 \neq 0$ and $B_x \neq 0$ with no restrictions on the other parameter-functions. A partial preliminary group classification of the class $\bar{\mathcal{F}}$ was carried out in [35]. Some subclasses of the class $\bar{\mathcal{F}}$ and separate equations were considered within the Lie-symmetry framework in [45, 70, 76, 138–140].

The equations $\mathcal{F}_0$ and $\mathcal{F}_2$ of the form (2.1) with $\beta = 0$ and $\beta = 2$ are distinguished by their symmetry properties not only in the narrow class $\mathcal{F}$ but also in the very wide class $\bar{\mathcal{F}}$. More specifically, the essential Lie invariance algebra of the equation $\mathcal{F}_0$ is eight-dimensional, which is the maximum of such dimensions among equations from the class $\bar{\mathcal{F}}$. Moreover, possessing the essential Lie invariance algebra of dimension eight is an attribute of the equation $\mathcal{F}_0$ that singles it out, up to the point equivalence, within the class $\bar{\mathcal{F}}$. This is why the second equation in the class $\mathcal{F}$ with eight-dimensional essential Lie invariance algebra, $\mathcal{F}_5$, is necessarily equivalent to $\mathcal{F}_0$ with respect to point transformations. The essential Lie invariance algebra of the equation $\mathcal{F}_2$ is five-dimensional and nonsolvable, and modulo the point equivalence, it is the only equation from the class $\bar{\mathcal{F}}$ with this property. Within the class $\mathcal{F}$, there are only two equations with five-dimensional essential Lie invariance algebras, $\mathcal{F}_2$ and $\mathcal{F}_3$,

and they are similar with respect to a point transformation. Because of this, we refer to the equations (2.1) with $\beta = 0$ and $\beta = 2$ as the *remarkable Fokker–Planck equation* and the *fine Kolmogorov backward equation*, respectively. The equation $\mathcal{F}_0$ is the canonical form of the ultraparabolic Fokker–Planck equations with constant diffusion and drift coefficients f and g, whose study was initiated by Kolmogorov in [68] as mentioned above. Moreover, it is shown in [10] that the Kolmogorov equations describing the price of the geometric and arithmetic average strike call option are reduced by point transformations to Eqs. (3.4) and (4.5) [10], which coincide with $\mathcal{F}_0$ and $\mathcal{F}_2$ up to alternating the sign of y, respectively.

Due to its special form, the remarkable Fokker–Planck equation can also be considered as a Kolmogorov backward equation. Its extended symmetry analysis was carried out in [70], where we computed its point-symmetry pseudogroup using the direct method and analyzed its structure, which allowed us to list one- and two-dimensional subalgebras of its essential and maximal Lie invariance algebra with a view toward the classification of the Lie reductions and finding Lie invariant solutions. Moreover, we have carried out peculiar generalized reductions that allowed us to construct three wide families of the solutions of the remarkable Fokker–Planck equation parameterized by an arbitrary finite number of solutions of the linear (1+1)-dimensional heat equation.

The subject of the present paper is a similar analysis for the fine Kolmogorov backward equation,

$$u_t + xu_y = x^2 u_{xx}. \tag{2.2}$$

The paper is organized as follows. In Section 2.2 we describe the structure and important properties of the maximal and essential Lie invariance algebras $\mathfrak{g}$ and $\mathfrak{g}^{\rm ess}$ of the equation (2.2). Using an advanced version of the direct method, which involves the known explicit form of the (usual) equivalence pseudogroup of the class $\bar{\mathcal{F}}$ and its normalization in the usual sense, in Section 2.3 we compute the point-symmetry pseudogroup G of the equation (2.2). We thoroughly analyze the structure of the pseudogroup G and modify its multiplication operation. This allows us to single out its essential subgroup $G^{\rm ess}$ and properly classify the discrete symmetries of (2.2). Section 2.4 is devoted to the classifications of subalgebras of the algebra $\mathfrak{g}^{\rm ess}$ up to the equivalences generated by the inner automorphisms of this algebra and by the action

of the Lie group $G^{\rm ess}$ on $\mathfrak{g}^{\rm ess}$ as its Lie algebra. The latter classification forms a basis for exhaustively classifying the Lie reductions of (2.2) in Section 2.5. As a result, we construct wide families of Lie invariant solutions therein, two families in terms of the general solution of the (1+1)-dimensional linear heat equation and one family in terms of the general solution of the (1+1)-dimensional linear heat equations with inverse square potentials, where the potential coefficient is an additional parameter. These solutions are used in Section 2.6 as seed solutions when generating new solutions of the equation (2.2) from known ones by acting with the point symmetries of this equation and with its Lie-symmetry operators. To the best of our knowledge, this is the first systematic consideration of the latter action in the literature. The subject of Section 2.7 is given by peculiar generalized reductions and their relation to generating solutions by Lie-symmetry operators. One of the obtained families of solutions is parameterized by an arbitrary finite number of arbitrary solutions of the (1+1)-dimensional linear heat equation. The results of the paper are summarized in Section 2.8.

For readers' convenience, the constructed exact solutions of the equation (2.2) are marked by the bullet symbol $\bullet$.

2.2 Lie invariance algebra

The maximal Lie invariance algebra $\mathfrak{g}$ of (2.2) is spanned by the vector fields

$$\mathcal{P}^y=\partial_y,\quad \mathcal{D}=x\partial_x+y\partial_y,\quad \mathcal{K}=2xy\partial_x+y^2\partial_y-xu\partial_u,$$
$$\mathcal{P}^t=\partial_t,\quad \mathcal{I}=u\partial_u,\quad \mathcal{Z}(f)=f(t,x,y)\partial_x,$$

where the parameter-function f of (t,x,y) ranges through the solution set of (2.2). We compute $\mathfrak{g}$ as well as other invariance algebras in the present paper using the packages DESOLV [27] and Jets [9] for Maple; the latter package is based on algorithms developed in [90].

The set $\mathfrak{g}^{\rm lin}:=\{\mathcal{Z}(f)\}$ is an infinite-dimensional abelian ideal of the algebra $\mathfrak{g}$ associated with the linear superposition of solutions. The subalgebra $\mathfrak{g}^{\rm ess}$ is supplemental

to the ideal $\mathfrak{g}^{\rm lin}$ in $\mathfrak{g}$,

$$\mathfrak{g}^{\rm ess} = \langle \mathcal{P}^y, \mathcal{D}, \mathcal{K}, \mathcal{P}^t, \mathcal{I} \rangle, \tag{2.3}$$

therefore $\mathfrak{g}$ is a semidirect product $\mathfrak{g}^{\rm ess} \in \mathfrak{g}^{\rm lin}$.

Up to the skew-symmetry of the Lie bracket, the nontrivial commutation relations are

$$[\mathcal{P}^y, \mathcal{D}] = \mathcal{P}^y, \quad [\mathcal{P}^y, \mathcal{K}] = 2\mathcal{D}, \quad [\mathcal{D}, \mathcal{K}] = \mathcal{K}.$$

The algebra $\mathfrak{g}^{\rm ess}$ is nonsolvable since it contains a subalgebra $\mathfrak{f} = \langle \mathcal{P}^y, \mathcal{D}, \mathcal{K} \rangle$ that is isomorphic to the algebra $\mathrm{sl}(2, \mathbb{R})$. A complementary subalgebra to $\mathfrak{f}$ in $\mathfrak{g}^{\rm ess}$ is $\langle \mathcal{P}^t, \mathcal{I} \rangle$, which is the center $\mathfrak{z}$ of the algebra $\mathfrak{g}^{\rm ess}$. Hence $\mathfrak{g}^{\rm ess} = \mathfrak{f} \oplus \mathfrak{z}$. Therefore, the entire algebra $\mathfrak{g}^{\rm ess}$ is isomorphic to the algebra $\mathrm{sl}(2, \mathbb{R}) \oplus 2A_1$, where $2A_1$ is an abelian two-dimensional Lie algebra, see [125] for notation. Moreover, since $\mathrm{sl}_2(\mathbb{R}) \oplus A_1 \simeq \mathrm{gl}(2, \mathbb{R})$, we have the decomposition $\mathfrak{g}^{\rm ess} = \mathfrak{h} \oplus \mathfrak{p}$, where $\mathfrak{h} := \langle \mathcal{P}^y, \mathcal{D}, \mathcal{K}, \mathcal{I} \rangle \simeq \mathrm{gl}(2, \mathbb{R})$ and $\mathfrak{p} := \langle \mathcal{P}^t \rangle \simeq A_1$.

The Iwasawa decomposition for the algebra $\mathrm{sl}(2, \mathbb{R})$ gives another basis for $\mathfrak{g}^{\rm ess}$, which is more convenient from many prospectives for analysis than the chosen one. This basis is given by the tuple $(\mathcal{Q}^+, \mathcal{D}, \mathcal{P}^y, \mathcal{P}^t, \mathcal{I})$, where $\mathcal{Q}^\pm := \mathcal{P}^y \pm \mathcal{K}$.

2.3 Point symmetry pseudogroup

The equation (2.2) corresponds to the value $(x, x^2, 0, 0, 0) =: \bar\theta_2$ of the arbitrary-element tuple $\bar\theta = (B, A^2, A^1, A^0, C)$ of the class $\bar{\mathcal{F}}$. To find the point symmetry pseudogroup G of (2.2), we use the advanced version of the direct method based on the known transformational properties of the class $\bar{\mathcal{F}}$ that were presented in [70]. (A similar approach was first applied in [16] involving some results from [124]. For the standard version of the direct method, see [63].) In particular, the class $\bar{\mathcal{F}}$ is normalized in the usual sense, i.e., its equivalence groupoid coincides with the action groupoid of its usual equivalence pseudogroup $G^\sim_{\bar{\mathcal{F}}}$, which consists of the point transformations with the components [70, Theorem 1], see also Theorem 3.2 in Chapter 3,

$$\tilde t = T(t,y), \quad \tilde x = X(t,x,y), \quad \tilde y = Y(t,y), \quad \tilde u = U^1(t,x,y)u + U^0(t,x,y), \tag{2.4a}$$

$$\tilde A^0 = \frac{-U^1}{T_t+BT_y} E \frac{1}{U^1}, \quad \tilde A^1 = A^1 \frac{X_x}{T_t+BT_y} - \frac{X_t+BX_y}{T_t+BT_y} + A^2 \frac{X_{xx}-2X_xU^1_x/U^1}{T_t+BT_y}, \tag{2.4b}$$

$$\tilde A^2 = A^2 \frac{X_x^2}{T_t+BT_y}, \quad \tilde B = \frac{Y_t+BY_y}{T_t+BT_y}, \quad \tilde C = \frac{U^1}{T_t+BT_y}\left(C - E\frac{U^0}{U^1}\right). \tag{2.4c}$$

Here T, X, Y, U^1 and U^0 are arbitrary smooth functions of their arguments with the nondegeneracy condition $(T_tY_y - T_yY_t)X_xU^1 \neq 0$, and $E := \partial_t + B\partial_y - A^2\partial_{xx} - A^1\partial_x - A^0$. The normalization of $\bar{\mathcal{F}}$ implies that the latter groupoid necessarily contains the vertex group $\mathcal{G}^{\sim}_{\bar{\mathcal{F}}}(\bar\theta_2,\bar\theta_2)$ of (2.2), which is the set of all admissible transformations with $\bar\theta_2$ as both their target and source, $\mathcal{G}^{\sim}_{\bar{\mathcal{F}}}(\bar\theta_2,\bar\theta_2) = \{(\bar\theta_2,\Phi,\theta_2) \mid \Phi \in G\}$. These arguments allow us to use the description of $G^{\sim}_{\bar{\mathcal{F}}}$ for computing G.

Theorem 2.1. *The point symmetry pseudogroup G of the fine Kolmogorov backward equation (2.2) consists of the point transformations of the form*

$$\begin{aligned} &\tilde t = t + \lambda, \quad \tilde x = \frac{\alpha\delta-\beta\gamma}{(\gamma y+\delta)^2}x, \quad \tilde y = \frac{\alpha y+\beta}{\gamma y+\delta}, \\ &\tilde u = \sigma \exp\left(-\frac{\gamma}{\gamma y+\delta}x\right)\big(u + f(t,x,y)\big), \end{aligned} \tag{2.5}$$

where α, β, γ and δ are arbitrary constants with $\alpha\delta-\beta\gamma = \pm1$ that are defined up to simultaneously alternating their signs; λ and σ are arbitrary constants with $\sigma \neq 0$, and f is an arbitrary solution of the equation (2.2).

Proof. We integrate the system (2.4), where both source and target arbitrary-element tuples are equal to $\bar\theta_2$, with respect to the parameter-functions T, X, Y and U. The first two equations from (2.4c) take the form

$$X = \frac{Y_t+xY_y}{T_t+xT_y}, \quad X^2(T_t+xT_y) = (xX_x)^2.$$

In view of the first of these equations, the second one reduces to the equation

$$(Y_t+xY_y)^2(T_t+xT_y)^3 - x^2(Y_yT_t - Y_tT_y)^2 = 0,$$

whose left-hand side is a polynomial of x with coefficients depending on t and y. Collecting the coefficients of x^5 and x^0, we obtain $T_y^3Y_y^2 = T_t^3Y_t^2 = 0$, i.e., in view of the nondegeneracy condition, either $T_t = Y_y = 0$ or $T_y = Y_t = 0$. In the first case, collecting of the coefficients of x^3 results in the equation $T_yY_t = 0$, which contradicts the nondegeneracy condition. Hence the only possible case is $T_y = Y_t = 0$, where collecting the coefficients of x^2 leads to the equation $T_t^2Y_y^2(T_t - 1) = 0$. In view of the inequality $T_tY_y \neq 0$, we derive the equation $T_t = 1$, which integrates to $T = t + \lambda$, where λ is an arbitrary constant.

Moreover, $T_y = Y_t = 0$ together with the first equation from (2.4c) give us that $X = xY_y$. Therefore, $X_t = X_{xx} = 0$. Then the second equation in (2.4b) simplifies to $U^1_x/U^1 = -Y_{yy}/(2Y_y)$, and thus it integrates to

$$U^1 := V(t,y)\exp\left(-\frac{Y_{yy}}{2Y_y}x\right),$$

where V is a nonvanishing smooth function of (t,y), the explicit expression for which will be derived below. Substituting the obtained expression for U^1 into the first equation in (2.4b) results in the equation

$$x^2\left(\frac{U^1_x}{U^1}\right)^2 + \frac{U^1_t + xU^1_y}{U^1} = 0,$$

whose left-hand side is a quadratic polynomial in x. Collecting the coefficients of x^2, we get the equation $Y_{yyy}Y_y - \frac{3}{2}Y_{yy}^2 = 0$, meaning that the Schwarzian derivative of Y is zero. Therefore, Y is a linear fractional function of y, $Y = (\alpha y + \beta)/(\gamma y + \delta)$. Since the constants α, β, γ and δ are defined up to a nonzero constant multiplier, we can assume that $\alpha\delta - \beta\gamma = \pm 1$. Collecting the coefficients of x^1 and x^0, we obtain the equations $V_y = V_t = 0$. Hence $V =: \sigma \in \mathbb{R}\setminus\{0\}$.

Finally, the third equation in (2.4c) takes the form

$$\left(\frac{U^0}{U^1}\right)_t + x\left(\frac{U^0}{U^1}\right)_y = x^2\left(\frac{U^0}{U^1}\right)_{xx},$$

and thus $U^0 = U^1 f$, where $f = f(t,x,y)$ is an arbitrary solution of (2.2).

Clearly, the set of transformations of the form (2.5) is closed under the composition, therefore G is a pseudogroup. Moreover, it is a Lie pseudogroup. □

Recall that solutions of (2.2) are considered locally (see definition at the end of Section 1.1), for instance, they may have different domains and domain restrictions. The natural maximal domain of a point transformation of the form (2.5) coincides with the relative complement of $M_{\gamma\delta} := \{(t,x,y,u) \in \mathbb{R}^4 \mid \gamma y+\delta = 0\}$ with respect to the set $\operatorname{dom} f \times \mathbb{R}_u$. Since $(\gamma,\delta) \neq (0,0)$, the set $M_{\gamma\delta}$ is either hyperplane in $\mathbb{R}^4_{t,x,y,u}$ defined by $t = -\delta/\gamma$ if $\gamma \neq 0$ or the empty set otherwise. To properly interpret the algebraic structure on the set of such transformations, we consider them on natural maximal domains together with all their restrictions as partial functions defined on the entire space $\mathbb{R}^4_{t,x,y,u}$ rather than functions defined on subsets of this space. Considering the composition of these partial functions as the operation on G, we endow G with the *pseudogroup* structure in the sense of Definition 1.8, which is a particular case of the more general algebraic structure called *inverse monoid*, see Section 1.3.1.

Since the y-component of any element of G is a linear fractional function in y, we follow the papers [70,74] and take the *modified transformation composition* $\Phi_1 \circ^{\rm m} \Phi_2$ of $\Phi_1, \Phi_2 \in G$ as the operation in G. We first consider transformations Φ_1 and Φ_2 of the form (2.5) having maximal domains for this form. The domain of their composition $\Phi_1 \circ \Phi_2 =: \tilde\Phi$ is standardly defined as the preimage of the domain of Φ_1 under Φ_2,

$$\operatorname{dom}\tilde\Phi = \Phi_2^{-1}(\operatorname{dom}\Phi_1) = (\operatorname{dom}\tilde f \times \mathbb{R}_u) \setminus (M_{\gamma_2\delta_2} \cup M_{\tilde\gamma\tilde\delta}),$$

where $\tilde\gamma = \gamma_1\alpha_2 + \delta_1\gamma_2$, $\tilde\delta = \gamma_1\beta_2 + \delta_1\delta_2$, $\operatorname{dom}\tilde f = \big((\pi_*\Phi_2)^{-1}\operatorname{dom} f^1\big) \cap \operatorname{dom} f^2$ with the natural projection $\pi\colon \mathbb{R}^4_{t,x,y,u} \twoheadrightarrow \mathbb{R}^3_{t,x,y}$, and the parameters with indices 1 and 2 and tildes correspond Φ_1, Φ_2 and $\tilde\Phi$, respectively. Modifying the composition $\Phi_1 \circ \Phi_2$ to $\Phi_1 \circ^{\rm m} \Phi_2$, we continuously extend $\Phi_1 \circ \Phi_2$ to the set

$$\operatorname{dom}^{\rm m}\tilde\Phi := (\operatorname{dom}\tilde f \times \mathbb{R}_u) \setminus M_{\tilde\gamma\tilde\delta},$$

i.e., the modified composition $\Phi_1 \circ^{\rm m} \Phi_2$ is the transformation of the form (2.5) with the same parameters as in $\Phi_1 \circ \Phi_2$ and with the natural domain, $\operatorname{dom}(\Phi_1 \circ^{\rm m} \Phi_2) = \operatorname{dom}^{\rm m}\tilde\Phi$. In other words, we redefine $\Phi_1 \circ \Phi_2$ on the set $(\operatorname{dom}\tilde f \times \mathbb{R}_u) \cap M_{\gamma_2\delta_2}$ if $\gamma_1\gamma_2 \neq 0$; otherwise $\operatorname{dom}^{\rm m}\tilde\Phi = \operatorname{dom}\tilde\Phi$ and the extension is trivial.

If transformations Φ_1 and Φ_2 from G are defined on open subsets of maximal domains of the corresponding forms (2.5), then we also redefine the domain of their composition in a natural way. More specifically, the set $\operatorname{dom}(\Phi_1 \circ \Phi_2) = \Phi_2^{-1}(\operatorname{dom}\Phi_1)$

is contained in $(\operatorname{dom}\tilde f\times\mathbb{R}_u)\setminus(M_{\gamma_2\delta_2}\cup M_{\tilde\gamma\tilde\delta})$ and if $\gamma_1\gamma_2\neq0$ as the modified composition $\Phi_1\circ^{\rm m}\Phi_2$ we take the continuous extension of $\Phi_1\circ\Phi_2$ to the set

$$\operatorname{int}\Big(\Phi_2^{-1}(\operatorname{dom}\Phi_1)\cup\big(\operatorname{cl}\big(\Phi_2^{-1}(\operatorname{dom}\Phi_1)\big)\cap M_{\gamma_2\delta_2}\cap(\operatorname{dom}\tilde f\times\mathbb{R}_u)\big)\Big).$$

In other words, we complete the set $\operatorname{dom}(\Phi_1\circ\Phi_2)$ by its limit points from $M_{\gamma_2\delta_2}$ that belong to the interior of the completed set. If $\gamma_1\gamma_2=0$ we simply set $\Phi_1\circ^{\rm m}\Phi_2=\Phi_1\circ\Phi_2$, thus $\operatorname{dom}\Phi_1\circ^{\rm m}\Phi_2=\Phi_2^{-1}(\operatorname{dom}\Phi_1)$. By this, it is clear that $\Phi_1\circ^m\Phi_2\in G$ for arbitrary $\Phi_1,\Phi_2\in G$, thus the set G with the modified composition $\circ^{\rm m}$ is indeed a pseudogroup, and, moreover, a Lie pseudogroup.

This allows us to appropriately describe the structure of the pseudogroup G. The set $G^{\rm lin}$ constituted by the point transformations of the form

$$\mathcal{Z}(f)\colon\quad \tilde t=t,\quad \tilde x=x,\quad \tilde y=y,\quad \tilde u=u+f(t,x,y),$$

where the function f of (t,x,y) ranges through the solution set of (2.2), is a normal pseudosubgroup of G; moreover, the pseudogroup G splits over $G^{\rm lin}$, $G=G^{\rm ess}\ltimes G^{\rm lin}$, see Section 1.3.1 for all the necessary definitions.

Here $G^{\rm ess}$ is the subgroup of G constituted by the transformations of the form (2.5) with $f=0$ and the natural domain of definition, and thus it is a five-dimensional Lie group. We call $G^{\rm ess}$ the *essential point symmetry group* of the equation (2.2). Under the modified composition, $G^{\rm ess}$ is equipped with a well-defined binary operation, enabling us to analyze its Lie group structure. For instance, we can identify the subgroup of linear fractional transformations and the subgroup of discrete symmetries of $G^{\rm ess}$ and correctly interpret them.

The pushforward π_*G of the pseudogroup G, where $\pi\colon\mathbb{R}^4_{t,x,y,u}\to\mathbb{R}_y$ denotes the natural projection, coincides with the pseudogroup of Möbius transformations of the real line. At the same time, Möbius transformations with their natural domains constitute a group with respect to the modified composition as the group operation [83]. In this sense, the above consideration extends the approach described in [83] to more general transformation groups like $G^{\rm ess}$ and analogous groups in [70, 74].

The group $G^{\rm ess}$ itself is the direct product, $G^{\rm ess}=F\times Z$, of its normal subgroups F and Z that are singled out by the constraints $\lambda=0$, $\sigma=1$ and $\alpha=\delta=1$, $\beta=\gamma=0$ and are isomorphic, as Lie groups, to the groups $\mathrm{PSL}^{\pm}(2,\mathbb{R})$

and $(\mathbb{R}^2,+)\times\mathbb{Z}_2$, respectively. Here $(\mathbb{R}^2,+)$ is the real two-dimensional connected torsion-free abelian Lie group. Recall that $\mathrm{PSL}^{\pm}(2,\mathbb{R})$ is by definition the quotient group $\mathrm{SL}^{\pm}(2,\mathbb{R})/\mathrm{Z}(\mathrm{SL}^{\pm}(2,\mathbb{R}))$, where $\mathrm{Z}(\mathrm{SL}^{\pm}(2,\mathbb{R}))=\{\pm\mathrm{diag}(1,1)\}$ is the center of the group $\mathrm{SL}^{\pm}(2,\mathbb{R})$, which coincides with the center of the group $\mathrm{SL}(2,\mathbb{R})$. The above isomorphisms are established by the correspondences

$$\varrho_1=\pm(\alpha,\beta,\gamma,\delta)_{\alpha\delta-\beta\gamma=\pm1}\mapsto\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}\mathrm{Z}(\mathrm{SL}(2,\mathbb{R})),$$
$$(\lambda,\sigma)_{\sigma>0}\mapsto(\lambda,\ln|\sigma|,\bar{0})\quad\text{and}\quad(\lambda,\sigma)_{\sigma<0}\mapsto(\lambda,\ln|\sigma|,\bar{1}),$$

where $\bar{0}:=2\mathbb{Z}$ and $\bar{1}:=1+2\mathbb{Z}$ constitute the group $\mathbb{Z}_2:=\mathbb{Z}/2\mathbb{Z}$. Summing up, we have that

$$G^{\mathrm{ess}}\simeq\mathrm{PSL}^{\pm}(2,\mathbb{R})\times(\mathbb{R}^2,+)\times\mathbb{Z}_2.$$

This decomposition corresponds to the Lie algebra decomposition $\mathfrak{g}^{\mathrm{ess}}=\mathfrak{f}\oplus\mathfrak{r}\simeq\mathrm{sl}(2,\mathbb{R})\oplus 2A_1$.

There is another factorization of the group G^{ess}. It contains a normal subgroup H, which is singled out by the constraints $\lambda=0$ and $\sigma>0$ and is isomorphic to the Lie group $\mathrm{GL}(2,\mathbb{R})$. The group-complement to H in the group G is also a normal subgroup P of G. It is singled out by the constraints $\sigma=\pm1$, $\alpha=\delta=1$ and $\beta=\gamma=0$ and is isomorphic to $(\mathbb{R},+)\times\mathbb{Z}_2$. The isomorphisms between H and $\mathrm{GL}(2,\mathbb{R})$ and between P and $(\mathbb{R},+)\times\mathbb{Z}_2$ are established by the correspondences

$$\varrho_2=(\alpha,\beta,\gamma,\delta,\sigma)_{\alpha\delta-\beta\gamma=\pm1,\sigma>0}\mapsto\sigma\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix},$$
$$(\lambda,\sigma)_{\sigma=\pm1}\mapsto\begin{cases}(\lambda,\bar{1})\text{ if }\sigma=-1,\\ (\lambda,\bar{0})\text{ if }\sigma=1.\end{cases}$$

In other words, the group G^{ess} admits the following decomposition:

$$G^{\mathrm{ess}}=H\times P\simeq\mathrm{GL}(2,\mathbb{R})\times(\mathbb{R},+)\times\mathbb{Z}_2.$$

The counterpart of this isomorphism in terms of Lie algebras is the Lie algebra decomposition $\mathfrak{g}^{\mathrm{ess}}=\mathfrak{h}\oplus\mathfrak{p}$, given in Section 2.2.

Note that the former and the latter decompositions of $G^{\rm ess}$ are consistent. Recall that the group $\mathrm{GL}(2,\mathbb{R})$ is isomorphic to $\mathrm{PGL}(2,\mathbb{R})\times(\mathbb{R},+)$, where $\mathrm{PGL}(2,\mathbb{R})$ is the rank-two real projective linear group $\mathrm{PGL}(2,\mathbb{R}):=\mathrm{GL}(2,\mathbb{R})/\mathrm{Z}(\mathrm{GL}(2,\mathbb{R}))$. Here the center $\mathrm{Z}(\mathrm{GL}(2,\mathbb{R}))$ of the group $\mathrm{GL}(2,\mathbb{R})$ coincides with the set of scalar matrices, $\mathrm{Z}(\mathrm{GL}(2,\mathbb{R}))=\{\lambda\,\mathrm{diag}(1,1)\mid\lambda\in\mathbb{R}^\times\}$, and as a Lie group it is isomorphic to the group $(\mathbb{R},+)\times\mathbb{Z}_2$. It is clear that $\mathrm{PGL}(2,\mathbb{R})\simeq\mathrm{PSL}^{\pm}(2,\mathbb{R})$, therefore $\mathrm{PSL}^{\pm}(2,\mathbb{R})\times(\mathbb{R},+)\simeq\mathrm{GL}(2,\mathbb{R})$.

The one-parameter subgroups of the group $G^{\rm ess}$ that are generated by the chosen basis elements of the algebra $\mathfrak{g}^{\rm ess}$, see (2.3), are of the form

$$
\begin{aligned}
&\mathcal{P}^y(\epsilon)\colon\ \tilde t=t,\quad &&\tilde x=x,\quad &&\tilde y=y+\epsilon,\quad &&\tilde u=u,\\
&\mathcal{D}(\epsilon)\colon\ \tilde t=t,\quad &&\tilde x=\mathrm{e}^{\epsilon}x,\quad &&\tilde y=\mathrm{e}^{\epsilon}y,\quad &&\tilde u=u,\\
&\mathcal{K}(\epsilon)\colon\ \tilde t=t,\quad &&\tilde x=\frac{x}{(1-\epsilon y)^2},\quad &&\tilde y=\frac{y}{1-\epsilon y},\quad &&\tilde u=u\mathrm{e}^{\frac{\epsilon x}{\epsilon y-1}},\\
&\mathcal{P}^t(\epsilon)\colon\ \tilde t=t+\epsilon,\quad &&\tilde x=x,\quad &&\tilde y=y,\quad &&\tilde u=u,\\
&\mathcal{I}(\epsilon)\colon\ \tilde t=t,\quad &&\tilde x=x,\quad &&\tilde y=y,\quad &&\tilde u=\mathrm{e}^{\epsilon}u
\end{aligned}
$$

with constant ϵ playing the role of group parameter. The one-parameter subgroup generated by the vector field $\mathcal{Q}^+=\mathcal{P}^t+\mathcal{K}$, which stems from the Iwasawa decomposition for the algebra $\mathrm{sl}(2,\mathbb{R})$, consists of the transformations

$$
\begin{aligned}
&\mathcal{Q}^+(\epsilon)\colon\ \tilde t=t,\quad \tilde x=\frac{x}{(\cos\epsilon-y\sin\epsilon)^2},\quad \tilde y=\frac{\sin\epsilon+y\cos\epsilon}{\cos\epsilon-y\sin\epsilon},\\
&\qquad\tilde u=u\exp\left(\frac{x\sin\epsilon}{y\sin\epsilon-\cos\epsilon}\right)
\end{aligned}
$$

parameterized by $\epsilon\in\mathbb{R}$ as well. This subgroup is distinguished among the others by its peculiar action on the Lie algebra $\mathfrak{g}^{\rm ess}$.

Corollary 2.16 from Section 2.A implies that the subgroup F is the semidirect product of its subgroups $F_{\rm id}$ and $F_{\rm d}$, $F=F_{\rm id}\rtimes F_{\rm d}$, where the former is singled out in F by the constraints $\alpha\delta-\beta\gamma=1$, and the latter is generated by the transformation $(\tilde t,\tilde x,\tilde y,\tilde u)=(t,-x,-y,u)$. It is also clear that the group Z is the direct product of its subgroups $Z_{\rm id}$ and $Z_{\rm d}$, where $Z_{\rm id}$ is singled out in Z by $\sigma>0$, and $Z_{\rm d}$ is generated by the transformation $(\tilde t,\tilde x,\tilde y,\tilde u)=(t,x,y,-u)$. Moreover, due to the isomorphisms $F_{\rm id}\simeq\mathrm{PSL}(2,\mathbb{R})$ and $Z_{\rm id}\simeq(\mathbb{R}^2,+)$ we have that the groups $F_{\rm id}$ and $Z_{\rm id}$ are connected.

Corollary 2.2. *The center of the group $G^{\rm ess}$ coincides with the normal subgroup Z,*

$\mathrm{Z}(G^{\rm ess}) = Z$.

Proof. Obviously $Z \subset \mathrm{Z}(G^{\rm ess})$. Since $G^{\rm ess}/Z \simeq \mathrm{PSL}^{\pm}(2,\mathbb{R})$, then $\mathrm{Z}(G^{\rm ess}/Z) \simeq \mathrm{Z}(\mathrm{PSL}^{\pm}(2,\mathbb{R})) = \{\pm\mathrm{diag}(1,1)\}$, which is the identity element of $\mathrm{PSL}^{\pm}(2,\mathbb{R})$. This means that the group $\mathrm{Z}(G^{\rm ess}/Z)$ is trivial. Observing that $\mathrm{Z}(G^{\rm ess})/Z \subset \mathrm{Z}(G^{\rm ess}/Z)$ we deduce $\mathrm{Z}(G^{\rm ess}) \subset Z$. □

Corollary 2.3. *The quotient group of $G^{\rm ess}$ by its identity component $G^{\rm ess}_{\rm id}$ is isomorphic to the Klein four-group $\mathbb{Z}_2^2$.*

Proposition 2.4. *A complete list of discrete point symmetry transformations of the fine Kolmogorov backward equation (2.2) that are independent up to combining with each other and with continuous point symmetry transformations of this equation is exhausted by the involution $\mathcal{I}'$ alternating the sign of u, the involution $\mathcal{J}'$ simultaneously alternating the signs of x and y,*

$$\mathcal{I}' := (t,x,y,u) \mapsto (t,x,y,-u), \quad \mathcal{J}' := (t,x,y,u) \mapsto (t,-x,-y,u),$$

and their composition. Thus, the group of discrete symmetries of (2.2) is isomorphic to the Klein four-group $\mathbb{Z}_2^2$.

Proof. Clearly, the entire pseudosubgroup $G^{\rm lin}$ is contained in the connected component of the identity transformation in G. Since both $F_{\rm id}$ and $Z_{\rm id}$ are connected, then without loss of generality we can assume that the discrete symmetry transformations constitute the group $F_{\rm d} \times Z_{\rm d}$, and as the generators of it the transformations $\mathcal{I}'$ and $\mathcal{J}'$ can be chosen. □

2.4 Classification of subalgebras

Subalgebras of a Lie algebra $\mathfrak{g}$ can be classified up to various equivalence relations. However, from the algebraic perspective, the most interesting and useful is the classification modulo the action of the inner automorphism group $\mathrm{Inn}(\mathfrak{g})$ of $\mathfrak{g}$. Moreover, when $\mathfrak{g}$ is a real (resp. complex) finite-dimensional Lie algebra, this action coincides with the adjoint action of the corresponding real (resp. complex) simply connected Lie group on $\mathfrak{g}$. Another natural equivalence for the subalgebra classification is generated

by the action of the entire automorphism group $\mathrm{Aut}(\mathfrak{g})$ of $\mathfrak{g}$. Note that it often happens that there is no big difference between the above two equivalences or, in more precise algebraic terms, $\mathrm{Inn}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g})$ or at least the index of $\mathrm{Inn}(\mathfrak{g})$ as a subgroup of $\mathrm{Aut}(\mathfrak{g})$ is finite and small.

In group analysis of differential equations, the classification of subalgebras of maximal (resp. essential) Lie invariance algebra $\mathfrak{g}$ of a system of differential equations $\mathcal{L}$ is required in the course of classifying its Lie reductions for the purpose of constructing group-invariant solutions. Since these solutions are naturally considered modulo their similarity, i.e., the equivalence generated by the action of the entire (resp. essential) point symmetry group G on solution set of the system $\mathcal{L}$, the most relevant subalgebra classification here is that up to the G-equivalence of subalgebras of $\mathfrak{g}$. If the group G is not connected, this classification may differ from the classification modulo the $\mathrm{Inn}(\mathfrak{g})$-equivalence.

For these reasons, we carry out the classification of subalgebras of the algebra $\mathfrak{g}^{\rm ess}$ up to the $G^{\rm ess}_{\rm id}$-equivalence, which coincides with the $\mathrm{Inn}(\mathfrak{g}^{\rm ess})$-equivalence, and then construct a complete list of $G^{\rm ess}$-inequivalent subalgebras of $\mathfrak{g}^{\rm ess}$. Since the group $G^{\rm ess}$ is generated by continuous symmetries $\mathcal{P}^y(\epsilon)$, $\mathcal{D}(\epsilon)$, $\mathcal{K}(\epsilon)$, $\mathcal{P}^t(\epsilon)$ and $\mathcal{I}(\epsilon)$, where $\epsilon \in \mathbb{R}$, together with the discrete symmetries $\mathcal{I}'$ and $\mathcal{J}'$, the action of $G^{\rm ess}$ on $\mathfrak{g}^{\rm ess}$ is completely defined by the actions of these generators on the chosen basis elements of $\mathfrak{g}^{\rm ess}$. The only nonidentity actions among them are the following:

$$
\begin{aligned}
&\mathcal{P}^y(\epsilon)_*\mathcal{D} = \mathcal{D} - 2\epsilon\mathcal{P}^y, && \mathcal{K}(\epsilon)_*\mathcal{D} = \mathcal{D} + 2\epsilon\mathcal{K}, && \mathcal{D}(\epsilon)_*\mathcal{P}^y = e^{2\epsilon}\mathcal{P}^y,\\
&\mathcal{P}^y(\epsilon)_*\mathcal{K} = \mathcal{K} - \epsilon\mathcal{D} + \epsilon^2\mathcal{P}^y, && \mathcal{K}(\epsilon)_*\mathcal{P}^y = \mathcal{P}^y + \epsilon\mathcal{D} + \epsilon^2\mathcal{K}, && \mathcal{D}(\epsilon)_*\mathcal{K} = e^{-2\epsilon}\mathcal{K},\\
&\mathcal{J}'_*\mathcal{P}^y = -\mathcal{P}^y, && \mathcal{J}'_*\mathcal{K} = -\mathcal{K}.
\end{aligned}
$$

It is also useful to consider the action of the one-parameter subgroup $\{\mathcal{Q}^+(\epsilon) \mid \epsilon \in \mathbb{R}\}$ on $\mathfrak{g}^{\rm ess}$,

$$
\begin{aligned}
&\mathcal{Q}^+(\epsilon)_*\mathcal{P}^y = \mathcal{K}\sin^2\epsilon + \mathcal{D}\sin 2\epsilon + \mathcal{P}^y\cos^2\epsilon,\\
&\mathcal{Q}^+(\epsilon)_*\mathcal{D} = \tfrac{1}{2}\mathcal{K}\sin 2\epsilon + \mathcal{D}\cos 2\epsilon - \tfrac{1}{2}\mathcal{P}^y\sin 2\epsilon,\\
&\mathcal{Q}^+(\epsilon)_*\mathcal{K} = \mathcal{K}\cos^2\epsilon - \mathcal{D}\sin 2\epsilon + \mathcal{P}^y\sin^2\epsilon.
\end{aligned}
$$

Lemma 2.5. *A complete list of $G^{\rm ess}_{\rm id}$-inequivalent proper subalgebras of the algebra $\mathfrak{g}^{\rm ess}$ is exhausted by the following subalgebras, where $\delta \in \{-1, 0, 1\}$, $\varepsilon = \pm 1$, $\mu, \mu' \in \mathbb{R}$,*

$\nu \gtrless 0$ and $\nu' > 0$:

$$
\begin{aligned}
&\text{1D}\colon\quad \mathfrak{s}^{\mu}_{1.1} = \langle \mathcal{P}^t + \mu\mathcal{I}\rangle,\quad \mathfrak{s}_{1.2} = \langle \mathcal{I}\rangle,\quad \mathfrak{s}^{\varepsilon,\mu}_{1.3} = \langle \mathcal{P}^y + \varepsilon\mathcal{P}^t + \mu\mathcal{I}\rangle,\\
&\qquad \mathfrak{s}^{\delta}_{1.4} = \langle \mathcal{P}^y + \delta\mathcal{I}\rangle,\quad \mathfrak{s}^{\nu',\mu}_{1.5} = \langle \mathcal{D} + \nu'\mathcal{P}^t + \mu\mathcal{I}\rangle,\quad \mathfrak{s}^{\nu}_{1.6} = \langle \mathcal{D} + \nu\mathcal{I}\rangle,\\
&\qquad \mathfrak{s}^{\mu,\mu'}_{1.7} = \langle \mathcal{P}^y + \mathcal{K} + \mu\mathcal{P}^t + \mu'\mathcal{I}\rangle,\\
&\text{2D}\colon\quad \mathfrak{s}_{2.1} = \langle \mathcal{P}^t, \mathcal{I}\rangle,\quad \mathfrak{s}^{\delta,\mu}_{2.2} = \langle \mathcal{P}^y + \delta\mathcal{I}, \mathcal{P}^t + \mu\mathcal{I}\rangle,\quad \mathfrak{s}^{\delta}_{2.3} = \langle \mathcal{P}^y + \delta\mathcal{P}^t, \mathcal{I}\rangle,\\
&\qquad \mathfrak{s}^{\nu,\mu}_{2.4} = \langle \mathcal{D} + \nu\mathcal{I}, \mathcal{P}^t + \mu\mathcal{I}\rangle,\quad \mathfrak{s}^{\nu}_{2.5} = \langle \mathcal{D} + \nu\mathcal{P}^t, \mathcal{I}\rangle,\\
&\qquad \mathfrak{s}^{\mu,\mu'}_{2.6} = \langle \mathcal{P}^y + \mathcal{K} + \mu\mathcal{I}, \mathcal{P}^t + \mu'\mathcal{I}\rangle,\quad \mathfrak{s}^{\mu}_{2.7} = \langle \mathcal{P}^y + \mathcal{K} + \mu\mathcal{P}^t, \mathcal{I}\rangle,\\
&\qquad \mathfrak{s}^{\nu',\mu}_{2.8} = \langle \mathcal{P}^y, \mathcal{D} + \nu'\mathcal{P}^t + \mu\mathcal{I}\rangle,\quad \mathfrak{s}^{\nu}_{2.9} = \langle \mathcal{P}^y, \mathcal{D} + \nu\mathcal{I}\rangle,\\
&\text{3D}\colon\quad \mathfrak{s}_{3.1} = \langle \mathcal{P}^y, \mathcal{P}^t, \mathcal{I}\rangle,\quad \mathfrak{s}_{3.2} = \langle \mathcal{D}, \mathcal{P}^t, \mathcal{I}\rangle,\quad \mathfrak{s}_{3.3} = \langle \mathcal{P}^y + \mathcal{K}, \mathcal{P}^t, \mathcal{I}\rangle,\\
&\qquad \mathfrak{s}^{\nu,\mu}_{3.4} = \langle \mathcal{P}^y, \mathcal{D} + \nu\mathcal{I}, \mathcal{P}^t + \mu\mathcal{I}\rangle,\quad \mathfrak{s}^{\nu}_{3.5} = \langle \mathcal{P}^y, \mathcal{D} + \nu\mathcal{P}^t, \mathcal{I}\rangle,\\
&\qquad \mathfrak{s}_{3.6} = \langle \mathcal{P}^y, \mathcal{D}, \mathcal{K}\rangle,\\
&\text{4D}\colon\quad \mathfrak{s}_{4.1} = \langle \mathcal{P}^y, \mathcal{D}, \mathcal{P}^t, \mathcal{I}\rangle,\quad \mathfrak{s}^{\mu}_{4.2} = \langle \mathcal{P}^y, \mathcal{D}, \mathcal{K}, \mathcal{P}^t + \mu\mathcal{I}\rangle,\quad \mathfrak{s}_{4.3} = \langle \mathcal{P}^y, \mathcal{D}, \mathcal{K}, \mathcal{I}\rangle.
\end{aligned}
$$

Proof. Recall that the algebra $\mathfrak{g}^{\rm ess}$ can be decomposed into the direct sum of its Levi factor $\mathfrak{f}$ and radical $\mathfrak{z}$, $\mathfrak{g}^{\rm ess} = \mathfrak{f} \oplus \mathfrak{z}$, where $\mathfrak{f} \simeq \mathrm{sl}(2,\mathbb{R})$ and $\mathfrak{z} \simeq 2A_1$, see Section 2.2. Moreover, $G^{\rm ess}_{\rm id} = F_{\rm id} \times Z_{\rm id}$ and $Z_{\rm id}$ is the center of $G^{\rm ess}_{\rm id}$, and thus the actions of $G^{\rm ess}_{\rm id}$ on $\mathfrak{z}$ and of $Z_{\rm id}$ on $\mathfrak{g}^{\rm ess}$ are trivial. To classify subalgebras of $\mathfrak{g}^{\rm ess}$ up to the $G^{\rm ess}_{\rm id}$-equivalence, we use the Goursat method, see [115] for the first presentation of this method and [150] for further details.

Since the algebras $\mathfrak{f}$ and $\mathfrak{z}$ are the realizations of the algebras $\mathrm{sl}(2,\mathbb{R})$ and $2A_1$, respectively, optimal lists of their proper subalgebras are as follows:

$$
\begin{aligned}
&\mathfrak{f}\colon\quad \langle \mathcal{P}^y\rangle,\quad \langle \mathcal{D}\rangle,\quad \langle \mathcal{P}^y + \mathcal{K}\rangle,\quad \langle \mathcal{P}^y, \mathcal{K}\rangle;\\
&\mathfrak{z}\colon\quad \langle \mathcal{P}^t + \mu\mathcal{I}\rangle_{\mu\in\mathbb{R}},\quad \langle \mathcal{I}\rangle.
\end{aligned}
$$

Let $\pi_{\mathfrak{f}}$ and $\pi_{\mathfrak{z}}$ denote the natural projections with respect to the vector space decomposition $\mathfrak{g}^{\rm ess} = \mathfrak{f} \dotplus \mathfrak{z}$. For any subalgebra $\mathfrak{s} \subset \mathfrak{g}$, we can set $\pi_{\mathfrak{f}}\mathfrak{s} \in \{\langle 0\rangle, \langle \mathcal{P}^y\rangle, \langle \mathcal{D}\rangle, \langle \mathcal{P}^y + \mathcal{K}\rangle, \langle \mathcal{P}^y, \mathcal{K}\rangle, \mathfrak{f}\}$, modulo the $F_{\rm id}$-equivalences. In addition, we have that $\pi_{\mathfrak{z}}\mathfrak{s}, \mathfrak{s}\cap\mathfrak{z} \in \{\{0\}, \langle \mathcal{P}^t + \mu\mathcal{I}\rangle_{\mu\in\mathbb{R}}, \langle \mathcal{I}\rangle, \mathfrak{z}\}$. It is clear that for any subalgebra $\mathfrak{s} \subset \mathfrak{g}^{\rm ess}$ the following values are invariants under the action of $G^{\rm ess}_{\rm id}$: $\dim\mathfrak{s}$, $[\pi_{\mathfrak{f}}\mathfrak{s}]_{F_{\rm id}}$, $\pi_{\mathfrak{z}}\mathfrak{s}$ and $\mathfrak{s}\cap\mathfrak{z}$, where $[\pi_{\mathfrak{f}}\mathfrak{s}]_{F_{\rm id}}$ denotes the equivalence class of $\pi_{\mathfrak{f}}\mathfrak{s}$ under the $F_{\rm id}$-equivalence. This means that

any two subalgebras with different corresponding tuples of the invariants introduced above cannot be equivalent. Let us denote the quadruple $\dim\mathfrak{s}$, $\dim\pi_{\mathfrak{f}}\mathfrak{s}$, $\dim\mathfrak{s}\cap\mathfrak{z}$ and $\dim\pi_{\mathfrak{z}}\mathfrak{s}$ by n, $\hat{n}$, $\check{n}$ and m, respectively. It is easy to see that $n=\hat{n}+\check{n}\leqslant\hat{n}+m$, and $\max(0,m-\hat{n})\leqslant\check{n}\leqslant m\leqslant 2$.

We use the above invariant objects for splitting the consideration into inequivalent cases.

$\hat{n}=0$. This means that $\mathfrak{s}\subseteq\mathfrak{z}$, and thus the subalgebra $\mathfrak{s}$ is equivalent to a subalgebra from the list of inequivalent subalgebras of $\mathfrak{z}$. Therefore, we obtain the subalgebras $\mathfrak{s}_{1.1}^{\mu}$, $\mathfrak{s}_{1.2}$, and $\mathfrak{s}_{2.1}=\mathfrak{z}$.

Before proceeding with classifying subalgebras $\mathfrak{s}$ with $\hat{n}=1$, we note several simple observations. Since $\hat{n}=1$, we have $n\leqslant 3$.

(i) If $n=1$, then $\check{n}=0$, i.e., $\mathfrak{s}\cap\mathfrak{z}=\{0\}$.

(ii) If $n=2$, then $\check{n}=1$, and hence $\mathfrak{s}\cap\mathfrak{z}$ is a proper subalgebra of $\mathfrak{s}$. Moreover, if in addition $m=1$, then $\mathfrak{s}\cap\mathfrak{z}=\pi_{\mathfrak{z}}\mathfrak{s}$, which in its turn implies $\mathfrak{s}=\pi_{\mathfrak{f}}\mathfrak{s}\oplus\pi_{\mathfrak{z}}\mathfrak{s}$.

(iii) If $n=3$, then $\check{n}=m=2$. Hence $\mathfrak{s}\cap\mathfrak{z}=\pi_{\mathfrak{z}}\mathfrak{s}=\mathfrak{z}$, which implies $\mathfrak{s}=\pi_{\mathfrak{f}}\mathfrak{s}\oplus\mathfrak{z}$.

$\pi_{\mathfrak{f}}\mathfrak{s}=\langle\mathcal{P}^y\rangle$. If $m=0$, then $\mathfrak{s}=\mathfrak{s}_{1.4}^{0}$. Further we assume $m>0$ and consider the cases with different values of n separately.

$n=1$. For $\pi_{\mathfrak{z}}\mathfrak{s}=\langle\mathcal{P}^t+\mu\mathcal{I}\rangle$, due to $\mathfrak{s}\cap\mathfrak{z}=\{0\}$ we obtain $\mathfrak{s}=\langle\mathcal{P}^y+\alpha(\mathcal{P}^t+\mu\mathcal{I})\rangle$ for some nonzero constant α. Acting on $\mathfrak{s}$ by $\mathcal{D}(\frac{1}{2}\ln|\alpha|)_*$, we can always set α to be ±1. For $\pi_{\mathfrak{z}}\mathfrak{s}=\langle\mathcal{I}\rangle$, the arguments are analogous.

$n=2$. For $\pi_{\mathfrak{z}}\mathfrak{s}=\langle\mathcal{P}^t+\mu\mathcal{I}\rangle$ or $\pi_{\mathfrak{z}}\mathfrak{s}=\langle\mathcal{I}\rangle$, claim ($ii$) implies $\mathfrak{s}=\pi_{\mathfrak{f}}\mathfrak{s}\oplus\pi_{\mathfrak{z}}\mathfrak{s}$. For $\pi_{\mathfrak{z}}\mathfrak{s}=\langle\mathcal{P}^t,\mathcal{I}\rangle$, the cases with $\mathfrak{s}\cap\mathfrak{z}=\langle\mathcal{P}^t+\mu\mathcal{I}\rangle$ and $\mathfrak{s}\cap\mathfrak{z}=\langle\mathcal{I}\rangle$ correspond to the subalgebras $\mathfrak{s}=\langle\mathcal{P}^y+\nu\mathcal{I},\mathcal{P}^t+\mu\mathcal{I}\rangle$ and $\mathfrak{s}=\langle\mathcal{P}^y+\nu\mathcal{P}^t,\mathcal{I}\rangle$, where $\nu\neq0$. Acting then on $\mathfrak{s}$ by $\mathcal{D}(\frac{1}{2}\ln|\nu|)_*$, we can set $\nu=\pm1$.

$n=3$. We obtain $\mathfrak{s}=\pi_{\mathfrak{f}}\mathfrak{s}\oplus\mathfrak{z}$ by claim (iii).

In total, this results in the families of subalgebras $\mathfrak{s}_{1.3}^{\varepsilon,\mu}$, $\mathfrak{s}_{1.4}^{\delta}$, $\mathfrak{s}_{2.2}^{\delta,\mu}$, $\mathfrak{s}_{2.3}^{\delta}$, $\mathfrak{s}_{3.1}$.

$\pi_{\mathfrak{f}}\mathfrak{s}=\langle\mathcal{D}\rangle$. If $m=0$, then $\mathfrak{s}=\mathfrak{s}_{1.6}^{0}$. Further we assume $m>0$ and separately consider cases with different values of n.

$n=1$. The condition $\pi_{\mathfrak{z}}\mathfrak{s}=\langle\mathcal{P}^t+\mu\mathcal{I}\rangle$ implies that $\check{n}=0$, and thus $\mathfrak{s}=\langle\mathcal{D}+\nu\mathcal{P}^t+\mu\mathcal{I}\rangle$, where $\nu\neq0$. Acting on $\mathfrak{s}$ by $\mathcal{Q}^+(\pi/2)_*$, we can set $\nu>0$. The consideration of the condition $\pi_{\mathfrak{z}}\mathfrak{s}=\langle\mathcal{I}\rangle$ is the same. These two cases result in the family of subalgebras $\mathfrak{s}_{1.5}^{\nu',\mu}$ and $\mathfrak{s}_{1.6}^{\nu'}$ with $\nu'>0$ and $\mu\in\mathbb{R}$.

$n=2$. If in addition $m=1$, then $\mathfrak{s}=\pi_{\mathfrak{f}}\mathfrak{s}\oplus\pi_{\mathfrak{z}}\mathfrak{s}$ due to claim (ii), and depending on $\pi_{\mathfrak{z}}\mathfrak{s}=\mathfrak{s}\cap\mathfrak{z}$, we get the subalgebras $\mathfrak{s}_{2.4}^{0,\mu}$ and $\mathfrak{s}_{2.5}^{0}$. If $\pi_{\mathfrak{z}}\mathfrak{s}=\langle\mathcal{P}^t,\mathcal{I}\rangle$, then the cases with $\mathfrak{s}\cap\mathfrak{z}=\langle\mathcal{P}^t+\mu\mathcal{I}\rangle$ and $\mathfrak{s}\cap\mathfrak{z}=\langle\mathcal{I}\rangle$ correspond to the families of subalgebras $\mathfrak{s}_{2.4}^{\nu,\mu}$ and $\mathfrak{s}_{2.5}^{\nu}$ with $\nu>0$ and $\mu\in\mathbb{R}$, respectively.

$n=3$. We straightforwardly obtain $\mathfrak{s}_{3.2}$.

$\pi_{\mathfrak{f}}\mathfrak{s}=\langle\mathcal{P}^y+\mathcal{K}\rangle$. This case is analogous to the previous cases $\pi_{\mathfrak{f}}\mathfrak{s}=\langle\mathcal{P}^y\rangle$ and $\pi_{\mathfrak{f}}\mathfrak{s}=\langle\mathcal{D}\rangle$, except that scaling parameters or gauging their signs are impossible. This gives subalgebras $\mathfrak{s}_{1.7}^{\mu,\mu'}$, $\mathfrak{s}_{2.6}^{\mu,\mu'}$, $\mathfrak{s}_{2.7}^{\mu}$, $\mathfrak{s}_{3.3}$ with $\mu,\mu'\in\mathbb{R}$.

$\pi_{\mathfrak{f}}\mathfrak{s}=\langle\mathcal{P}^y,\mathcal{D}\rangle$. It is then clear that $2\leqslant n\leqslant4$. If $m=0$, then $\mathfrak{s}=\mathfrak{s}_{2.9}^{0}$. Further $m>0$. We again consider cases with different values of n separately.

$n=2$. Then $\check{n}=0$. If $\pi_{\mathfrak{z}}\mathfrak{s}=\langle\mathcal{P}^t+\mu\mathcal{I}\rangle$, then $\mathfrak{s}=\langle Q^1,Q^2\rangle$, where $Q^1=\mathcal{P}^y+\alpha(\mathcal{P}^t+\mu\mathcal{I})$ and $Q^2=\mathcal{D}+\beta(\mathcal{P}^t+\mu\mathcal{I})$. Since $[Q^1,Q^2]=\mathcal{P}^y\in\langle Q^1,Q^2\rangle$, we derive $\alpha=0$, which leads to the family of subalgebras $\mathfrak{s}_{2.8}^{\nu,\mu}$, where we can additionally set $\nu>0$. If $\pi_{\mathfrak{z}}\mathfrak{s}=\langle\mathcal{I}\rangle$, then using analogous arguments, we get the family $\mathfrak{s}_{2.9}^{\nu}$ with $\nu>0$. The case $\pi_{\mathfrak{z}}\mathfrak{s}=\langle\mathcal{P}^t,\mathcal{I}\rangle$ is impossible due to the commutation relations between involved vector fields.

$n=3$. Hence $\check{n}=1$. If $m=1$, then $\mathfrak{s}\cap\mathfrak{z}=\pi_{\mathfrak{z}}\mathfrak{s}$, i.e., the algebra $\mathfrak{s}$ is the direct sum of its projections on $\mathfrak{f}$ and $\mathfrak{z}$, which results in the subalgebras $\mathfrak{s}_{3.4}^{0,\mu}$ and $\mathfrak{s}_{3.5}^{0}$ for $\pi_{\mathfrak{z}}\mathfrak{s}=\langle\mathcal{P}^t+\mu\mathcal{I}\rangle$ and $\pi_{\mathfrak{z}}\mathfrak{s}=\langle\mathcal{I}\rangle$, respectively.

If $\pi_{\mathfrak{z}}\mathfrak{s}=\langle\mathcal{P}^t,\mathcal{I}\rangle$, we still have two options for $\mathfrak{s}\cap\mathfrak{z}$, either $\langle\mathcal{P}^t+\mu\mathcal{I}\rangle$ or $\langle\mathcal{I}\rangle$. The corresponding subalgebras are $\mathfrak{s}=\langle\mathcal{P}^y,\mathcal{D}+\nu\mathcal{I},\mathcal{P}^t+\mu\mathcal{I}\rangle$ or $\mathfrak{s}=\langle\mathcal{P}^y,\mathcal{D}+\nu\mathcal{P}^t,\mathcal{I}\rangle$ with $\nu\neq0$, where ν can be set greater than zero. By this we obtain the subalgebras $\mathfrak{s}_{3.4}^{\nu,\mu}$ and $\mathfrak{s}_{3.5}^{\nu}$, where $\nu>0$.

$n=4$. It is obvious that then $\mathfrak{s}=\pi_{\mathfrak{f}}\mathfrak{s}\oplus\mathfrak{z}$. Hence we obtain the subalgebra $\mathfrak{s}_{4.1}$.

$\pi_{\mathfrak{f}}\mathfrak{s}=\langle\mathcal{P}^y,\mathcal{D},\mathcal{K}\rangle$. If $n=3$, then the subalgebra $\mathfrak{s}$ is a Levi factor of $\mathfrak{g}^{\rm ess}$. Therefore, by the Levi–Malcev theorem, it is equivalent with respect to an inner automorphism to the subalgebra $\mathfrak{s}_{3.6}=\mathfrak{f}$.

In the case $n=4$ and $m=1$, we clearly have the factorization $\mathfrak{s}=\pi_{\mathfrak{f}}\mathfrak{s}\oplus\pi_{\mathfrak{z}}\mathfrak{s}$ and obtain two families of inequivalent subalgebras, $\mathfrak{s}_{4.2}^{\mu}$ and $\mathfrak{s}_{4.3}$. The case with $n=4$ and $m=2$ is impossible due to the commutation relations between involved vector fields.

Each two subalgebra families among those listed in lemma's statement correspond to different quadruples $(\dim\mathfrak{s},[\pi_{\mathfrak{f}}\mathfrak{s}]_{F_{\rm id}},\pi_{\mathfrak{z}}\mathfrak{s},\mathfrak{s}\cap\mathfrak{z})$ of $G^{\rm ess}_{\rm id}$-invariants, which means that there are no conjugations among subalgebras from different families. The absence of conjugated subalgebras within each of the families follows from the above consideration. □

Remark 2.6. Due to the isomorphisms $\mathfrak{g}^{\rm ess}\simeq\mathrm{sl}(2,\mathbb{R})\oplus 2A_1\simeq\mathrm{gl}(2,\mathbb{R})\oplus A_1$, Lemma 2.5 also presents the classifications of subalgebras of the algebras $\mathrm{sl}(2,\mathbb{R})\oplus 2A_1$ and $\mathrm{gl}(2,\mathbb{R})\oplus A_1$ up to their inner automorphism groups. The second decomposition also suggests another way of proving Lemma 2.5 that is based on the known list of inequivalent subalgebras of the algebra $\mathrm{gl}(2,\mathbb{R})$ [114, 125].

Corollary 2.7. *A complete list of $G^{\rm ess}$-inequivalent proper subalgebras of the algebra $\mathfrak{g}^{\rm ess}$ is exhausted by the following subalgebras, where $\delta'\in\{0,1\}$, $\mu\in\mathbb{R}$, $\nu\geqslant 0$, $\nu'>0$:*

$$
\begin{aligned}
&\text{1D:}\quad \mathfrak{s}_{1.1}^{\mu},\quad \mathfrak{s}_{1.2},\quad \mathfrak{s}_{1.3}^{1,\mu},\quad \mathfrak{s}_{1.4}^{\delta'},\quad \mathfrak{s}_{1.5}^{\nu',\mu},\quad \mathfrak{s}_{1.6}^{\nu},\quad \mathfrak{s}_{1.7}^{\nu',\mu},\quad \mathfrak{s}_{1.7}^{0,\nu},\\
&\text{2D:}\quad \mathfrak{s}_{2.1},\quad \mathfrak{s}_{2.2}^{\delta',\mu},\quad \mathfrak{s}_{2.3}^{\delta'},\quad \mathfrak{s}_{2.4}^{\nu,\mu},\quad \mathfrak{s}_{2.5}^{\nu},\quad \mathfrak{s}_{2.6}^{\nu,\mu},\quad \mathfrak{s}_{2.7}^{\nu},\quad \mathfrak{s}_{2.8}^{\nu',\mu},\quad \mathfrak{s}_{2.9}^{\nu},\\
&\text{3D:}\quad \mathfrak{s}_{3.1},\quad \mathfrak{s}_{3.2},\quad \mathfrak{s}_{3.3},\quad \mathfrak{s}_{3.4}^{\nu,\mu},\quad \mathfrak{s}_{3.5}^{\nu},\quad \mathfrak{s}_{3.6},\\
&\text{4D:}\quad \mathfrak{s}_{4.1},\quad \mathfrak{s}_{4.2}^{\mu},\quad \mathfrak{s}_{4.3}.
\end{aligned}
$$

Proof. The group $G^{\rm ess}$ is the semidirect product of its normal subgroup $G^{\rm ess}_{\rm id}$ and its finite subgroup $G^{\rm ess}_{\rm d}$ generated by the independent discrete symmetries $\mathcal{I}'$ and $\mathcal{J}'$ of the equation (2.2), which are given in Proposition 2.4. If subalgebras are $G^{\rm ess}_{\rm id}$-equivalent, then they are clearly $G^{\rm ess}$-equivalent. Hence a complete list of $G^{\rm ess}$-inequivalent proper subalgebras of the algebra $\mathfrak{g}^{\rm ess}$ is contained within the list presented in Lemma 2.5. Moreover, subalgebras from different families in the latter list are $G^{\rm ess}$-inequivalent since the quadruple of $G^{\rm ess}_{\rm id}$-invariants $(\dim\mathfrak{s},[\pi_{\mathfrak{f}}\mathfrak{s}]_{F_{\rm id}},\pi_{\mathfrak{z}}\mathfrak{s},\mathfrak{z}\cap\mathfrak{s})$ is invariant under the action of $G^{\rm ess}$ as well. Among the transformations $\mathcal{I}'$ and $\mathcal{J}'$, the only pushforward of $\mathcal{J}'$ acts nontrivially on $\mathfrak{g}^{\rm ess}$, and this action is nonidentity only for the basis elements $\mathcal{P}^y$ and $\mathcal{K}$, $\mathcal{J}'_*\mathcal{P}^y=-\mathcal{P}^y$ and $\mathcal{J}'_*\mathcal{K}=-\mathcal{K}$. Hence acting by $G^{\rm ess}_{\rm d}$ on the subalgebras listed

in Lemma 2.5 establishes the following $G^{\rm ess}$-equivalences:

$$\mathfrak{s}_{1.3}^{-1,-\mu}\sim\mathfrak{s}_{1.3}^{1,\mu},\quad \mathfrak{s}_{1.4}^{-1}\sim\mathfrak{s}_{1.4}^{1},\quad \mathfrak{s}_{1.7}^{-\mu,-\mu'}\sim\mathfrak{s}_{1.7}^{\mu,\mu'},$$
$$\mathfrak{s}_{2.2}^{-1,\mu}\sim\mathfrak{s}_{2.2}^{1,\mu},\quad \mathfrak{s}_{2.3}^{-1}\sim\mathfrak{s}_{2.3}^{1},\quad \mathfrak{s}_{2.6}^{-\mu,\mu'}\sim\mathfrak{s}_{2.6}^{\mu,\mu'},\quad \mathfrak{s}_{2.7}^{-\mu}\sim\mathfrak{s}_{2.7}^{\mu}.$$

For each of the above pairs, we exclude one of its elements from the list in Lemma 2.5, which gives a complete list of $G^{\rm ess}$-inequivalent proper subalgebras of the essential Lie invariance algebra $\mathfrak{g}^{\rm ess}$. □

2.5 Lie reductions

Throughout this section, the subscripts 1 and 2 of the dependent variable w denote the derivatives with respect to z_1 and z_2, respectively. We use the complex label $d.i^*$ to mark the Lie reduction corresponding to the subalgebra $\mathfrak{s}_{d.i}^*$ from Corollary 2.7, where d, i and $*$ are the dimension of this subalgebra, its number in the list of d-dimensional subalgebras of the algebra $\mathfrak{g}^{\rm ess}$, and the values of subalgebra parameters for a subalgebra family, respectively. We omit the superscript in the label whenever it is not essential.

Codimension one. All the inequivalent one-dimensional subalgebras of $\mathfrak{g}^{\rm ess}$ that are listed in Corollary 2.7, except the subalgebra $\mathfrak{s}_{1.2}$, satisfy the transversality condition and thus are appropriate for using for Lie reductions of the equation (2.2). (The space of $\mathfrak{s}_{1.2}$-invariant solutions of (2.2) is exhausted by the trivial identically zero solution.) For each of these subalgebras, we construct the corresponding Lie ansatz, i.e., the expression for u in terms of the new unknown function $w(z_1,z_2)$ of two invariant independent variables z_1 and z_2, and derive the associated reduced equation. We merge the subalgebra families $\{\mathfrak{s}_{1.5}^{\nu',\mu}\}$ and $\{\mathfrak{s}_{1.6}^{\nu}\}$ to $\{\mathfrak{s}_{1.5}^{\nu,\mu}\}$, assuming that $\mu\geqslant 0$ if $\nu=0$. The results are presented in Table 2.1.

The reduced equations are linear homogeneous second-order evolution equations in two independent variables. The group classification of these equations is well known and was first obtained by Sophus Lie himself over $\mathbb{C}$ in [86], see also [127, Section 2] for the modern view on the problem. We exploit this classification for constructing point transformations that map the reduced equations to the canonical representatives of their equivalence class, which are (1+1)-dimensional linear heat equations with

Table 2.1: $G^{\rm ess}$-inequivalent codimension-one Lie reductions of the equation (2.2)

no.	u	z_1	z_2	Reduced equation
1.1^{μ}	$\mathrm{e}^{\mu t}w$	y	x	$z_2w_1=z_2^2w_{22}-\mu w$
$1.3^{1,\mu}$	$\mathrm{e}^{\mu t}w$	$t-y$	x	$(1-z_2)w_1=z_2^2w_{22}-\mu w$
$1.4^{\delta'}$	$\mathrm{e}^{\delta' y}w$	t	x	$w_1=z_2^2w_{22}-\delta' z_2w$
$1.5^{\nu,\mu}$	$\lvert y\rvert^{\mu}w$	$t-\nu\ln\lvert y\rvert$	x/y	$(1-\nu z_2)w_1=z_2^2w_{22}+z_2^2w_2-\mu z_2w$
$1.7^{\nu,\mu}$	$\mathrm{e}^{\mu\arctan(y)-\frac{xy}{y^2+1}}w$	$t-\nu\arctan y$	$\dfrac{x}{y^2+1}$	$(1-\nu z_2)w_1=z_2^2w_{22}+z_2(z_2-\mu)w$

Here $\delta'\in\{0,1\}$, $\mu\in\mathbb{R}$, $\nu\geqslant 0$, and $\mu\geqslant 0$ in Cases 1.5 and 1.7 with $\nu=0$; $w=w(z_1,z_2)$ is the new unknown function of the new independent variables (z_1,z_2).

potentials.

For each of the reduced equations, $\mathcal{R}$, its maximal Lie invariance algebra $\mathfrak{g}_{\mathcal{R}}$ contains the infinite-dimensional abelian ideal $\{h(z_1,z_2)\partial_w\}$, where h runs through the solution set of $\mathcal{R}$. This ideal is merely associated with the linear superposition of solutions of $\mathcal{R}$ and thus can be assumed to be a trivial part of the algebra $\mathfrak{g}_{\mathcal{R}}$. Moreover, the algebra $\mathfrak{g}_{\mathcal{R}}$ splits over it, and its complement in $\mathfrak{g}_{\mathcal{R}}$ is a finite-dimensional subalgebra $\mathfrak{a}_{\mathcal{R}}$, called the essential Lie invariance algebra of $\mathcal{R}$. The algebras $\mathfrak{a}_{\mathcal{R}}$ for specific reduced equations $\mathcal{R}$ are the following:

$$\mathfrak{a}^{\mu}_{1.3}=\mathfrak{a}^{1}_{1.4}=\mathfrak{a}^{\nu,\mu}_{1.5}=\mathfrak{a}^{\nu,\mu}_{1.7}=\langle\partial_{z_1},\ w\partial_w\rangle,$$

$$\mathfrak{a}^{\mu}_{1.1}=\langle\partial_{z_1},\ z_1\partial_{z_1}+z_2\partial_{z_2},\ z_1^2\partial_{z_1}+2z_1z_2\partial_{z_2}-wz_2\partial_w,\ w\partial_w\rangle\quad\text{if}\quad\mu\neq-\tfrac{3}{16},$$

$$\begin{aligned}\mathfrak{a}^{\mu_0}_{1.1}=\big\langle&\partial_{z_1},\ z_1\partial_{z_1}+z_2\partial_{z_2}-\tfrac{1}{2}w\partial_w,\ z_1^2\partial_{z_1}+2z_1z_2\partial_{z_2}-z_2w\partial_w,\\&\lvert z_2\rvert^{-\frac{1}{2}}\big(4z_2\partial_{z_2}+w\partial_w\big),\ \lvert z_2\rvert^{-\frac{1}{2}}\big(4z_1z_2\partial_{z_2}+(z_1-4z_2)w\partial_w\big),\ w\partial_w\big\rangle\\&\text{with}\quad\mu_0:=-\tfrac{3}{16},\end{aligned}$$

$$\begin{aligned}\mathfrak{a}^{0}_{1.4}=\big\langle&\partial_{z_1},\ 4z_1\partial_{z_1}+2z_2\ln\lvert z_2\rvert\partial_{z_2}-(z_1-\ln\lvert z_2\rvert)w\partial_w,\\&4z_1^2\partial_{z_1}+4z_1z_2\ln\lvert z_2\rvert\partial_{z_2}-\big((z_1-\ln\lvert z_2\rvert)^2+2z_1\big)w\partial_w,\\&2z_1z_2\partial_{z_2}+(z_1-\ln\lvert z_2\rvert)w\partial_w,\ z_2\partial_{z_2},\ w\partial_w\big\rangle.\end{aligned}$$

Since $\dim\mathfrak{a}^{\mu_0}_{1.1}=\dim\mathfrak{a}^{0}_{1.4}>\dim\mathfrak{g}^{\rm ess}$, the equation (2.2) definitely admits hidden symmetries related to Lie reductions 1.1^{μ_0} and 1.4^{0}. Recall that a hidden symmetry of a differential equation is a symmetry of its reduced equation that is not induced

by a symmetry of the original equation. To find hidden symmetries, one usually describes the subalgebra of induced symmetries in the Lie invariance algebra of the reduced equation. The algebra of induced essential Lie symmetries of reduced equation $1.i$ is isomorphic to the quotient algebra $\mathrm{N}_{\mathfrak{g}^{\rm ess}}(\mathfrak{s}_{1.i})/\mathfrak{s}_{1.i}$. Hence to check which Lie symmetries of reduced equation $1.i$ are induced by Lie symmetries of (2.2), we compute the normalizer $\mathrm{N}_{\mathfrak{g}^{\rm ess}}(\mathfrak{s}_{1.i})$ of the subalgebra $\mathfrak{s}_{1.i}$ in the algebra $\mathfrak{g}^{\rm ess}$,

$$\begin{aligned}
&\mathrm{N}_{\mathfrak{g}^{\rm ess}}(\mathfrak{s}^{\mu}_{1.1})=\mathfrak{g}^{\rm ess},\quad \mathrm{N}_{\mathfrak{g}^{\rm ess}}(\mathfrak{s}^{0}_{1.4})=\langle\mathcal{P}^y,\mathcal{D},\mathcal{P}^t,\mathcal{I}\rangle,\\
&\mathrm{N}_{\mathfrak{g}^{\rm ess}}(\mathfrak{s}^{1,\mu}_{1.3})=\mathrm{N}_{\mathfrak{g}^{\rm ess}}(\mathfrak{s}^{1}_{1.4})=\langle\mathcal{P}^y,\mathcal{P}^t,\mathcal{I}\rangle,\\
&\mathrm{N}_{\mathfrak{g}^{\rm ess}}(\mathfrak{s}^{\nu,\mu}_{1.5})=\langle\mathcal{D},\mathcal{P}^t,\mathcal{I}\rangle,\quad \mathrm{N}_{\mathfrak{g}^{\rm ess}}(\mathfrak{s}^{\nu,\mu}_{1.7})=\langle\mathcal{P}^y+\mathcal{K},\mathcal{P}^t,\mathcal{I}\rangle.
\end{aligned}$$

The subalgebras of essential induced symmetries in $\mathfrak{a}^{\mu}_{1.1}$, $\mathfrak{a}_{1.2}$, \dots, $\mathfrak{a}^{\nu,\mu}_{1.7}$ are

$$\begin{aligned}
&\tilde{\mathfrak{a}}^{\mu}_{1.1}=\langle\partial_{z_1},z_1\partial_{z_1}+z_2\partial_{z_2},2z_1z_2\partial_{z_2}+z_1^2\partial_{z_1}-z_2w\partial_w,w\partial_w\rangle,\\
&\tilde{\mathfrak{a}}^{0}_{1.4}=\langle\partial_{z_1},z_2\partial_{z_2},w\partial_w\rangle,\\
&\tilde{\mathfrak{a}}^{1,\mu}_{1.3}=\tilde{\mathfrak{a}}^{1}_{1.4}=\tilde{\mathfrak{a}}^{\nu,\mu}_{1.5}=\tilde{\mathfrak{a}}^{\nu,\mu}_{1.7}=\langle\partial_{z_1},w\partial_w\rangle.
\end{aligned}$$

Hence the genuine hidden symmetries of (2.2) are contained only in the set complements $\mathfrak{a}^{\mu_0}_{1.1}\setminus\tilde{\mathfrak{a}}^{\mu_0}_{1.1}$ and $\mathfrak{a}^{0}_{1.4}\setminus\tilde{\mathfrak{a}}^{0}_{1.4}$. This means that the further Lie reductions of the reduced equations, except 1.1^{μ_0} and 1.4^0 lead only to solutions of (2.2) that can be obtained by codimensions-two Lie reductions of (2.2).

The dimension of the essential Lie invariance algebras of each of the reduced equations is greater than one. Therefore, we can map these equations by point transformations to (1+1)-dimensional linear heat equations with potentials, which are functions of the new variable $\tilde z_2$, i.e., to the equations of the form $\tilde w_{\tilde z_1}=\tilde w_{\tilde z_2\tilde z_2}+V(\tilde z_2)\tilde w$. Here and in what follows the subscripts 1 and 2 of $\tilde w$ denote the derivatives of $\tilde w$ with respect to $\tilde z_1$ and $\tilde z_2$, respectively. The transformations and the mapped equations are

$$1.1^{\mu}\textbf{:}\quad \tilde z_1=\varepsilon z_1,\quad \tilde z_2=2\sqrt{|z_2|},\quad \tilde w(\tilde z_1,\tilde z_2)=|z_2|^{-\frac14}w,\quad \tilde w_1=\tilde w_{22}-\frac{4\mu+3/4}{\tilde z_2^2}\tilde w;$$

$$1.3^{\mu},\ z_2<1\textbf{:}\quad \tilde z_1=z_1,\quad \tilde z_2=2\sqrt{1-z_2}+\ln\left|\frac{\sqrt{1-z_2}-1}{\sqrt{1-z_2}+1}\right|,\quad \tilde w=\frac{(1-z_2)^{1/4}}{|z_2|^{1/2}}w,$$
$$\tilde w_1=\tilde w_{22}-\frac{1}{16}\left(\frac{16\mu+3}{z_2-1}-\frac{6}{(z_2-1)^2}-\frac{5}{(z_2-1)^3}\right)\tilde w;$$

1.3^{μ}, $z_2 > 1$: $\tilde{z}_1 = -z_1$, $\tilde{z}_2 = 2\sqrt{z_2 - 1} - 2\arctan\sqrt{z_2 - 1}$, $\tilde{w} = \dfrac{(z_2-1)^{1/4}}{z_2^{1/2}} w$,

$$\tilde{w}_1 = \tilde{w}_{22} + \frac{1}{16}\left(\frac{16\mu + 3}{z_2 - 1} - \frac{6}{(z_2-1)^2} - \frac{5}{(z_2-1)^3}\right)\tilde{w};$$

1.4^{δ}: $\tilde{z}_1 = z_1$, $\tilde{z}_2 = \ln|z_2|$, $\tilde{w} = \mathrm{e}^{z_1/4}|z_2|^{-1/2} w$, $\tilde{w}_1 = \tilde{w}_{22} - \delta\tilde{\varepsilon}\mathrm{e}^{\tilde{z}_2}\tilde{w}$ with $\tilde{\varepsilon} = \operatorname{sgn} z_2$;

$1.5^{\nu',\mu}$, $\nu' z_2 < 1$: $\tilde{z}_1 = z_1$, $\tilde{z}_2 = 2\sqrt{1 - \nu' z_2} + \ln\left|\dfrac{\sqrt{1-\nu' z_2} - 1}{\sqrt{1-\nu' z_2} + 1}\right|$,

$$\tilde{w} = \frac{\mathrm{e}^{z_2/2}}{|z_2|^{1/2}}(1 - \nu' z_2)^{1/4} w,$$

$$\tilde{w}_1 = \tilde{w}_{22} + \frac{1}{16}\left(\frac{4z_2^2 + 16\mu z_2 + 3}{\nu' z_2 - 1} - \frac{6}{(\nu' z_2 - 1)^2} - \frac{5}{(\nu' z_2 - 1)^3}\right)\tilde{w};$$

$1.5^{\nu',\mu}$, $\nu' z_2 > 1$: $\tilde{z}_1 = -z_1$, $\tilde{z}_2 = 2\sqrt{\nu' z_2 - 1} - 2\arctan\sqrt{\nu' z_2 - 1}$,

$$\tilde{w} = \frac{\mathrm{e}^{z_2/2}}{z_2^{1/2}}(\nu' z_2 - 1)^{1/4} w,$$

$$\tilde{w}_1 = \tilde{w}_{22} - \frac{1}{16}\left(\frac{4z_2^2 + 16\mu z_2 + 3}{\nu' z_2 - 1} - \frac{6}{(\nu' z_2 - 1)^2} - \frac{5}{(\nu' z_2 - 1)^3}\right)\tilde{w};$$

$1.5^{0,\nu}$: $\tilde{z}_1 = z_1$, $\tilde{z}_2 = \ln|z_2|$, $\tilde{w} = \mathrm{e}^{z_1/4 + z_2/2}|z_2|^{-1/2} w$,

$$\tilde{w}_1 = \tilde{w}_{22} - (\tilde{\varepsilon}\nu\mathrm{e}^{\tilde{z}_2} + \tfrac{1}{4}\mathrm{e}^{2\tilde{z}_2})\tilde{w} \quad \text{with} \quad \tilde{\varepsilon} = \operatorname{sgn} z_2;$$

$1.7^{\nu',\mu}$, $\nu' z_2 < 1$: $\tilde{z}_1 = z_1$, $\tilde{z}_2 = 2\sqrt{1 - \nu' z_2} + \ln\left|\dfrac{\sqrt{1-\nu' z_2} - 1}{\sqrt{1-\nu' z_2} + 1}\right|$, $\tilde{w} = \dfrac{(1-\nu' z_2)^{1/4}}{|z_2|^{1/2}} w$,

$$\tilde{w}_1 = \tilde{w}_{22} + \frac{1}{16}\left(\frac{16 z_1(\mu - z_2) + 3}{\nu z_2 - 1} - \frac{6}{(\nu z_2 - 1)^2} - \frac{5}{(\nu z_2 - 1)^3}\right)\tilde{w};$$

$1.7^{\nu',\mu}$, $\nu' z_2 > 1$: $\tilde{z}_1 = z_1$, $\tilde{z}_2 = 2\sqrt{\nu' z_2 - 1} - 2\arctan\sqrt{\nu' z_2 - 1}$, $\tilde{w} = \dfrac{(\nu' z_2 - 1)^{1/4}}{z_2^{1/2}} w$,

$$\tilde{w}_1 = \tilde{w}_{22} - \frac{1}{16}\left(\frac{16 z_1(\mu - z_2) + 3}{\nu z_2 - 1} - \frac{6}{(\nu z_2 - 1)^2} - \frac{5}{(\nu z_2 - 1)^3}\right)\tilde{w};$$

$1.7^{0,\nu}$: $\tilde{z}_1 = z_1$, $\tilde{z}_2 = \ln|z_2|$, $\tilde{w} = \dfrac{\mathrm{e}^{z_1/4}}{|z_2|^{1/2}} w$, $\tilde{w}_1 = \tilde{w}_{22} + (\mathrm{e}^{2\tilde{z}_2} - \tilde{\varepsilon}\nu\mathrm{e}^{\tilde{z}_2})\tilde{w}$ with $\tilde{\varepsilon} := \operatorname{sgn} z_2$.

The potentials in Cases 1.3^{μ}, $1.5^{\nu',\mu}$ and $1.7^{\nu',\mu}$ are defined as expressions with $z_2 = \tilde{Z}^2(\tilde{z}_2)$, where $\tilde{Z}^2$ is the inverse of the function in the z_2-component $\tilde{z}_2 = Z^2(z_2)$ of the corresponding transformation.

In several cases, where the dimension of the essential Lie invariance algebras of the associated reduced equations is even greater than two, the corresponding mapped equations are of especially nice form.

Thus, the essential Lie invariance algebras $\mathfrak{a}^{\mu_0}_{1.1}$ and $\mathfrak{a}^{0}_{1.4}$ are isomorphic to the special Galilei algebra, which agrees with the fact that the mapped equations in Cases 1.1^{μ_0} and 1.4^{0} coincide with the (1+1)-dimensional linear heat equation. This leads to the solutions of the equation (2.2)

- $u = \mathrm{e}^{-\frac{3}{16}t}|x|^{1/4}\theta(\varepsilon y, 2\sqrt{|x|}),$ (2.6)
- $u = \mathrm{e}^{-\frac{1}{4}t}|x|^{1/2}\theta(t, \ln|x|),$ (2.7)

where θ is an arbitrary solution of the equation $\theta_1 = \theta_{22}$. An enhanced complete collection of inequivalent Lie invariant solutions of this equation was presented in [147, Section A], following Examples 3.3 and 3.17 in [103].

The essential Lie invariance algebra of the equation 1.1^{μ} with $\mu \neq -\frac{3}{16}$ is four-dimensional. Hence the corresponding mapped equation is the (1+1)-dimensional linear heat equation with inverse square potential. This gives wide families of solutions of the equation (2.2) that are of the form

- $u = \mathrm{e}^{\mu t}|x|^{1/4}\vartheta^{4\mu+\frac{3}{4}}(\varepsilon y, 2\sqrt{|x|}),$ (2.8)

where $\vartheta^{\tilde\mu}(\tilde z_1, \tilde z_2)$ is an arbitrary solution of the equation $\tilde w_1 = \tilde w_{22} - \tilde\mu \tilde z_2^{-2}\tilde w$ with $\tilde\mu \neq 0$. Wide families of solutions for such equations were found in [70, Section A].

Codimensions two and three. Among the inequivalent two-dimensional subalgebras listed in Corollary 2.7 the subalgebras $\mathfrak{s}_{2.1}$, $\mathfrak{s}^{\delta'}_{2.3}$, $\mathfrak{s}^{\nu}_{2.5}$, and $\mathfrak{s}^{\nu}_{2.7}$ are inappropriate for constructing a Lie ansatz since they do not satisfy the local transversality condition. Each of the other listed two-dimensional subalgebras contains the subalgebra $\mathfrak{s}^{\mu}_{1.1}$ or the subalgebra $\mathfrak{s}^{0}_{1.4}$. Therefore, each of the solutions constructed using the corresponding Lie codimension-two reductions is $G^{\rm ess}$-equivalent to a solution from one of the families (2.6)–(2.8).

An analogous claim is true for reductions to algebraic equations. The only listed three-dimensional subalgebras that are suitable for Lie reductions are $\mathfrak{s}^{\nu,\mu}_{3.4}$ and $\mathfrak{s}_{3.6}$.

However, any of them contains the subalgebra $\mathfrak{s}^0_{1.4}$. Hence, the corresponding reduction gives no essentially new solution in addition to the already constructed ones.

2.6 Generating new solutions from known ones

Point symmetries of the fine Kolmogorov backward equation (2.2) can also be exploited in the course of generating new solutions of this equation from the known ones. Recall that by definition a point symmetry of the system of differential equations is a point transformation that preserves its solution set. Thus, in the notation of Theorem 2.1, pulling back a given solution $u = h(t, x, y)$ of (2.2) by an arbitrary element of the group G gives

$$\tilde{u} = \sigma \exp\left(-\frac{\gamma}{\alpha - \gamma y}x\right) h\left(t - \lambda, \frac{\alpha\delta - \beta\gamma}{(\alpha - \gamma y)^2}x, \frac{\delta y - \beta}{\alpha - \gamma y}\right) + f(t, x, y),$$

which is the most general form of the G-equivalent counterparts of the seed solution $u = h(t, x, y)$.

Due to the linearity of the equation (2.2), there is one more way of generating its solutions from the known ones within the framework of symmetry analysis, which uses linear generalized symmetries of (2.2). To the best of our knowledge, the study of generalized symmetries of the equation (2.2) has not been initiated in the literature. The thorough investigation of the generalized symmetry algebra of this equation involves cumbersome and sophisticated computations and can become a topic of a separate research. Nevertheless, there are a number of arguments on the description of this algebra to be expected. See [103, Chapter 5], especially pp. 305–306, and [73] for required definitions, notation and related results. In particular, we refer to the differential operators in total derivatives that are associated with the linear Lie-symmetry vector fields of the equation (2.2) as to the *Lie-symmetry operators* of this equation. Thus, the Lie-symmetry vector fields $-\mathcal{P}^y$, $-\mathcal{D}$, $-\mathcal{K}$, $-\mathcal{P}^t$ and $\mathcal{I}$ correspond to the operators

$$\mathrm{P}^y := \mathrm{D}_y, \quad \mathrm{D} := x\mathrm{D}_x + y\mathrm{D}_y, \quad \mathrm{K} := 2xy\mathrm{D}_x + y^2\mathrm{D}_y + x, \quad \mathrm{P}^t := \mathrm{D}_t$$

and the identity operator. The equation (2.2) is also extraordinary by the fact that its operator $L := \mathrm{D}_t + x\mathrm{D}_y - x^2\mathrm{D}_x^2$ can be represented in terms of its Lie-symmetry

operators,

$$L = \mathrm{P}^t - \mathrm{D}^2 + \frac{1}{2}(\mathrm{P}^y\mathrm{K} + \mathrm{K}\mathrm{P}^y).$$

The operator P^t is equivalent on the solutions of (2.2) to the element

$$\hat{\mathrm{P}}^t := \mathrm{D}^2 - \frac{1}{2}(\mathrm{P}^y\mathrm{K} + \mathrm{K}\mathrm{P}^y) = \mathrm{D}^2 - \mathrm{D} - \mathrm{K}\mathrm{P}^y$$

of the associative algebra $\Upsilon_{\mathfrak{f}}$ generated by P^y, D and K since $\mathrm{P}^t - \hat{\mathrm{P}}^t = L$. Moreover, the operator $\hat{\mathrm{P}}^t$ is a Casimir operator of the Levi factor $\mathfrak{f} = \langle \mathcal{P}^y, \mathcal{D}, \mathcal{K}\rangle$ of the algebra $\mathfrak{g}^{\rm ess}$, see Section 2.2.

Based on a preliminary analysis in [73], we conjecture the following.

Conjecture 2.8. *Up to the equivalence of generalized symmetries and neglecting the Lie symmetries associated with the linear superposition of solutions, generalized symmetries of the fine Kolmogorov backward equation* (2.2) *are exhausted by its reduced linear generalized symmetries. The Lie algebra* $\hat{\Lambda}$ *of these symmetries is generated from the simplest linear generalized symmetry* $u\partial_u$ *by acting with the recursion operators* P^y, D *and* K *of the equation* (2.2),

$$\hat{\Lambda} = \{(\mathrm{Q}u)\partial_u \mid \mathrm{Q} \in \Upsilon_{\mathfrak{f}}\}.$$

In fact, the operators P^y and K suffice for generating the algebra $\Upsilon_{\mathfrak{f}}$ since $\mathrm{D} = \frac{1}{2}(\mathrm{P}^y\mathrm{K} - \mathrm{K}\mathrm{P}^y)$. However, considering $\{\mathrm{P}^y, \mathrm{D}, \mathrm{K}\}$ as the canonical generating set for $\Upsilon_{\mathfrak{f}}$ is natural and beneficial since fixing then an ordering in this set, we get the associated canonical basis of $\Upsilon_{\mathfrak{f}}$.

Lemma 2.9. *The monomials* $\mathbf{Q}^\alpha := (\mathrm{Q}^1)^{\alpha_1}(\mathrm{Q}^2)^{\alpha_2}(\mathrm{Q}^3)^{\alpha_3}$, *where* $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$ *and* $(\mathrm{Q}^1, \mathrm{Q}^2, \mathrm{Q}^3)$ *is any fixed ordering of* P^y, D *and* K, *constitute a basis of the algebra* $\Upsilon_{\mathfrak{f}}$.

Proof. The algebra $\Upsilon_{\mathfrak{f}}$ has a canonical filtration associated with the order $\operatorname{ord}\mathrm{Q}$ of the differential operators Q,

$$\Upsilon_{\mathfrak{f}} = \bigcup_{k=0}^{\infty} \Upsilon_{\mathfrak{f},k} \quad \text{with} \quad \Upsilon_{\mathfrak{f},k} = \{\mathrm{Q} \in \Upsilon_{\mathfrak{f}} \mid \operatorname{ord}\mathrm{Q} \leqslant k\}, \quad \Upsilon_{\mathfrak{f},k}\Upsilon_{\mathfrak{f},l} \subset \Upsilon_{\mathfrak{f},k+l}, \quad k, l \in \mathbb{N}_0.$$

It suffices to show that the monomials $\mathbf{Q}^\alpha$ with $|\alpha| := \alpha_1 + \alpha_2 + \alpha_3 \leqslant k$ constitute a basis of $\Upsilon_{\mathfrak{f},k}$. It is clear that these monomials span $\Upsilon_{\mathfrak{f},k}$ for any $k \in \mathbb{N}_0$, since any monomial $\mathrm{S}^1 \cdots \mathrm{S}^l$ with $\mathrm{S}^1, \dots, \mathrm{S}^k \in \{\mathrm{P}^y, \mathrm{D}, \mathrm{K}\}$ can be represented as a linear combination of the monomials $\mathbf{Q}^\alpha$ with $|\alpha| \leqslant l$ due to the commutation relations between operators in $\{\mathrm{P}^y, \mathrm{D}, \mathrm{K}\}$.

It remains to establish the linear independence of the monomials $\mathbf{Q}^\alpha$ with $|\alpha| \leqslant k$, which we will do by the induction with respect to the order k. The induction basis with $k = 0$ is obvious. Suppose that $\mathbf{Q}^\alpha$, $|\alpha| \leqslant k$ are linearly independent and

$$\mathrm{S} := \sum_{|\alpha| \leqslant k+1} \lambda_\alpha \mathbf{Q}^\alpha = 0 \tag{2.9}$$

for some $\lambda_\alpha \in \mathbb{R}$, $|\alpha| \leqslant k+1$. Then $\mathrm{e}^{-\mu x - \nu y} \mathrm{S} \mathrm{e}^{\mu x + \nu y}$ is an identically vanishing polynomial in (x, y, μ, ν). The part of this polynomial that includes all the terms of order $k+1$ with respect to (μ, ν), i.e., $\sum_{|\alpha|=k+1} \lambda_\alpha z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}$, where $z_1 := \nu$, $z_2 := \mu x + \nu y$ and $z_3 := \nu y^2 + 2\mu x y + x$, also vanishes. Since z_1, z_2 and z_3 are functionally independent as functions of (x, y, μ, ν), this implies that $\lambda_\alpha = 0$ if $|\alpha| = k+1$. Invoking the induction hypothesis, the equality (2.9) implies $\lambda_\alpha = 0$ for every α with $|\alpha| \leqslant k+1$. □

Remark 2.10. In Lemma 2.9, the operators P^y, D and K can be replaced by linear combinations of the form $a_{i1}\mathrm{P}^y + a_{i2}\mathrm{D} + a_{i3}\mathrm{K} + a_{i0}$, where $\det(a_{ij})_{i,j=1,2,3} \neq 0$. A more sophisticated basis is convenient for a part of the further consideration. We split the space $\Upsilon_{\mathfrak{f}}$ into two subspaces Γ_1 and Γ_2 that are spanned by the monomials $\mathrm{K}^{\alpha_1}(\mathrm{P}^y)^{\alpha_2}\mathrm{D}^{\alpha_3}$ with $\alpha_1 \leqslant \alpha_2$ and $\alpha_1 > \alpha_2$, respectively. In view of the relations

$$\mathrm{K}\mathrm{P}^y = \mathrm{D}^2 - \mathrm{D} - \hat{\mathrm{P}}^t \quad \text{and} \quad \mathrm{P}^y\mathrm{D} = (\mathrm{D}+1)\mathrm{P}^y,$$

the subspace Γ_1 coincides with the space of polynomials in $(\hat{\mathrm{P}}^t, \mathrm{P}^y, \mathrm{D})$ and the subspace Γ_2 possesses the bases constituted by the monomials $\mathrm{K}^{\alpha_1}(\hat{\mathrm{P}}^t)^{\alpha_2}\mathrm{D}^{\alpha_3}$ with $\alpha_1 > 0$.

Corollary 2.11. *The algebra $\Upsilon_{\mathfrak{f}}$ is isomorphic to the universal enveloping algebra $\mathfrak{U}(\mathrm{sl}(2,\mathbb{R}))$ of the real rank-two special linear Lie algebra $\mathrm{sl}(2,\mathbb{R})$.*

Proof. The correspondence $-\mathcal{P}^y \mapsto \mathrm{P}^y$, $-\mathcal{D} \mapsto \mathrm{D}$ and $-\mathcal{K} \mapsto \mathrm{K}$ by linearity extends to the Lie algebra homomorphism φ from $\mathfrak{f}$ to the Lie algebra $\Upsilon_{\mathfrak{f}}^{(-)}$ associated with the associative algebra $\Upsilon_{\mathfrak{f}}$, $\varphi\colon \mathfrak{f} \to \Upsilon_{\mathfrak{f}}^{(-)}$. By the universal property of the universal

enveloping algebra $\mathfrak{U}(\mathfrak{f})$, the Lie algebra homomorphism φ extends to the (unital) associative algebra homomorphism $\hat{\varphi}\colon \mathfrak{U}(\mathfrak{f}) \to \Upsilon_{\mathfrak{f}}$, i.e., $\varphi = \hat{\varphi} \circ \iota$ as homomorphisms of vector spaces, where $\iota\colon \mathfrak{f} \to \mathfrak{U}(\mathfrak{f})$ is the canonical embedding of $\mathfrak{f}$ in $\mathfrak{U}(\mathfrak{f})$. Since the algebra $\Upsilon_{\mathfrak{f}}$ is generated by $\varphi(\mathfrak{f})$, the homomorphism $\hat{\varphi}$ is surjective. Fixed an ordering of the basis elements of $\mathfrak{f}$, in view of the Poincaré–Birkhoff–Witt theorem, the homomorphism $\hat{\varphi}$ maps the corresponding Poincaré–Birkhoff–Witt basis of $\mathfrak{U}(\mathfrak{f})$ to a basis of $\Upsilon_{\mathfrak{f}}$. Therefore, $\hat{\varphi}$ is an isomorphism. Thus $\mathfrak{f} \simeq \mathrm{sl}(2,\mathbb{R})$ implies $\mathfrak{U}(\mathfrak{f}) \simeq \mathfrak{U}(\mathrm{sl}(2,\mathbb{R}))$. $\square$

Given a linear system of differential equations $\mathcal{L}$ and its linear generalized symmetry with the associated linear differential operator Q, we have that *$\mathrm{Q}h$ is a solution of $\mathcal{L}$ whenever h is*. This can be interpreted as generating solutions from known ones using linear generalized symmetries [73]; see [136, 137] for first examples of such generation. In view of Conjecture 2.8 and Lemma 2.9, the most general formula for generating solutions of the fine Kolmogorov backward equation (2.2) using its linear generalized symmetries from the seed solution $u = h(t,x,y)$ takes the form

$$\tilde{u} = \sum_{\alpha \in \mathbb{N}_0^3} \lambda_\alpha \mathrm{Q}^\alpha h(t,x,y). \tag{2.10}$$

Here all but finitely many (real) constants λ_α are equal to zero, and any ordering in $\{\mathcal{P}^y, \mathcal{D}, \mathcal{K}\}$ can be fixed.

However, depending on h and λ_α, $\alpha \in \mathbb{N}_0^3$, this formula can result in a known or even the zero solution. Let us analyze which solutions generated according (2.10) from Lie invariant solutions of (2.2) may be of interest. It suffices to consider only the solutions that are invariant with respect to one of the inequivalent one-dimensional subalgebras of $\mathfrak{g}^{\mathrm{ess}}$ listed in Corollary 2.7. Such solutions were described in Section 2.5. We exclude the subalgebra $\mathfrak{s}_{1.2}$ as that admitting a trivial zero space of invariant solutions.

In view of Corollary 2.11, the operator $\hat{\mathrm{P}}^t$ generates the center Z of the algebra $\Upsilon_{\mathfrak{f}}$. Therefore, for any subalgebra $\mathfrak{s}$ of $\mathfrak{g}^{\mathrm{ess}}$, any polynomial of $\hat{\mathrm{P}}^t$ maps the space of $\mathfrak{s}$-invariant solutions of (2.2) to itself. It is also clear that given an element Q of $\mathfrak{g}^{\mathrm{ess}}$ and the corresponding operator Q from $\Upsilon_{\mathfrak{f}}$, the action by any element from the principal left ideal $(\mathrm{Q}) := \Upsilon_{\mathfrak{f}}\mathrm{Q}$ on any $\langle Q\rangle$-invariant solution of (2.2) results in the zero solution. More generally, any element of $(\mathrm{Q}) + \mathrm{C}(\mathrm{Q})$, i.e., the sum of any elements from the principal left ideal (Q) and from the centralizer $\mathrm{C}(\mathrm{Q})$ of Q in $\Upsilon_{\mathfrak{f}}$ (obviously

containing Z) maps the space of $\langle Q\rangle$-invariant solutions of (2.2) to itself.

1.1^{μ}. For the above reason, the Lie-symmetry operator $\hat{\mathrm{P}}^t-\mu$, which corresponds to the Lie-symmetry vector field $-\mathcal{P}^t-\mu\mathcal{I}$ of (2.2), commutes with any monomial $\mathbf{Q}^{\alpha}$. Hence the action by any element from $\Upsilon_{\mathfrak{f}}$ maps the set of $\mathfrak{s}^{\mu}_{1.1}$-invariant solutions to itself. This implies that the solution families (2.6) and (2.8) of the equation (2.2) cannot be extended using the generation of its solutions by the action of its Lie-symmetry operators according to (2.10). Another manifestation of the above phenomenon is that each Lie-symmetry vector field of (2.6) induces a Lie-symmetry vector field of reduced equation 1.1^{μ}. Hence the algebra $\Upsilon_{\mathfrak{f}}$ induces the entire algebra generated by Lie-symmetry operators of reduced equation 1.1^{μ} if $\mu\neq-\frac{3}{16}$ and a proper subalgebra of the latter algebra otherwise.

The rest of the cases are more complicated to analyze. We make only particular remarks on them.

$1.3^{1,\mu}$. For the same reason as above, the operator $\mathrm{P}^y+\hat{\mathrm{P}}^t-\mu$ commutes with any polynomial in P^y and $\hat{\mathrm{P}}^t$. This is why the action by such polynomials maps the space of $\mathfrak{s}^{1,\mu}_{1.3}$-invariant solutions to itself. It can be shown that the centralizer of P^y in $\Upsilon_{\mathfrak{f}}$ and thus the centralizer of $\mathrm{P}^y+\hat{\mathrm{P}}^t-\mu$ in $\Upsilon_{\mathfrak{f}}$ coincides with the space of these polynomials.

$1.4^{\delta'}$. It is clear that for any $\delta'\in\{0,1\}$ the operator $\mathrm{P}^y-\delta'$ commutes with P^y. Hence the space of $\mathfrak{s}^{\delta'}_{1.4}$-invariant solutions is mapped to itself at least by the action by any polynomial in P^y and $\hat{\mathrm{P}}^t$.

Let $\delta'=0$. In view of Remark 2.10, we have the representation $\Upsilon_{\mathfrak{f}}=\Gamma_1\dotplus\Gamma_2$. Since Γ_1 is the space of polynomials in $(\hat{\mathrm{P}}^t,\mathrm{P}^y,\mathrm{D})$, any element of it acts within the space of $\mathfrak{s}^{0}_{1.4}$-invariant solutions. This is why to generate new solutions of (2.2) from solutions of the form (2.7) within the framework discussed, up to linearly combining solutions, it suffices to iteratively act by the operator K, see Section 2.7 for explicit formulas and the relation to generalized reductions.

$1.5^{\nu,\mu}$. Recall that this case corresponds to the two subalgebra families listed in Corollary 2.7, $\{\mathfrak{s}^{\nu',\mu}_{1.5}\}$ and $\{\mathfrak{s}^{\nu}_{1.6}\}$, see the beginning of Section 2.5. Since $\mathrm{KD}=(\mathrm{D}-1)\mathrm{K}$ and $\mathrm{P}^y\mathrm{D}=(\mathrm{D}+1)\mathrm{P}^y$, we have

$$\mathrm{K}^{\alpha_1}(\mathrm{P}^y)^{\alpha_2}\mathrm{D}^{\alpha_3}(\mathrm{D}+\nu\hat{\mathrm{P}}^t-\mu)=(\mathrm{D}+\nu\hat{\mathrm{P}}^t-\mu-\alpha_1+\alpha_2)\mathrm{K}^{\alpha_1}(\mathrm{P}^y)^{\alpha_2}\mathrm{D}^{\alpha_3}.$$

In other words, the monomial $\mathrm{K}^{\alpha_1}(\mathrm{P}^y)^{\alpha_2}\mathrm{D}^{\alpha_3}$ maps the space of $\{\mathfrak{s}_{1.5}^{\nu,\mu}\}$-invariant solutions to the space of $\{\mathfrak{s}_{1.5}^{\nu,\tilde{\mu}}\}$-invariant solutions, where $\tilde{\mu}=\mu+\alpha_1-\alpha_2$. Hence any element of $\Upsilon_{\mathfrak{f}}$ maps the space spanned by the $\{\mathfrak{s}_{1.5}^{\nu,\tilde{\mu}}\}$-invariant solutions with μ running through $\mathbb{R}$ into itself.

$1.7^{\nu,\mu}$. We can only state that the action by each polynomial in $\mathrm{P}^y+\mathrm{K}$ and $\hat{\mathrm{P}}^t$ maps the space of $\mathfrak{s}_{1.7}^{\nu,\mu}$-invariant solutions to itself since such polynomials commute with the operator $\mathrm{P}^y+\mathrm{K}+\nu\hat{\mathrm{P}}^t-\mu$.

2.7 Generalized reductions

Linear generalized symmetries of a linear system $\mathcal{L}$ of differential equations in the unknown functions $u=(u^1,\dots,u^m)$ can be used not only for generating solutions from known ones that are obtained by other methods but also for constructing solutions from scratch. The basis of the method of linear generalized reduction is the fact that if Q is a symmetry operator of $\mathcal{L}$, then the differential constraint $\mathrm{Q}u=0$ is compatible with $\mathcal{L}$. Nevertheless, the procedure of finding generalized invariant solutions is not as simple as its counterpart for Lie-invariant solutions at all. To construct a generalized ansatz, it is necessary to construct the general solution of the system $\mathrm{Q}u=0$ in an explicit form, which is possible only in particular cases. Moreover, it often happens that the solutions invariant with respect to a generalized symmetry can be obtained in a much simpler way by generation from known solutions.

We present two peculiar examples of generalized reductions of the equation (2.2), and the first series of generalized invariant solutions is related to the results of Section 2.6.

The equation (2.2) admits generalized symmetries $\big(\mathrm{D}_y^n u\big)\partial_u$, $n\in\mathbb{N}$, which are generated from its elementary Lie symmetry $u\partial_u$ by acting with the recursion operators $\mathrm{P}^y=\mathrm{D}_y$ associated with its Lie symmetry $-\mathcal{P}^y=-\partial_y$. This choice of a Lie symmetry for generating generalized symmetries is justified by the consideration of codimension-one Lie reduction 1.4^0 in Section 2.5. The differential constraint $\mathrm{D}_y^n u=0$ can be easily integrated, which gives the corresponding generalized ansatz

$$u=\sum_{s=0}^{n-1} w^s(t,x)\frac{y^s}{s!}. \tag{2.11}$$

Here and in what follows w^s are sufficiently smooth functions of their arguments, the index s runs from 0 to $n-1$, and $\epsilon = \operatorname{sgn} x$.

Substituting the generalized ansatz (2.11) into (2.2) and splitting the resulting equation with respect to y, we obtain the system $w^s_t = x^2 w^s_{xx} - x w^{s+1}$ with $w^n := 0$. The point transformation $z_1 = t$, $z_2 = \ln|x|$, $\tilde w^s = \mathrm{e}^{\frac14 t}|x|^{-1/2} w^s$ maps it to the system

$$\mathcal{H}_n\colon \quad \tilde w^s_1 = \tilde w^s_{22} - \epsilon \mathrm{e}^{z_2}\tilde w^{s+1}$$

with $\tilde w^n := 0$, which is easier for integrating since its last equation is the (1+1)-dimensional linear heat equation. This is why the general solution of the system $\mathcal{H}_n$ can be expressed in terms of n arbitrary solutions of this equation. We can obtain a particular solution of the system $\mathcal{H}_n$ by prolonging any solution of $\mathcal{H}_{n'}$ with $n' < n$ by zeros to $\tilde w^s$, $s = n', \dots, n-1$, and hence such solutions can be considered as inessential.

Lemma 2.12. *The solutions of $\mathcal{H}_n$ that are essential up to subtracting the prolonged solutions of $\mathcal{H}_{n'}$, $n' = 1, \dots, n-1$, and thus have $\tilde w^{n-1} \neq 0$, take the form*

$$\tilde w^s = \epsilon^s \mathrm{e}^{(n-s-1)z_2}\left(\prod_{k=n-s}^{n-1}(2k\partial_2 + k^2)\right)\theta(z_1, z_2),$$

where $\theta = \theta(z_1, z_2)$ is an arbitrary solution of the heat equation.

Proof. In fact, we should construct a particular solution of the system of $n-1$ first equations of $\mathcal{H}_n$ for an arbitrary solution of the last equation of $\mathcal{H}_n$. We use the induction with respect to the parameter n. The induction base $n = 0$ is clear.

Let now $n > 0$. We reduce the construction of the solution to that for the system $\mathcal{H}_{n-1}$. We substitute an ansatz $\tilde w^0 = \mathrm{e}^{(n-1)z_2}\theta(z_1, z_2)$ with a solution θ of the heat equation $\theta_1 = \theta_{22}$ to the first equation of the system $\mathcal{H}_n$ and thus derive the representation $\tilde w^1 = \mathrm{e}^{(n-2)z_2}\tilde\theta$, where $\tilde\theta = \epsilon(2(n-1)\partial_2 + (n-1)^2)\theta$. Since $\epsilon(2(n-1)\partial_2 - (n-1)^2\theta\partial_\theta)$ is a Lie-symmetry vector field of the heat equation $\theta_1 = \theta_{22}$, the function $\tilde\theta$ is a solution of this equation as well. The subsystem of the $n-1$ last equations of $\mathcal{H}_n$ coincides with the system $\mathcal{H}_{n-1}$ up to the index shift by 1, $s' = s-1$. By the induction hypothesis,

this subsystem has the particular solution

$$\tilde{w}^s=\epsilon^{s-1}\mathrm{e}^{(n-1-s)z_2}\left(\prod_{k=n-s}^{n-2}(2k\partial_2+k^2)\right)\tilde{\theta}(z_1,z_2),$$

where s ranges through the set $\{1,\dots,n-1\}$. Substituting the expression of $\tilde{\theta}$ in terms of θ into this solution and completing the range of s by zero in view of the ansatz for $\tilde{w}^0$, we obtain the required representation for particular solutions of the system $\mathcal{H}_n$, where the component $\tilde{w}^{n-1}=\epsilon^{n-1}\big(\prod_{k=1}^{n-1}(2k\partial_2+k^2)\big)\theta$ runs through the solution set of the linear heat equation if the function θ does, see [74, Lemma 7]. □

Therefore, the essential solutions of the equation (2.2) that are associated with the generalized ansatz (2.11) take the form

- $$u=\mathrm{e}^{-\frac{1}{4}t}|x|^{n-\frac{1}{2}}\sum_{s=0}^{n-1}\frac{y^sx^{-s}}{s!}\left(\prod_{k=n-s}^{n-1}(2kx\partial_x+k^2)\right)\theta(t,\ln|x|),\tag{2.12}$$

where $\theta=\theta(z_1,z_2)$ is an arbitrary solution of the heat equation $\theta_1=\theta_{22}$.

The solutions of the form (2.12) admit an interpretation in terms of the generation of new solutions by the action of Lie-symmetry operators on Lie-invariant solutions.

Lemma 2.13. *For any* $n\in\mathbb{N}$, $\mathrm{D}_y^{n+1}\mathrm{K}^n=\left(\prod_{k=1}^{n}(\mathrm{D}_y\mathrm{K}+2k\mathrm{D}_y+k^2+k)\right)\mathrm{D}_y$.

Lemma 2.13 implies that K^nu satisfies the equation $\mathrm{D}_y^{n+1}u=0$ if u is a solution of the equation $\mathrm{D}_yu=0$. There is even a more precise assertion.

Theorem 2.14. *For any* $n\in\mathbb{N}$, *the family of solutions of the form* (2.12) *that are invariant with respect to the generalized symmetries* $(\mathrm{D}_y^nu)\partial_u$ *can be expressed in terms of the iterative action by the Lie-symmetry operator* K *on* $\mathfrak{s}_{1.4}^0$*-invariant solutions of* (2.2), $u=\mathrm{K}^{n-1}\big(\mathrm{e}^{-\frac{1}{4}t}|x|^{1/2}\theta(t,\ln|x|)\big)$.

This theorem is proved by induction with respect to the parameter n, which involves cumbersome computations.

The second example of generalized reductions of the equation (2.2) is related to its recursion operator $\mathrm{D}_t-\mu$, which is associated with the Lie-symmetry vector field $-\partial_t-\mu u\partial_u$. Thus, this equation admits the generalized symmetries $((\mathrm{D}_t-\mu)^nu)\partial_u$,

$n\in\mathbb{N}$. We can expect a nontrivial output of applying the generalized reductions with respect to these symmetries due to their association with Lie reduction 1.1^{μ}. The corresponding generalized ansatz is

$$u=\mathrm{e}^{\mu t}\sum_{s=0}^{n-1}w^s(x,y)\frac{t^s}{s!}.$$

Substituting it into (2.2) and splitting the resulting equation with respect to t, we obtain the system $xw^s_y=x^2w^s_{xx}-\mu w^s-w^{s+1}$ with $w^n:=0$. It is mapped by the point transformation $z_1=\epsilon y$, $z_2=2|x|^{\frac12}$, $\tilde w^s=|x|^{-\frac14}w^s$ to the system

$$\mathcal{S}^{\mu}_n\colon\quad \tilde w^s_1=\tilde w^s_{22}-\frac{4\mu+3/4}{z_2^2}\tilde w^s-\frac{4}{z_2^2}\tilde w^{s+1}$$

with $\tilde w^n:=0$, whose last equation is the linear $(1+1)$-dimensional heat equation with the inverse square potential $(4\mu+3/4)z_2^{-2}$.

The Lie-symmetry vector field $-\mathcal{P}^t-\mu\mathcal{I}$ belongs to the center $\mathfrak{z}$ of the algebra $\mathfrak{g}^{\rm ess}$ and thus the associated Lie-symmetry operator $\hat{\rm P}^t-\mu$ commutes with all elements of $\Upsilon_{\mathfrak{f}}$. This is why the solutions of (2.2) that are invariant with respect to $((\mathrm{D}_t-\mu)^nu)\partial_u$ cannot be related to generating solutions with iterative action of Lie-symmetry operators, cf. paragraph 1.1^{μ} in Section 2.6.

Consider the particular case $\mu=0$ and $n=2$. The system $\mathcal{S}^0_2$ takes the form

$$\tilde w^0_1-\tilde w^0_{22}+\frac{3}{4z_2^2}\tilde w^0=-\frac{4}{z_2^2}\tilde w^1,\quad \tilde w^1_1-\tilde w^1_{22}+\frac{3}{4z_2^2}\tilde w^1=0.$$

We consider the first equation of $\mathcal{S}^0_2$ as an inhomogeneous linear equation with respect to w^0 and look for its particular solution in the form $\tilde w^0=-2z_2^{-1}v$, where the new unknown function $v=v(z_1,z_2)$ satisfies the equation

$$v_1-v_{22}-\frac14z_2^{-2}v=0.$$

Substituting this ansatz into the first equation of $\mathcal{S}^0_2$, we derive the representation $\tilde w^1=v_2-\frac12z_2^{-1}v$, which means that $\tilde w^1$ is the Darboux transformation of v with respect to the solution $v=|z_2|^{1/2}$, see [91, 126] for the required definitions and results. This Darboux transformation exactly relates the equation for v and the second equation of $\mathcal{S}^0_2$. As a result, up to linearly combining with $\mathfrak{s}^0_{1.1}$-invariant solutions, we construct

the following family of solutions of the equation (2.2):

- $u = |x|^{1/4}\Big(t\vartheta_2^{3/4}(\varepsilon y, 2\sqrt{|x|}) - (\frac{1}{4}t-1)|x|^{-1/2}\vartheta^{3/4}(\varepsilon y, 2\sqrt{|x|})\Big)$, (2.13)

where $\vartheta^{3/4} = \vartheta^{3/4}(\tilde z_1, \tilde z_2)$ is an arbitrary solution of the equation $\vartheta_1 = \vartheta_{22} - \frac{3}{4}\tilde z_2^{-2}\vartheta$, and $\vartheta_2^{3/4}$ is its derivative with respect to its second argument. Recall that wide families of solutions for the latter equation were obtained in [70, Section A].

2.8 Conclusion

The symmetry properties of the fine Kolmogorov backward equation (2.2) are prominent within both the general class $\bar{\mathcal{F}}$ of $(1+2)$-dimensional ultraparabolic linear partial differential equations and the subclass $\mathcal{F}$ of such equations with the standard ultraparabolic part and power diffusivity, see Section 2.1. As a Kolmogorov backward equation, it also appears in various applications. These facts motivated us to thoroughly carry out the extended symmetry analysis of the equation (2.2).

The initial point of this study is the computation of the maximal Lie invariance algebra $\mathfrak{g}$ of the equation (2.2) in Section 2.2. The algebra $\mathfrak{g}$ splits over its infinite-dimensional ideal $\mathfrak{g}^{\rm lin}$, which is associated with the linear superposition of solutions of (2.2), $\mathfrak{g} = \mathfrak{g}^{\rm ess} \in \mathfrak{g}^{\rm lin}$. The subalgebra $\mathfrak{g}^{\rm ess}$ is five-dimensional and reductive. It is isomorphic to the algebra $\mathrm{sl}(2,\mathbb{R})\oplus 2A_1$ and thus to $\mathrm{gl}(2,\mathbb{R})\oplus A_1$. Up to the equivalence with respect to point transformation, the above symmetry properties completely single out the equation (2.2) among the members of the class $\bar{\mathcal{F}}$.

In Section 2.3 we have computed the point symmetry pseudogroup G of the equation (2.2) using the direct method reinforced by the our knowledge of transformational properties of the entire class $\bar{\mathcal{F}}$. According to [70, Theorem 1], the class $\bar{\mathcal{F}}$ is normalized in the usual sense. Under the identification of the pseudogroup G with the vertex group of (2.2) in the equivalence groupoid $\mathcal{G}^{\sim}_{\bar{\mathcal{F}}}$ of $\bar{\mathcal{F}}$, this implies that the pseudogroup G is contained in the natural projection of the equivalence group $G^{\sim}_{\bar{\mathcal{F}}}$ of $\bar{\mathcal{F}}$ to the space with the coordinates (t,x,y,u). The application of the reinforced direct method has led to a significant simplification of the system of determining equations for the transformation components, which is still a coupled overdetermined nonlinear

system of partial differential equations. We have successfully found the general solution to this complicated system and constructed a nice representation of the elements of the group G in Theorem 2.1.

Furthermore, after redefining the multiplication in the pseudogroup G following the approaches from [70, 74], we have exhaustively studied its structure. The pseudogroup G splits over its normal pseudosubgroup $G^{\rm lin}$ that is associated with the linear superposition of solutions of (2.2), $G = G^{\rm ess} \leftthreetimes G^{\rm lin}$, and due to using the modified multiplication, we can choose the first component $G^{\rm ess}$ of the splitting to be a group, which is, moreover, a five-dimensional Lie group. This analysis required some notions from the general theory of pseudogroups of transformations, which we also have revisited and enhanced in Section 1.3.1. The group $G^{\rm ess}$ admits the factorization $G^{\rm ess} = F \times Z$, where $F \simeq {\rm PSL}^{\pm}(2,\mathbb{R})$ and $Z \simeq (\mathbb{R}^2,+)\times\mathbb{Z}_2$, which corresponds to the Lie algebra decomposition $\mathfrak{g}^{\rm ess} = \mathfrak{f}\oplus\mathfrak{z}$. Another convenient factorization for the study of the structure of $G^{\rm ess}$ is $G^{\rm ess} = H \times P$, where $H \simeq {\rm GL}(2,\mathbb{R})$ and $P \simeq (\mathbb{R},+)\times\mathbb{Z}_2$. In the process of constructing the decompositions of $G^{\rm ess}$, we used the results given in Section 2.A about the structure of the two-by-two matrix groups ${\rm SL}_2^{\pm}(\mathbb{R})$, ${\rm PSL}_2^{\pm}(\mathbb{R})$ and ${\rm GL}_2(\mathbb{R})$.

The accurate description of the structure of $G^{\rm ess}$ has allowed us to correctly classify the subalgebras of the algebra $\mathfrak{g}^{\rm ess}$ in Section 2.4 modulo two equivalences. One equivalence is induced by the action of the inner automorphism group ${\rm Inn}(\mathfrak{g}^{\rm ess})$, which is the same as the action of $G^{\rm ess}_{\rm id}$ on $\mathfrak{g}^{\rm ess}$ by the adjoint representation. The other equivalence is defined by the action of the Lie group $G^{\rm ess}$ on its Lie algebra $\mathfrak{g}^{\rm ess}$.

The classification of $G^{\rm ess}$-inequivalent subalgebras of $\mathfrak{g}^{\rm ess}$ allowed us to exhaustively describe Lie reductions of the equation (2.2) in Section 2.5. Due to the comprehensive study of hidden symmetries of (2.2), we have shown that it suffices to carry out only inequivalent Lie reductions of codimension one. For each of these reductions, the corresponding reduced equation is a linear homogeneous second-order evolution equation in two independent variables, which has been mapped to a linear (1+1)-dimensional heat equation with a potential depending only on the space-like independent variable. The peculiar reductions occurred when the corresponding potential is trivial or the inverse square one led to constructing wide families of explicit solutions (2.6)–(2.8) of (2.2).

Section 2.6 is devoted to generating solutions of the equation (2.2) using its Lie and generalized symmetries. We have discussed various situations where the (repeated)

action by Lie-symmetry operators may or may not give new solutions, which is the first discussion of this kind in the literature. An important role in the discussion was played by the nonobvious fact that the operator associated with the right-hand side of (2.2) can be interpreted as a Casimir operator of the Levi factor $\mathfrak{f} = \langle \mathcal{P}^y, \mathcal{D}, \mathcal{K} \rangle$ of $\mathfrak{g}^{\rm ess}$. This, in particular, has allowed us to express the equation (2.2) in terms of its Lie-symmetry operators. This is a non-generic feature; for instance, even the remarkable Fokker–Planck equation, which is very singular from the symmetry perspective, lacks this property. This factorisability, along with the structure of the symmetry algebra $\mathfrak{g}$, have led us to Conjecture 2.8 on the structure of the algebra of generalized symmetries of (2.2).

Carrying out some particular generalized reductions of the equation (2.2) in Section 2.7, we have constructed wide families of its solutions parameterized by an arbitrary finite number of arbitrary solutions of the (1+1)-dimensional linear heat equation (2.12) or one or two arbitrary solutions of (1+1)-dimensional linear heat equations with inverse square potentials (2.13). To the best of our knowledge, these are the first exact solutions of the equation (2.2) in the literature, and they can be important for testing numerical solvers for this equation. We have treated the above results in the context of those on generating solutions.

We reasonably conjectured the structure of the algebra of generalized symmetries of the equation (2.2) in [73]. Nevertheless, we expect that the proof of this conjecture is sophisticated and cumbersome and it should be the subject of a separate study. The classification of reduction modules of the equation (2.2) is also of interest, especially singular ones in view of its ultraparabolicity. Besides, one could study potential symmetries of this equation although up to now there are no meaningful examples of potential symmetries of multidimensional partial differential equations (both linear and nonlinear) in the literature.

Appendix 2.A Two-by-two matrix groups as semi-direct products

In this section we present a proof that the group $\mathrm{SL}^{\pm}(2,\mathbb{R})$ of two-by-two matrices with determinant ±1 is a semidirect product of its subgroup $\mathrm{SL}(2,\mathbb{R})$ and the subgroup

isomorphic to $\mathbb{Z}_2$, and as a corollary we obtain the analogous statement for the groups $\mathrm{PSL}^{\pm}(2,\mathbb{R})$ and $\mathrm{GL}(2,\mathbb{R})$. Despite the fact that the statement looks classical, we did not find the proof in the literature.

Proposition 2.15. $\mathrm{SL}^{\pm}(2,\mathbb{R}) \simeq \mathrm{SL}(2,\mathbb{R}) \rtimes_{\varphi} \mathbb{Z}_2$, *where* φ *is the homomorphism from* $\mathbb{Z}_2$ *to the group of automorphisms of* $\mathrm{SL}(2,\mathbb{R})$, $\varphi\colon \mathbb{Z}_2 \to \mathrm{Aut}(\mathrm{SL}(2,\mathbb{R}))$, *whose value* $\varphi(\bar{1})$ *on the generator* $\bar{1}$ *of* $\mathbb{Z}_2$ *is the involution*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

Proof. Consider the group homomorphism $\det\colon \mathrm{SL}^{\pm}(2,\mathbb{R}) \to \mathrm{U}(\mathbb{Z})$, where $\mathrm{U}(\mathbb{Z})$ denotes the group of units of the ring $\mathbb{Z}$, $\mathrm{U}(\mathbb{Z}) = \{1,-1\}$, i.e., the set of all invertible elements of the ring $\mathbb{Z}$ with the operation given by the multiplication in $\mathbb{Z}$. The group $\mathrm{U}(\mathbb{Z})$ is isomorphic to $\mathbb{Z}_2$. The following sequence is exact

$$\mathrm{SL}(2,\mathbb{R}) \overset{\iota}{\hookrightarrow} \mathrm{SL}^{\pm}(2,\mathbb{R}) \overset{\det}{\twoheadrightarrow} \mathrm{U}(\mathbb{Z}),$$

where ι is the inclusion homomorphism.

We need to show that the above sequence splits. In other words, there exists a right section of det, $\nu\colon \mathrm{U}(\mathbb{Z}) \to \mathrm{SL}^{\pm}(2,\mathbb{R})$, such that $\mathrm{SL}^{\pm}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})\nu(\mathrm{U}(\mathbb{Z}))$ and $\det \circ \nu = \mathrm{id}_{\mathrm{U}(\mathbb{Z})}$. To satisfy the first condition, we should require that $\det \nu(-1) = -1$, and since $\nu(-1)^2 = \mathrm{diag}(1,1)$, the matrix $\nu(-1)$ is diagonalizable with the eigenvalues ± 1, so the Jordan normal form for $\nu(-1)$ is $\mathrm{diag}(1,-1)$.

Taking the map $\nu\colon \mathrm{U}(\mathbb{Z}) \to \mathrm{SL}^{\pm}(2,\mathbb{R})$ defined on the generator of $\mathrm{U}(\mathbb{Z})$ by $-1 \mapsto \mathrm{diag}(1,-1)$ as the canonical homomorphism, we get that $\mathrm{SL}^{\pm}(2,\mathbb{R})$ is the semidirect product of its subgroups $\mathrm{SL}(2,\mathbb{R})$ and $\nu(\mathrm{U}(\mathbb{Z}))$, $\mathrm{SL}^{\pm}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R}) \rtimes \nu(\mathrm{U}(\mathbb{Z}))$. The conjugation in $\mathrm{SL}(2,\mathbb{R})$ by the matrix $\nu(-1)$ gives the desired involution $\varphi(\bar{1})$. □

Corollary 2.16. $\mathrm{PSL}^{\pm}(2,\mathbb{R}) \simeq \mathrm{PSL}(2,\mathbb{R}) \rtimes_{\varphi} \mathbb{Z}_2$, *where* φ *is a homomorphism from* $\mathbb{Z}_2$ *to the automorphism group of* $\mathrm{PSL}(2,\mathbb{R})$, $\varphi\colon \mathbb{Z}_2 \to \mathrm{Aut}(\mathrm{PSL}(2,\mathbb{R}))$, *that is given (on the generator of* $\mathbb{Z}_2$*) by the involution*

$$\varphi(\bar{1}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathrm{Z}(\mathrm{SL}(2,\mathbb{R})) \mapsto \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \mathrm{Z}(\mathrm{SL}(2,\mathbb{R})).$$

Corollary 2.17. $\mathrm{GL}(2,\mathbb{R}) \simeq \mathrm{GL}^+(2,\mathbb{R}) \rtimes_\varphi \mathbb{Z}_2$, *where* φ *is a homomorphism from* $\mathbb{Z}_2$ *to the automorphism group of* $\mathrm{GL}^+(2,\mathbb{R})$, $\varphi\colon \mathbb{Z}_2 \to \mathrm{Aut}(\mathrm{GL}^+(2,\mathbb{R}))$, *that is given (on the generator of* $\mathbb{Z}_2$*) by the involution*

$$\varphi(\bar{1}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

Chapter 3

Surprising symmetry properties and exact solutions of Kolmogorov backward equations with power diffusivity

3.1 Introduction

The Kolmogorov backward equations originated from the foundational work by Kolmogorov [67], where he extended the theory of discrete-time Markov processes to its continuous counterpart. Among the latter, Kolmogorov distinguished diffusion processes and, for describing their evolution, derived a pair of classes of equations, usually referred to as Kolmogorov forward (or Fokker–Planck) and Kolmogorov backward equations. Since these equations capture fundamental features of diffusion processes, they became the cornerstones of many branches of physics, natural sciences and even economics. For instance, in physics these equations are instrumental in modeling lasers and other nonlinear systems far from thermal equilibrium [130, Chapter 12]. Specifically for the Kolmogorov backward equations, they are applied in evolutionary biology, ecology and environmental science to model and predict the dynamics of ecosystems and populations.

In the present paper, we carry out extended symmetry analysis of the ultra-parabolic (1+2)-dimensional Kolmogorov backward equations of the form

$$\mathcal{F}_{\alpha\beta}\colon\ u_t + xu_y = |x-\alpha|^{\beta}u_{xx}, \tag{3.1}$$

which were formally introduced in [151].[1] Here α and β are arbitrary real parameters. The more general Kolmogorov backward equations $u_t + xu_y = a(x)u_{xx} + b(x)u_x$, appeared in [131, Eq. (30)] as models for describing the evolution of cell populations, and then only the values $a = \text{const}$ and $b = 0$ were selected as the simplest ones [131, Eq. (32)]. Equations of the form (3.1) with $\alpha = 0$ also arise in the course of considering path-dependent options in a constant elasticity variance environment [10, Section 5.1]. It is also shown in [10] that the Kolmogorov equations modeling the price of the geometric and arithmetic average strike call option are reduced by point transformations to the above equations, where in addition $\beta = 0$ and $\beta = 2$, respectively.

Furthermore, the equations of the form (3.1) are important beyond their practical applications. They also constitute a family of interesting objects for theoretical studying within the field of symmetry analysis of differential equations. The study of their symmetries provides powerful insights into the dynamics of real-world phenomena these equations model. Moreover, it offers the potential for constructing their exact solutions.

Switching to the established rigorous framework of group analysis of differential equations, the main object of the present paper is the class of differential equations of the form (3.1) denoted by $\mathcal{F}$ with the tuple of arbitrary elements $\theta := (\alpha, \beta)$, whose components as constants run through the solution set of the auxiliary system of differential equations $\alpha_t = \alpha_x = \alpha_y = 0$, $\beta_t = \beta_x = \beta_y = 0$. See [12, 17, 111, 126, 146] and references therein for the definition of classes of differential equations, a theoretical background on their transformational properties and related computational techniques. In contrast to the preliminary consideration of the class $\mathcal{F}$ in [151], we compute the (usual) equivalence group $G^{\sim}_{\mathcal{F}}$ and the equivalence groupoid $\mathcal{G}^{\sim}_{\mathcal{F}}$ of this

[1] The formally adjoint to the Kolmogorov forward equation $u_t + xu_y = (|x-\alpha|^{\beta}u)_{xx}$ is the (true) Kolmogorov backward equation $-u_t - xu_y = |x-\alpha|^{\beta}u_{xx}$. For convenience, we simultaneously change the signs of t and y, obtaining the equation (3.1). This change is not essential from the point of view of symmetry analysis of differential equations, and we preserve the name "Kolmogorov backward equation" for the equations of the form (3.1).

class and thus find out unobvious discrete equivalences between equations from $\mathcal{F}$. Moreover, we easily show that any equation from the class $\mathcal{F}$ is $G^{\sim}_{\mathcal{F}}$-equivalent to the equation

$$\mathcal{F}'_{\beta}\colon \quad u_t + xu_y = |x|^{\beta}u_{xx} \tag{3.2}$$

with the same β. The equations $\mathcal{F}'_{\beta}$ constitute the class $\mathcal{F}'$, where β remains the sole arbitrary element. The above similarity between equations from the classes $\mathcal{F}$ and $\mathcal{F}'$ means that these classes are *weakly similar* [111, Definition 3]. This is why their group classifications modulo their equivalence groupoids are in fact the same, but this is not the case for the analogous classifications modulo their equivalence groups. To relate the latter classifications, we perform more thorough analysis. As a result, we solve the complete group classification problems for the classes $\mathcal{F}$ and $\mathcal{F}'$ modulo both of the above kinds of equivalences. Developing results of [70, 75, 120], we describe other symmetry properties of equations from $\mathcal{F}$ and construct wide families of their exact solutions in closed form. Even though the arbitrary elements of the classes $\mathcal{F}'$ and $\mathcal{F}$ are just constant parameters, symmetry analysis of these classes and their single elements is highly nontrivial.

More specifically, we begin solving the equivalence problem for the classes $\mathcal{F}$ and $\mathcal{F}'$ by describing properties of their equivalence groupoids $\mathcal{G}^{\sim}_{\mathcal{F}}$ and $\mathcal{G}^{\sim}_{\mathcal{F}'}$ in Section 3.2 using the advanced version of the direct method. A disadvantage of the direct method is that its straightforward application to the classes $\mathcal{F}$ and $\mathcal{F}'$ leads to tedious and cumbersome computations. An additional complication is provided by the fact that these classes are not normalized in the usual sense. Nevertheless, the problem complexity can be reduced in the following way. We first embed the classes $\mathcal{F}$ and $\mathcal{F}'$ in the class $\bar{\mathcal{F}}$ of the linear (1+2)-dimensional second-order ultraparabolic equations of the form

$$\begin{aligned} &u_t + B(t,x,y)u_y = A^2(t,x,y)u_{xx} + A^1(t,x,y)u_x + A^0(t,x,y)u + C(t,x,y)\\ &\text{with} \quad A^2 \neq 0, \quad B_x \neq 0, \end{aligned} \tag{3.3}$$

which is normalized and whose equivalence pseudogroup $G^{\sim}_{\bar{\mathcal{F}}}$ and thus equivalence groupoid $\mathcal{G}^{\sim}_{\bar{\mathcal{F}}}$ are known [70,73]. The tuple $\bar{\theta} := (B, A^2, A^1, A^0, C)$ of arbitrary elements of the class $\bar{\mathcal{F}}$ runs through the solution set of the system of the inequalities $A^2 \neq 0$ and $B_x \neq 0$ with no restrictions on A^0, A^1 and C. The class embeddings lead to the

corresponding groupoid embeddings $\mathcal{G}^{\sim}_{\mathcal{F}} \hookrightarrow \mathcal{G}^{\sim}_{\bar{\mathcal{F}}}$ and $\mathcal{G}^{\sim}_{\mathcal{F}'} \hookrightarrow \mathcal{G}^{\sim}_{\bar{\mathcal{F}}}$. We use the description of $\mathcal{G}^{\sim}_{\bar{\mathcal{F}}}$ to derive basic properties of the equivalence groupoids $\mathcal{G}^{\sim}_{\mathcal{F}}$ and $\mathcal{G}^{\sim}_{\mathcal{F}'}$ in Section 3.2, which allows us to simplify the computation of the equivalence groups $G^{\sim}_{\mathcal{F}}$ and $G^{\sim}_{\mathcal{F}'}$ and the equivalence algebras $\mathfrak{g}^{\sim}_{\mathcal{F}}$ and $\mathfrak{g}^{\sim}_{\mathcal{F}'}$ of the classes $\mathcal{F}$ and $\mathcal{F}'$ in Sections 3.3 and 3.4, respectively. We show that the equivalence group $G^{\sim}_{\mathcal{F}'}$ contains the distinguished discrete equivalence transformation

$$\mathcal{J}'\colon \quad \tilde{t} = y\operatorname{sgn} x, \quad \tilde{x} = \frac{1}{x}, \quad \tilde{y} = t\operatorname{sgn} x, \quad \tilde{u} = \frac{u}{x}, \quad \tilde{\beta} = 5-\beta, \tag{3.4}$$

which turns out to be the only point equivalence transformation that is essential for carrying out the group classification of the class $\mathcal{F}'$. Due to this transformation, we can impose the gauge $\beta \leqslant 5/2$.

Remark 3.1. Instead of the class $\mathcal{F}'$, we can consider the subset of its equations, where the constant arbitrary element β takes only rational values with odd denominators, and replace such equations by their counterparts without the absolute value sign,

$$u_t + xu_y = x^{\beta}u_{xx}. \tag{3.5}$$

Then the equivalence transformation (3.4) is replaced by the simpler transformation

$$\tilde{t} = y, \quad \tilde{x} = \frac{1}{x}, \quad \tilde{y} = t, \quad \tilde{u} = \frac{u}{x}, \quad \tilde{\beta} = 5-\beta \tag{3.6}$$

between equations of the form (3.6). For the specific values $\beta = -1, 1, 3, 5$, when they are considered separately from the generic values of β, we present exact solutions of the corresponding equation (3.5) but not (3.2).

In Section 3.A, we also compute the equivalence groups $G^{\sim}_{\mathcal{F}}$ and $G^{\sim}_{\mathcal{F}'}$ using the automorphism-based version of the algebraic method, which was originally proposed by Hydon [48–51] for finding the point symmetry groups of single systems of differential equations and extended to finding the equivalence groups of classes of such systems in [13]. This gives the first example of such computation in the literature. Since the dimensions of the corresponding equivalence algebras $\mathfrak{g}^{\sim}_{\mathcal{F}}$ and $\mathfrak{g}^{\sim}_{\mathcal{F}'}$ are relatively high, the straightforward construction of their automorphism groups is complicated, perhaps even impossible. To overcome this challenge, we first obtain a sufficient number of

megaideals of the algebras $\mathfrak{g}^{\sim}_{\mathcal{F}}$ and $\mathfrak{g}^{\sim}_{\mathcal{F}'}$, which allowed us, after fixing bases consistent with the constructed megaideal hierarchies, to preliminarily constraint automorphism matrices for proceeding with their computation in an efficient way. In total, we observe that the construction of the groups $G^{\sim}_{\mathcal{F}}$ and $G^{\sim}_{\mathcal{F}'}$ by the algebraic method is more involved than that by the advanced version of the direct method. Nevertheless, this serves as an excellent cross-verification of the correctness of the obtained results. In addition, to the best of our knowledge, there has been no comparison in the literature of the capabilities of these two methods when applied to a particular class of differential equations.

The results of the group classifications of the classes $\mathcal{F}'$ and $\mathcal{F}$ are presented in Section 3.5. We prove that there are only two $G^{\sim}_{\mathcal{F}'}$-inequivalent cases of essential Lie symmetry extensions in the class $\mathcal{F}'$, with $\beta = 0$ and $\beta = 2$. The corresponding essential Lie invariance algebras $\mathfrak{g}^{\rm ess}_0$ and $\mathfrak{g}^{\rm ess}_2$ are eight- and five-dimensional, respectively. The essential Lie invariance algebra $\mathfrak{g}^{\rm ess}_\beta$ of the equation $\mathcal{F}'_\beta$ in the generic case, where $\beta \in \mathbb{R} \setminus \{0, 2, 3, 5\}$, is four-dimensional and is isomorphic to the algebra $A^a_{3.4} \oplus A_1$ with $a = (\beta - 2)/(\beta - 3)$ in the notation of [125]. Among these Lie algebras with $\beta \in (-\infty, 5/2] \setminus \{0, 2\}$, there are no pairwise isomorphic ones. Hence the corresponding equations are definitely $G^{\sim}_{\mathcal{F}'}$-inequivalent. The above implies that $\mathcal{G}^{\sim}_{\mathcal{F}}$-equivalent cases of essential Lie symmetry extensions within the class $\mathcal{F}$ are exhausted by the equations $\mathcal{F}_{00}$ and $\mathcal{F}_{02}$. Due to the inconsistency of the groups $G^{\sim}_{\mathcal{F}'}$ and $G^{\sim}_{\mathcal{F}}$, the classification list for the class $\mathcal{F}$ up to the $G^{\sim}_{\mathcal{F}}$-equivalence should also include the cases $\beta = 5$ and $\beta = 3$ with $\alpha = 0$, which are $G^{\sim}_{\mathcal{F}'}$-equivalent to the cases $\beta = 0$ and $\beta = 2$, respectively.

The equations $\mathcal{F}'_0$ and $\mathcal{F}'_2$ are also singled out within the class $\bar{\mathcal{F}}$ by their exceptional Lie symmetry properties. The dimension of $\mathfrak{g}^{\rm ess}_0$ is maximum for the essential Lie invariance algebras of equations from this class. Moreover, the equation $\mathcal{F}'_0$ is the unique equation with this property in $\bar{\mathcal{F}}$ modulo the point equivalence. The algebra $\mathfrak{g}^{\rm ess}_2$ is five-dimensional and nonsolvable and up to the point equivalence, the equation $\mathcal{F}'_2$ is the unique equation in the class $\bar{\mathcal{F}}$ that satisfies this property. This is why the equations $\mathcal{F}'_0$ and $\mathcal{F}'_2$ were referred to as the *remarkable Fokker–Planck equation* and the *fine Kolmogorov backward equation*, and their extended symmetry analysis was carried out in [70] and [75], respectively.

In Sections 3.6 and 3.7, we outline and complete the results of [70, 75, 120] on the

equations $\mathcal{F}'_0$ and $\mathcal{F}'_2$, including the solution generation for these equations using their recursion operators related to their Lie symmetries. We also generate new solution families of the equation $\mathcal{F}'_2$ using Darboux transformations of its codimension-one submodels, which are derived in Section 3.B. All the known results on the equations $\mathcal{F}'_0$ and $\mathcal{F}'_2$ are extended to their $G^{\sim}_{\mathcal{F}'}$-counterparts $\mathcal{F}'_5$ and $\mathcal{F}'_3$, respectively. In particular, this provides the point symmetry pseudogroups and wide families of exact solutions of $\mathcal{F}'_5$ and $\mathcal{F}'_3$. To be precise, we actually consider the equations of the form (3.5) with $\beta = 5$ and $\beta = 3$ instead of $\mathcal{F}'_5$ and $\mathcal{F}'_3$ and use the transformation (3.6) but not (3.4).

Section 3.8 is devoted to the extended symmetry analysis of the equations $\mathcal{F}'_\beta$ with $\beta \in \mathbb{R}\setminus\{0,2,3,5\}$, which constitute the case of zero essential Lie symmetry extension within the class $\mathcal{F}'$. An original combination of the direct and algebraic methods is applied in Section 3.8.1 for constructing the point symmetry pseudogroups of these equations. Jointly with the obtained descriptions of the analogous pseudogroups for the other values of β, $\beta \in \{0,2,3,5\}$, it provides the exhaustive description of the fundamental groupoid $\mathcal{G}^{\rm f}_{\mathcal{F}'}$ of the class $\mathcal{F}'$. Hence, combining it with results of Section 3.2, we obtain the complete descriptions of the equivalence groupoids $\mathcal{G}^{\sim}_{\mathcal{F}'}$ and $\mathcal{G}^{\sim}_{\mathcal{F}}$, which are presented in Section 3.9. After listing inequivalent subalgebras of the essential Lie invariance algebra $\mathfrak{g}^{\rm ess}_\beta$ of the equation $\mathcal{F}'_\beta$ for each value of β from $\mathbb{R}\setminus\{0,2,3,5\}$ in Section 3.8.2, we classify its Lie reductions of codimension one and two in Sections 3.8.3 and 3.8.4 up to the $G^{\sim}_{\mathcal{F}'}$-equivalence. In the latter section, we also show that Lie reductions of codimension three do not lead to new solutions of $\mathcal{F}'_\beta$. Involving the $G^{\sim}_{\mathcal{F}'}$-equivalence allows us to reduce the number of cases to be considered. We have found hidden symmetries and constructed wide families of exact solutions of the equations $\mathcal{F}'_\beta$ with $\beta \in \mathbb{R}\setminus\{0,2,3,5\}$ that are associated with the codimension-one Lie reductions to the linear $(1+1)$-dimensional heat equations with the zero or the inverse square potentials. The most prominent among them are the solutions associated with the value $\beta = 1$ and their $G^{\sim}_{\mathcal{F}'}$-counterparts for $\beta = 4$. Another prominent case is given by some codimension-two Lie reductions for $\beta = -1$ and their $G^{\sim}_{\mathcal{F}'}$-counterparts for $\beta = 6$, which results in Whittaker equations. What is particularly surprising is that these are both the prominent cases, where constructed families of solutions can be significantly extended using the action by recursion operators of $\mathcal{F}'_\beta$ associated with Lie symmetries of $\mathcal{F}'_\beta$, which is the subject of Section 3.8.5. In total, Section 3.8 on its own gives an exhaustive symmetry analysis of the equations $\mathcal{F}'_\beta$ for the generic case of β.

Summarizing the results of the present paper in Section 3.9, we highlight discovered

remarkable and surprising properties of the classes $\mathcal{F}'$ and $\mathcal{F}$ and their single elements, emphasize novel features of our study and outline research perspectives regarding these classes as well as the superclass $\bar{\mathcal{F}}$.

For readers' convenience, the constructed exact solutions of the equations of the form (3.2) are marked by the bullet symbol $\bullet$. Given a class of differential equations, π denotes the natural projection of the joint space of the variables, and the arbitrary elements onto the space of the variables only.

3.2 Equivalence groupoids

We first compute the equivalence groupoids and the equivalence groups of the classes $\mathcal{F}$ and $\mathcal{F}'$ using the advanced version of the direct method. See, e.g., [63, 146] and references therein for required notions and techniques. Recall that the *direct method* is based on the definition of admissible transformations of a class of differential equations. Let $\mathcal{L}|_{\mathcal{S}} = \{\mathcal{L}_\theta \mid \theta \in \mathcal{S}\}$ be a class of systems of differential equations $\mathcal{L}_\theta$ for unknown functions $u = (u^1, \dots, u^m)$ of independent variables $x = (x_1, \dots, x_n)$ with the arbitrary-element tuple $\theta = (\theta^1, \dots, \theta^k)$ running through the solution set $\mathcal{S}$ of an auxiliary system of differential equations and inequalities. Suppose that a point transformation Φ in the space with the coordinates (x, u), Φ: $\tilde{x} = X(x, u)$, $\tilde{u} = U(x, u)$ with nonzero Jacobian $|\partial(X, U)/\partial(x, u)|$, relates systems $\mathcal{L}_\theta$ and $\mathcal{L}_{\tilde{\theta}}$ from the class $\mathcal{L}|_{\mathcal{S}}$, $\Phi_*\mathcal{L}_\theta = \mathcal{L}_{\tilde{\theta}}$. Expressing the required derivatives of $\tilde{u}$ with respect to $\tilde{x}$ in terms of the original derivatives of u with respect to x using the chain rule, we substitute the derived expressions into the system $\mathcal{L}_{\tilde{\theta}}$, obtaining the system $(\Phi^{-1})_*\mathcal{L}_{\tilde{\theta}}$, which should be identically satisfied on the solutions of $\mathcal{L}_\theta$. The splitting of the obtained condition with respect to the parametric derivatives of the system $\mathcal{L}_\theta$ usually results in an overdetermined coupled system of nonlinear differential equations, which is highly difficult to solve if possible at all. However, in the particular case of the classes $\mathcal{F}$ and $\mathcal{F}'$, we are able to perform the computations both efficiently and elegantly using the embedding of these classes in the class $\bar{\mathcal{F}}$, which is constituted by the equations of the form (3.3), and the explicit description of its known equivalence groupoid. Embedding a class $\mathcal{L}|_{\mathcal{S}}$ in a superclass with known or easier computed equivalence groupoid for deriving principal constraints for admissible transformations within $\mathcal{L}|_{\mathcal{S}}$, which simplifies the further required computations, is the essence of the advanced version of the direct method.

A good choice for such a superclass is a normalized class of differential equations.

Theorem 3.2. *The class $\bar{\mathcal{F}}$ is normalized. Its (usual) equivalence pseudogroup $G^{\sim}_{\bar{\mathcal{F}}}$ consists of the point transformations with the components*

$$\tilde{t}=T(t,y),\quad \tilde{x}=X(t,x,y),\quad \tilde{y}=Y(t,y),\quad \tilde{u}=U^1(t,x,y)u+U^0(t,x,y),\tag{3.7a}$$

$$\tilde{A}^0=\frac{-U^1}{T_t+BT_y}E\frac{1}{U^1},\ \tilde{A}^1=A^1\frac{X_x}{T_t+BT_y}-\frac{X_t+BX_y}{T_t+BT_y}+A^2\frac{X_{xx}-2X_xU^1_x/U^1}{T_t+BT_y},\tag{3.7b}$$

$$\tilde{A}^2=A^2\frac{X_x^2}{T_t+BT_y},\quad \tilde{B}=\frac{Y_t+BY_y}{T_t+BT_y},\quad \tilde{C}=\frac{U^1}{T_t+BT_y}\left(C-E\frac{U^0}{U^1}\right),\tag{3.7c}$$

where T, X, Y, U^1 and U^0 are arbitrary smooth functions of their arguments with $(T_tY_y-T_yY_t)X_xU^1\neq 0$, and $E:=\partial_t+B\partial_y-A^2\partial_{xx}-A^1\partial_x-A^0$.

There are the following embeddings for $\mathcal{F}'$, $\mathcal{F}$ and $\bar{\mathcal{F}}$ considered as sets of differential equations: $\mathcal{F}'\hookrightarrow\mathcal{F}\hookrightarrow\bar{\mathcal{F}}$. These embeddings arise from the embeddings $\mathcal{S}_{\mathcal{F}'}\hookrightarrow\mathcal{S}_{\mathcal{F}}\hookrightarrow\mathcal{S}_{\bar{\mathcal{F}}}$ of the corresponding sets $\mathcal{S}_{\mathcal{F}'}$, $\mathcal{S}_{\mathcal{F}'}$ and $\mathcal{S}_{\bar{\mathcal{F}}}$ of arbitrary elements, which are respectively defined via the mappings

$$\mathcal{S}_{\mathcal{F}'}\ni(\beta)\mapsto(0,\beta)\in\mathcal{S}_{\mathcal{F}},\quad \mathcal{S}_{\mathcal{F}}\ni(\alpha,\beta)\mapsto(x,0,0,0,|x-\alpha|^\beta)\in\mathcal{S}_{\bar{\mathcal{F}}},$$

which can be treated as class reparameterizations. Thus, the embedding $\mathcal{F}'\hookrightarrow\bar{\mathcal{F}}$ corresponds to their composition $\mathcal{S}_{\mathcal{F}'}\hookrightarrow\mathcal{S}_{\bar{\mathcal{F}}}$. The above maps also induce embeddings on the level of equivalence groupoids, $\mathcal{G}^{\sim}_{\mathcal{F}'}\hookrightarrow\mathcal{G}^{\sim}_{\mathcal{F}}\hookrightarrow\mathcal{G}^{\sim}_{\bar{\mathcal{F}}}$.

Under the successive embeddings, the class $\mathcal{F}'$ can be considered as a subclass of $\mathcal{F}$ and a subclass $\bar{\mathcal{F}}$, which are singled out by the additional auxiliary constraints $\alpha=0$ and $A^1=A^0=C=0$, $B=x$, $A^2_t=A^2_y=0$, $(x\ln|x|)A^2_x=A^2\ln A^2$, respectively. At the same time, even after the embedding $\mathcal{F}\hookrightarrow\bar{\mathcal{F}}$, the class $\mathcal{F}$ cannot be interpreted as a subclass of $\bar{\mathcal{F}}$ according to the formal definition of subclass given in [126] since the set of A^2 of the form $A^2=|x-\alpha|^\beta$, where α and β are arbitrary constants, is the union of the solution sets of the systems $A^2=1$ and $A^2_t=A^2_y=0$, $\Phi_{xx}=-\operatorname{sgn}(x-\alpha)\Phi_x\exp(\Phi\Phi_{xx}/\Phi_x^2)$ with $\Phi=\ln A^2$ but not the solution set of a regular system of differential equations and differential inequalities. We can overcome this obstacle via extending the notion of a class of differential equations by allowing arbitrary class elements to run the union of solution sets of auxiliary systems of differential equations and differential inequalities.

The aforementioned arguments allow us to use Theorem 3.2 in the course of computing the equivalence groupoids $\mathcal{G}^{\sim}_{\mathcal{F}}$ and $\mathcal{G}^{\sim}_{\mathcal{F}'}$ and the equivalence groups $G^{\sim}_{\mathcal{F}}$ and $G^{\sim}_{\mathcal{F}'}$ of the classes $\mathcal{F}$ and $\mathcal{F}'$. For this computation, the following elementary lemma is useful.

Lemma 3.3. *If the equality* $C_1|x-\mu_1|^{\nu_1}|x-\mu_2|^{\nu_2}=C_2|x-\mu_3|^{\nu_3}$ *holds, where* C_1, C_2, ν_1, ν_2, ν_3, μ_1, μ_2, μ_3 *are constants with* $C_1C_2\neq 0$, *then* $\mu_1=\mu_2=\mu_3$ *and* $\nu_3=\nu_1+\nu_2$, $C_1=C_2$.

Theorem 3.4. *(i) The point transformations* $\mathcal{S}(c_1)\colon (\tilde t,\tilde x,\tilde y,\tilde u,\tilde\beta,\tilde\alpha)=(t,x+c_1,y+c_1t,u,\beta,\alpha+c_1)$ *where* c_1 *is an arbitrary constant, constitute a one-parameter group of equivalence transformations of the class* $\mathcal{F}$.

(ii) The wide family of admissible transformations $\mathcal{S}_{\alpha\beta}:=\big((\alpha,\beta),\pi_*\mathcal{S}(-\alpha),(0,\beta)\big)$ *of the class* $\mathcal{F}$ *from the action groupoid of its equivalence group maps this class onto the class* $\mathcal{F}'$ *interpreted as a subclass of* $\mathcal{F}$.

(iii) The point transformation $\mathcal{J}'$ *given in* (3.4) *is a (discrete) equivalence transformation of the class* $\mathcal{F}'$.

(iv) The class $\mathcal{F}'$ *is semi-normalized with respect to the discrete equivalence subgroup generated by* $\mathcal{J}'$. *In other words, the equivalence groupoid* $\mathcal{G}^{\sim}_{\mathcal{F}'}$ *of* $\mathcal{F}'$ *is the Frobenius product of the action groupoid of this subgroup and the fundamental equivalence groupoid* $\mathcal{G}^{\rm f}_{\mathcal{F}'}$ *of* $\mathcal{F}'$.

Proof. Since items (i)–(iii) are straightforwardly verified, except the discreteness of $\mathcal{J}'$, which follows from item (iv), it suffices to prove only item (iv).

The determining equations for admissible transformations of the class $\mathcal{F}'$ can be derived by restricting their counterparts for the class $\bar{\mathcal{F}}$ in view of the embedding $\mathcal{F}'$ in $\bar{\mathcal{F}}$. Setting $(A^0,A^1,A^2,B,C)=(0,0,|x|^\beta,x,0)$ and $(\tilde A^0,\tilde A^1,\tilde A^2,\tilde B,\tilde C)=(0,0,|\tilde x|^{\tilde\beta},\tilde x,0)$ in (3.7b) and (3.7c), we obtain

$$\begin{aligned}
&X=\frac{Y_t+xY_y}{T_t+xT_y},\quad |X|^{\tilde\beta}=|x|^\beta\frac{(T_tY_y-T_yY_t)^2}{(T_t+xT_y)^5},\\
&X_t+xX_y=|x|^\beta\left(X_{xx}-2X_x\frac{U^1_x}{U^1}\right),\\
&\left(\frac{1}{U^1}\right)_t+x\left(\frac{1}{U^1}\right)_y=|x|^\beta\left(\frac{1}{U^1}\right)_{xx},\quad \left(\frac{U^0}{U^1}\right)_t+x\left(\frac{U^0}{U^1}\right)_y=|x|^\beta\left(\frac{U^0}{U^1}\right)_{xx}.
\end{aligned}\tag{3.8}$$

In view of Lemma 3.3, the consequence

$$|Y_t + xY_y|^{\tilde\beta} = (T_tY_y - T_yY_t)^2|x|^\beta|T_t + xT_y|^{\tilde\beta-5}\operatorname{sgn}(T_t + xT_y) \tag{3.9}$$

of the first two equations of (3.8) implies that $\tilde\beta = \beta$ if $T_y = 0$ or $\tilde\beta = 5 - \beta$ if $T_y \neq 0$ and $Y_y = 0$ or $\beta = 5$ and $\tilde\beta \in \{0, 5\}$ if $T_yY_y \neq 0$. Up to composing any admissible transformation of $\mathcal{F}'$ with elements of the discrete equivalence subgroup generated by $\mathcal{J}'$, we can assume that $\tilde\beta = \beta$, which means that this admissible transformation belongs to $\mathcal{G}^{\rm f}_{\mathcal{F}'}$. □

Corollary 3.5. *(i) Different equations $\mathcal{F}'_\beta$ and $\mathcal{F}'_{\tilde\beta}$ are similar with respect to point transformations if and only if $\beta + \tilde\beta = 5$.*

(ii) Equations $\mathcal{F}_{\alpha\beta}$ and $\mathcal{F}_{\tilde\alpha\tilde\beta}$ are similar with respect to point transformations if and only if either $\tilde\beta = \beta$ or $\beta + \tilde\beta = 5$.

(iii) The equivalence groupoid $\mathcal{G}^{\sim}_{\mathcal{F}}$ of $\mathcal{F}$ is generated by admissible transformations $\mathcal{S}_{\alpha\beta}$ and elements of $\mathcal{G}^{\sim}_{\mathcal{F}'}$. More specifically, for each admissible transformation $\big((\alpha,\beta),\Phi,(\tilde\alpha,\tilde\beta)\big)$ of $\mathcal{F}$, we have $\Phi = \pi_*\mathcal{S}(\tilde\alpha)\circ\breve\Phi\circ\pi_*\mathcal{S}(-\alpha)$ for some point transformation $\breve\Phi$ with $(\beta,\breve\Phi,\tilde\beta)\in\mathcal{G}^{\sim}_{\mathcal{F}'}$.

3.3 Equivalence group and equivalence algebra of the original class

We compute the equivalence group of the class $\mathcal{F}$ of Kolmogorov backward equations with power diffusivity using two fundamentally different approaches, the *direct method* in this section and the *algebraic method* in Section 3.A.1. While each of them has its advantages and drawbacks as well as its scope of applicability, they complement each other and provide a robust cross-verification of the correctness of computations when used in conjunction. Moreover, in the literature, no one has compared the capabilities of these two methods when applied to constructing the equivalence group of a specific class; see [16] for a similar comparison in the case of computing the point symmetry group of a single system of differential equations. The class $\mathcal{F}$ can serve as an excellent instructive example for this particular purpose.

Theorem 3.6. *The equivalence group $G^{\sim}_{\mathcal{F}}$ consists of the point transformations*

$$\begin{aligned}
&\tilde t = t + c_0, \quad \tilde x = \epsilon x + c_1, \quad \tilde y = \epsilon y + c_1 t + c_2,\\
&\tilde u = c_3 u + c_4(tx - y) + c_5 x + c_6,\\
&\tilde\alpha = \epsilon\alpha + c_1, \quad \tilde\beta = \beta,
\end{aligned} \tag{3.10}$$

where $\epsilon = \pm 1$, c_0, $\dots$, c_6 are arbitrary constants with $c_3 \neq 0$.

Proof. We apply the advanced version of the direct method. In view of the embedding $\mathcal{F} \hookrightarrow \bar{\mathcal{F}}$, the transformational parts of admissible transformations of the class $\mathcal{F}$ takes the form (3.7a), where the involved parameter functions satisfy the equations (3.7b) and (3.7c) specified for this class as a subclass of $\bar{\mathcal{F}}$,

$$\begin{aligned}
&X = \frac{Y_t + xY_y}{T_t + xT_y}, \quad |X - \tilde\alpha|^{\tilde\beta} = |x - \alpha|^{\beta}\frac{(T_tY_y - T_yY_t)^2}{(T_t + xT_y)^5},\\
&X_t + xX_y = |x - \alpha|^{\beta}\left(X_{xx} - 2X_x\frac{U^1_x}{U^1}\right),\\
&\left(\frac{1}{U^1}\right)_t + x\left(\frac{1}{U^1}\right)_y = |x - \alpha|^{\beta}\left(\frac{1}{U^1}\right)_{xx},\\
&\left(\frac{U^0}{U^1}\right)_t + x\left(\frac{U^0}{U^1}\right)_y = |x - \alpha|^{\beta}\left(\frac{U^0}{U^1}\right)_{xx}.
\end{aligned} \tag{3.11}$$

The first two equations of (3.11) imply

$$\begin{aligned}
&\left|Y_t - \tilde\alpha T_t + x(Y_y - \tilde\alpha T_y)\right|^{\tilde\beta}\\
&\qquad = (Y_yT_t - Y_tT_y)^2|x - \alpha|^{\beta}|T_t + xT_y|^{\tilde\beta - 5}\operatorname{sgn}(T_t + xT_y).
\end{aligned} \tag{3.12}$$

When looking for the equivalence transformation of $\mathcal{F}$, if only one of the pairs (α, β) or $(\tilde\alpha, \tilde\beta)$ appears in a derived equation, we can split it with respect to this pair. In view of item (ii) of Corollary 3.5, we have at most two cases for the β-components of equivalence transformation of $\mathcal{F}$, $\tilde\beta = \beta$ and $\tilde\beta = 5 - \beta$. In the latter case, Lemma 3.3, the equation (3.12) and the nondegeneracy condition $T_tY_y - T_yY_t \neq 0$ successively imply under varying β that $T_y = 0$, $T_t > 0$, $Y_y \neq 0$, $|Y_y|^{\beta-2} = T_t^{\beta-3}$ and $\tilde\alpha T_t - Y_t = \alpha Y_y$. Differentiating the last equation with respect to y and splitting the result with respect to α, we obtain $Y_{ty} = Y_{yy} = 0$. Then we differentiate the same equation with respect to t and split the result with respect to $\tilde\alpha$ to derive $T_{tt} = Y_{tt} = 0$. Now the equation

$|Y_y|^{\beta-2} = T_t^{\beta-3}$ splits with respect to β into $T_t = 1$ and $Y_y = \epsilon := \pm 1$. Therefore, $T = t + c_0$, $Y = \epsilon y + c_1 t + c_2$, and thus $X = \epsilon x + c_1$ for some constants c_0, c_1 and c_2. The third equation of (3.11) reduces to the equation $U^1_x = 0$, in view of which the fourth equation of (3.11) splits with respect to x into $U^1_t = U^1_y = 0$. Since U^1 is a constant, say c_3, the last equation of (3.11) splits with respect to β into $U^0_{xx} = 0$ and $U^0_y + xU^0_y = 0$, i.e., $U^0 = U^{01}(t,y)x + U^{00}(t,y)$, where $U^{00}_t = U^{01}_y = U^{01}_t + U^{00}_y = 0$. Integrating these equations, we obtain $U^0 = c_4(tx-y)+c_5x+c_6$ for some constants c_4, c_5 and c_6. □

Remark 3.7. The transformations of the form (3.10) with $\epsilon = 1$ and $c_1 = 0$ constitute a normal subgroup $\hat{G}^{\cap}_{\mathcal{F}}$ of $G^{\sim}_{\mathcal{F}}$, which is isomorphic, via the natural projection onto the space $\mathbb{R}^4_{t,x,y,u}$, to the kernel point symmetry group $G^{\cap}_{\mathcal{F}}$ of the class $\mathcal{F}$, cf. [36, Proposition 3]. The quotient of $G^{\sim}_{\mathcal{F}}$ with respect to $\hat{G}^{\cap}_{\mathcal{F}}$ is isomorphic to the group $\mathbb{R} \times \mathbb{Z}_2$ and can be naturally identified with the subgroup of $G^{\sim}_{\mathcal{F}'}$ that is generated by the transformations $\mathcal{J}'$ and

$$\mathcal{S}(c_1)\colon \quad \tilde{t} = t, \quad \tilde{x} = \epsilon x + c_1, \quad \tilde{y} = \epsilon y + c_1 t, \quad \tilde{u} = u, \quad \tilde{\alpha} = \epsilon\alpha + c_1, \quad \tilde{\beta} = \beta.$$

These transformations can be considered as the only essential equivalence transformations within the class $\mathcal{F}$, which establish relations between different elements of $\mathcal{F}$.

Knowing the equivalence group $G^{\sim}_{\mathcal{F}}$, we can easily construct the equivalence algebra $\mathfrak{g}^{\sim}_{\mathcal{F}}$ of this class as the set of generators of one-parameter subgroups of $G^{\sim}_{\mathcal{F}}$.

Corollary 3.8. *The equivalence algebra $\mathfrak{g}^{\sim}_{\mathcal{F}}$ of the class $\mathcal{F}$ is spanned by the vector fields ∂_t, $\partial_x + t\partial_y + \partial_\alpha$, ∂_y, $u\partial_u$, $(tx-y)\partial_u$, $x\partial_u$ and ∂_u.*

Corollary 3.9. *The usual equivalence group $G^{\sim}_{\mathcal{F}}$ of the class $\mathcal{F}$ is generated by the one-parameter groups associated with the basis vector fields of the equivalence algebra $\mathfrak{g}^{\sim}_{\mathcal{F}}$ of this class and its two discrete equivalence transformations,*

$$\mathcal{I}_u\colon (t,x,y,u,\alpha,\beta) \mapsto (t,x,y,-u,\alpha,\beta) \quad \textit{and}$$
$$\mathcal{I}_{\rm s}\colon (t,x,y,u,\alpha,\beta) \mapsto (t,-x,-y,u,-\alpha,\beta).$$

The algebra $\mathfrak{g}^{\sim}_{\mathcal{F}}$ can be represented as the semidirect sum $\mathfrak{g}^{\sim}_{\mathcal{F}} = \langle \partial_t + t\partial_y + \partial_\alpha \rangle \in \hat{\mathfrak{g}}^{\cap}_{\mathcal{F}}$, where the ideal $\hat{\mathfrak{g}}^{\cap}_{\mathcal{F}}$ spanned by the vector fields ∂_t, ∂_y, $u\partial_u$, $(tx-y)\partial_u$, $x\partial_u$ and ∂_u is isomorphic to the kernel invariance algebra $\mathfrak{g}^{\cap}_{\mathcal{F}}$ of equations from the class $\mathcal{F}$, see

Corollary 3.17. Both the algebras $\mathfrak{g}^{\sim}_{\mathcal{F}}$ and $\hat{\mathfrak{g}}^{\cap}_{\mathcal{F}}$ are indecomposable, two-step solvable and not nilpotent. The nilradicals of the algebras $\mathfrak{g}^{\sim}_{\mathcal{F}}$ and $\hat{\mathfrak{g}}^{\cap}_{\mathcal{F}}$ respectively are

$$\mathfrak{n}^{\sim}_{\mathcal{F}} = \langle \partial_x + t\partial_y + \partial_\alpha, \partial_t, \partial_y, (tx-y)\partial_u, x\partial_u, \partial_u \rangle \quad \text{and} \quad \mathfrak{n}^{\cap}_{\mathcal{F}} = \langle \partial_t, \partial_y, (tx-y)\partial_u, x\partial_u, \partial_u \rangle.$$

The nilradical $\mathfrak{n}^{\cap}_{\mathcal{F}}$ and the algebra $\hat{\mathfrak{g}}^{\cap}_{\mathcal{F}}$ are isomorphic to the nilpotent algebra L^1_5 [96] (or $\mathrm{g}_{5.1}$ in the notation of [97]) and the solvable algebra $g_{6.54}$ with $\lambda = \gamma = 1$ [98], respectively. The nilradical $\mathfrak{n}^{\sim}_{\mathcal{F}}$ is indecomposable and six-dimensional with the four-dimensional maximal abelian ideal $\langle \partial_u, x\partial_u, \partial_t, \partial_y \rangle$ and the one-dimensional center $\langle \partial_u \rangle$. The lower central series of $\mathfrak{n}^{\sim}_{\mathcal{F}}$ is $\mathfrak{n}^{\sim}_{\mathcal{F}} \supset \langle \partial_u, x\partial_u, \partial_y \rangle \supset \langle \partial_u \rangle$. These structural properties of the algebra $\mathfrak{n}^{\sim}_{\mathcal{F}}$ help us to identify it. It is isomorphic to the algebra B.14 with $\gamma = -1$ from Morozov's classification of six-dimensional nilpotent algebras [96].

3.4 Equivalence group and equivalence algebra of the gauged class

The elements of the group $G^{\sim}_{\mathcal{F}}$ that preserve the subclass $\mathcal{F}_0$ constitute a proper subgroup H of $G^{\sim}_{\mathcal{F}}$. Let ϖ denote the natural projection from the space with coordinates $(t, x, y, u, \alpha, \beta)$ onto the space with coordinates (t, x, y, u, β). The pushforward $\varpi_* H$ of the group H by ϖ is a subgroup of $G^{\sim}_{\mathcal{F}'}$. Nonetheless, since the class $\mathcal{F}$ is not normalized, we cannot guarantee that $\varpi_* H = G^{\sim}_{\mathcal{F}'}$. Moreover, we prove that $\varpi_* H \subsetneq G^{\sim}_{\mathcal{F}'}$. We cannot also use the embedding of the groupoid $\mathcal{G}^{\sim}_{\mathcal{F}'}$ in the groupoid $\mathcal{G}^{\sim}_{\mathcal{F}}$ induced by the embedding of $\mathcal{F}'$ in $\mathcal{F}$ since the description of $\mathcal{G}^{\sim}_{\mathcal{F}}$ in Corollary 3.5(iii) is based on that of $\mathcal{G}^{\sim}_{\mathcal{F}'}$. This is why we proceed with computing the equivalence group $G^{\sim}_{\mathcal{F}'}$ similarly to as it is done in Section 3.4. We apply the advanced version of the direct method to the class $\mathcal{F}'$, which is based on the embedding of the groupoid $\mathcal{G}^{\sim}_{\mathcal{F}'}$ in the groupoid $\mathcal{G}^{\sim}_{\bar{\mathcal{F}}}$; see Section 3.A.2 for the same computation using the algebraic method.

Theorem 3.10. *The usual equivalence group*[2] $G^{\sim}_{\mathcal{F}'}$ *of the class* $\mathcal{F}'$ *consists of the*

[2]Formally, the set of transformations $G^{\sim}_{\mathcal{F}'}$ together with the standard transformation composition form a *pseudogroup* rather than a group, see [75, Section A] for details. However, $G^{\sim}_{\mathcal{F}'}$ can be turned into a group in a simple way following the approach from [70,74,75]. It is clear that the composition of two transformations of the form (3.13) is also of this form and its natural domain is the entire space $\mathbb{R}^5 := \mathbb{R}^5_{t,x,y,u,\beta}$. Similarly, composing (3.13) with (3.14) results in a transformation of the form (3.14) with the natural domain $\mathbb{R}^5 \setminus M_0$, where $M_0 \subset \mathbb{R}^5$ is the hyperplane defined by the equation $x = 0$. However, the composition of two transformations of the form (3.14) is of the form (3.13) and its

transformations of the following two forms:

$$\begin{aligned}&\tilde t=t+c_0,\quad \tilde x=\epsilon x,\quad \tilde y=\epsilon y+c_1,\\&\tilde u=c_2u+c_3(tx-y)+c_4x+c_5,\quad \tilde\beta=\beta,\end{aligned}\tag{3.13}$$

$$\begin{aligned}&\tilde t=\varepsilon' y+c_0',\quad \tilde x=\frac{\epsilon'}{x},\quad \tilde y=\epsilon'\varepsilon' t+c_1',\\&\tilde u=\frac{c_2'}{x}u+c_3'\left(t-\frac{y}{x}\right)+\frac{c_4'}{x}+c_5',\quad \tilde\beta=5-\beta,\end{aligned}\tag{3.14}$$

where $\epsilon,\epsilon'\in\{\pm1\}$, $\varepsilon'=\operatorname{sgn}x$, $c_0,\dots,c_5$ *and* $c_0',\dots,c_5'$ *are arbitrary constants with* $c_2c_2'\neq0$.

Proof. Items (iii) and (iv) of Theorem 3.4 essentially simplify the construction of the equivalence group $G^{\sim}_{\mathcal{F}'}$ of the class $\mathcal{F}'$. Up to composing with the discrete equivalence subgroup generated by $\mathcal{J}'$, it suffices to find only the equivalence transformations with the identity β-components. Since β varies when considering equivalence transformations and $\tilde\beta=\beta$, the equation (3.9), Lemma 3.3 and the inequality $T_tY_y-T_yY_t\neq0$ implies $T_y=Y_t=0$, $|Y_y|^{\beta-2}=|T_t|^{\beta-3}\operatorname{sgn}T_t$ and thus T_t and Y_y are positive and nonzero constants, respectively. Then the system (3.8) reduces to

$$X=\frac{Y_y}{T_t}x,\quad |Y_y|^{\beta-2}=T_t^{\beta-3},\quad U^1_x=U^1_t=U^1_y=0,\quad U^0_t+xU^0_y=|x|^\beta U^0_{xx}$$

with $T_t>0$. Its second and fourth equations further split with respect to β to $T_t=|Y_y|=1$, $U^0_{xx}=0$ and $U^0_t+xU^0_y=0$. Therefore, $U^0=U^{01}(t,y)x+U^{00}(t,y)$, where $U^{00}_t=U^{01}_y=U^{01}_t+U^{00}_y=0$. The solution set of the derived equations corresponds to the family of transformations (3.13). Its complement in $G^{\sim}_{\mathcal{F}'}$ is obtained by composing its elements with $\mathcal{J}'$, which leads to the family of transformations (3.14). □

Remark 3.11. The transformations of the form (3.13) constitute a normal subgroup $\hat G^{\cap}_{\mathcal{F}'}$ of $G^{\sim}_{\mathcal{F}'}$, which is isomorphic, via the natural projection to the space $\mathbb{R}^4_{t,x,y,u}$, to the kernel point symmetry group $G^{\cap}_{\mathcal{F}'}$ of the class $\mathcal{F}'$, cf. [36]. The quotient of $G^{\sim}_{\mathcal{F}'}$ with respect to $\hat G^{\cap}_{\mathcal{F}'}$ is isomorphic to $\mathbb{Z}_2$ and can be naturally identified with the subgroup of $G^{\sim}_{\mathcal{F}'}$ constituted by the identity transformation of (t,x,y,u,β) and the transformation (3.4). Hence the latter transformation can be considered as the only

domain is the set $\mathbb{R}^5\setminus M_0$. We can modify this operation as follows: $\Phi_1\circ^{\rm mod}\Phi_2=\Phi_1\circ\Phi_2$ if Φ_1 and Φ_2 are both of the form (3.13) or of the forms (3.13) and (3.14). If Φ_1 and Φ_2 are both of the form (3.14), then $\Phi_1\circ^{\rm mod}\Phi_2$ is the extension by continuity of $\Phi_1\circ\Phi_2$ to the set $\mathbb{R}^5$ and is of the form (3.13). In this way, the set $G^{\sim}_{\mathcal{F}'}$ is a group with the modified composition $\circ^{\rm mod}$.

essential equivalence transformation within the class $\mathcal{F}'$, which establishes a relation between different elements of $\mathcal{F}'$.

Remark 3.12. Modulo $G^{\sim}_{\mathcal{F}'}$-equivalence, it suffices to consider a single value of β in each pair (β_1, β_2) with $\beta_1 + \beta_2 = 5$. As the canonical representative of such a pair, we choose the value $\beta \in (-\infty, 5/2]$.

Corollary 3.13. *The equivalence algebra $\mathfrak{g}^{\sim}_{\mathcal{F}'}$ of the class $\mathcal{F}'$ is spanned by the vector fields ∂_t, ∂_y, $u\partial_u$, $(tx-y)\partial_u$, $x\partial_u$ and ∂_u and is isomorphic to the kernel Lie invariance algebra of the class $\mathcal{F}'$ via the natural projection to the space $\mathbb{R}^4_{t,x,y,u}$.*

Corollary 3.14. *The usual equivalence group $G^{\sim}_{\mathcal{F}'}$ of the class $\mathcal{F}'$ is generated by the one-parameter groups associated with the basis vector fields of the equivalence algebra $\mathfrak{g}^{\sim}_{\mathcal{F}'}$ of this class and its discrete equivalence transformations*

$$\mathcal{I}'_u\colon (t,x,y,u,\beta) \mapsto (t,x,y,-u,\beta),$$
$$\mathcal{I}'_{\rm s}\colon (t,x,y,u,\beta) \mapsto (t,-x,-y,u,\beta),$$
$$\mathcal{J}'\colon (t,x,y,u,\beta) \mapsto (y\operatorname{sgn} x, 1/x, t\operatorname{sgn} x, u/x, 5-\beta).$$

In fact, there are only two independent[3] discrete equivalence transformations within the class $\mathcal{F}'$, e.g., $\mathcal{I}'_u$ and $\mathcal{I}'_{\rm s}\circ\mathcal{J}'$.

Corollary 3.15. *The quotient group of the group $G^{\sim}_{\mathcal{F}'}$ with respect to its identity component is isomorphic to the degree-four dihedral group D_4. As the canonical subgroup of discrete equivalence transformations, one can take those generated by $\mathcal{I}'_u$, $\mathcal{I}'_{\rm s}$ and $\mathcal{J}'$, and a minimal set of generators is $\{\mathcal{I}'_u, \mathcal{I}'_{\rm s}\circ\mathcal{J}'\}$.*

Proof. By π we denote the natural projection of the group $G^{\sim}_{\mathcal{F}'}$ onto the quotient group of this group by its identity component. The quotient group $\pi(G^{\sim}_{\mathcal{F}'})$ is generated by the cosets $\pi(\mathcal{I}'_u)$, $\pi(\mathcal{I}'_{\rm s})$ and $\pi(\mathcal{J}')$. Using the power-commutator presentation $\langle a_1, a_2, a_3 \mid a_1^2 = a_2^2 = a_3^2 = e,\ [a_1,a_2] = a_3,\ [a_1,a_3] = [a_2,a_3] = e\rangle$ of the group D_4, we define an isomorphism between $\pi(G^{\sim}_{\mathcal{F}'})$ and D_4 by the generator correspondences $\pi(\mathcal{I}'_u) \mapsto a_3$, $\pi(\mathcal{I}'_{\rm s}) \mapsto a_1$ and $\pi(\mathcal{J}') \mapsto a_2$. Then the canonical presentation $\langle a, b \mid a^4 = b^2 =$

[3]Throughout the paper, the independence of discrete elements in a transformation (pseudo)group is understood up to composing with each other and with elements from the identity component of this group. The independence of genuinely hidden Lie-symmetries vector fields is understood in a similar sense, up to linearly combining with each other and with induced Lie-symmetries vector fields of the corresponding reduced system.

$e, bab^{-1} = a^{-1}\rangle$ of D_4, where $a = a_1a_2$ and $b = a_3$ implies that a minimal set of generators of $\pi(G^{\sim}_{\mathcal{F}'})$ is constituted by $\pi(\mathcal{I}'_{\rm s} \circ \mathcal{J}')$ and $\pi(\mathcal{I}'_u)$. $\square$

It is clear that the kernel invariance algebras $\mathfrak{g}^{\cap}_{\mathcal{F}}$ and $\mathfrak{g}^{\cap}_{\mathcal{F}'}$ of equations from the classes $\mathcal{F}$ and $\mathcal{F}'$, respectively, coincide, see Theorem 3.16 and Corollary 3.17. At the same time, the corresponding kernel point symmetry groups $G^{\cap}_{\mathcal{F}}$ and $G^{\cap}_{\mathcal{F}'}$ are different, $G^{\cap}_{\mathcal{F}} \subsetneq G^{\cap}_{\mathcal{F}'}$. More specifically, the group $G^{\cap}_{\mathcal{F}'}$ is generated by elements of $G^{\cap}_{\mathcal{F}}$ and the transformation $(t, x, y, u) \mapsto (t, -x, -y, u)$.

Since $\mathfrak{g}^{\sim}_{\mathcal{F}'} \subsetneq \varpi_*\mathfrak{g}^{\sim}_{\mathcal{F}}$ and $G^{\sim}_{\mathcal{F}'} \not\subset \varpi_*G^{\sim}_{\mathcal{F}}$, we can interpret $G^{\sim}_{\mathcal{F}'}$ as a *nontrivial conditional equivalence group* [126, Section 2.6] of $\mathcal{F}$ associated with the constraint $\alpha = 0$, but the algebra $\mathfrak{g}^{\sim}_{\mathcal{F}'}$ is trivial when interpreted as a conditional equivalence algebra of $\mathcal{F}$.

3.5 Classification of Lie symmetries

In view of Corollary 3.5, the $\mathcal{G}^{\sim}_{\mathcal{F}}$- and $\mathcal{G}^{\sim}_{\mathcal{F}'}$-equivalences within the classes $\mathcal{F}$ and $\mathcal{F}'$, respectively, are consistent. More specifically, the equations from the class $\mathcal{F}'$ are $\mathcal{G}^{\sim}_{\mathcal{F}'}$-equivalent if and only if their counterparts in $\mathcal{F}$ under the embedding $\mathcal{F}' \hookrightarrow \mathcal{F}$ are $\mathcal{G}^{\sim}_{\mathcal{F}}$-equivalent. The class $\mathcal{F}'$ is semi-normalized in the usual sense, i.e., the $\mathcal{G}^{\sim}_{\mathcal{F}'}$-equivalence coincide with the $G^{\sim}_{\mathcal{F}'}$-equivalence. This is why it is convenient to first carry out the group classification of the class $\mathcal{F}'$ and then apply Corollary 3.5, Theorem 3.6 and Remark 3.11 to the obtained results for solving the group classification problem for the class $\mathcal{F}'$.

According to the classical Lie algorithm of computing continuous point symmetries [20,103], a Lie symmetry vector field Q of the equation $\mathcal{F}'_\beta$ takes the general form

$$Q = \tau(t, x, y, u)\partial_t + \xi^x(t, x, y, u)\partial_x + \xi^y(t, x, y, u)\partial_y + \eta(t, x, y, u)\partial_u$$

with components that can be found from the infinitesimal invariance criterion

$$\begin{aligned} &Q^{(2)}\big(u_t + xu_y - |x|^\beta u_{xx}\big)\big|_{\mathcal{F}'_\beta} \\ &\quad = \big(\eta^t + x\eta^y + \xi^x u_y - |x|^\beta \eta^{xx} - \beta\xi^x x^{-1}|x|^\beta u_{xx}\big)\big|_{\mathcal{F}'_\beta} = 0. \end{aligned}$$

Here $Q^{(2)}$ is the second-order prolongation of the vector field Q, $\eta^t = \mathrm{D}_t Q[u] + \tau u_{tt} + \xi^x u_{tx} + \xi^y u_{ty}$, $\eta^y = \mathrm{D}_y Q[u] + \tau u_{ty} + \xi^x u_{xy} + \xi^y u_{yy}$, $\eta^{xx} = \mathrm{D}_x^2 Q[u] + \tau u_{txx} + \xi^x u_{xxx} + \xi^y u_{xxy}$, the characteristic $Q[u]$ of the vector field Q is defined as $Q[u] := \eta - \tau u_t - \xi^x u_x - \xi^y u_y$, D_t, D_x, D_y are the total derivative operators with respect to t, x and y, respectively, $\mathrm{D}_t = \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + u_{tx}\partial_{u_x} + u_{ty}\partial_{u_y} + \cdots$ and similarly for D_x and D_y. The linear space of such vector fields with Lie bracket of vector fields is the maximal Lie invariance algebra $\mathfrak{g}_\beta$ of the equation $\mathcal{F}'_\beta$.

The form of admissible transformations (3.7a) implies the principal part of the determining equations for the components of the vector field Q,

$$\tau_u = \tau_x = 0, \quad \xi^x_u = 0, \quad \xi^y_u = \xi^y_x = 0, \quad \eta_{uu} = 0. \tag{3.15a}$$

The other determining equations, including the classifying ones, follow from the infinitesimal invariance criterion jointly with (3.15a),

$$\begin{aligned} &\eta_t + x\eta_y - |x|^\beta \eta_{xx} = 0,\\ &-2|x|^\beta \eta_{ux} = \xi^x_t + x\xi^x_y - |x|^\beta \xi^x_{xx},\\ &\xi^y_t + x\xi^y_y = 2x\xi^x_x + (1-\beta)\xi^x,\\ &\xi^y_t + x\xi^y_y = \tau_t x + \tau_y x^2 + \xi^x. \end{aligned} \tag{3.15b}$$

Unlike the group classification in [151], we carry out the group classification of the class $\mathcal{F}'$ up to the $G^{\sim}_{\mathcal{F}'}$-equivalence, which coincides with the $\mathcal{G}^{\sim}_{\mathcal{F}'}$-equivalence in view of the semi-normalization of this class. In other words, without loss of generality we can impose the gauge $\beta \leqslant 5/2$, which reduces the number of cases we need to consider.

Theorem 3.16. *The kernel Lie invariance algebra $\mathfrak{g}^{\cap}_{\mathcal{F}'}$ of the equations from the class $\mathcal{F}'$ is*

$$\mathfrak{g}^{\cap}_{\mathcal{F}'} = \langle \mathcal{P}^t, \mathcal{P}^y, \mathcal{I}, (tx-y)\partial_u, x\partial_u, \partial_u\rangle, \quad \text{where} \quad \mathcal{P}^t := \partial_t, \quad \mathcal{P}^y := \partial_y, \quad \mathcal{I} := u\partial_u.$$

Any equation $\mathcal{F}'_\beta$ from $\mathcal{F}'$ is invariant with respect to the algebra

$$\mathfrak{g}^{\mathrm{gen}}_\beta = \langle \mathcal{P}^t, \mathcal{P}^y, \mathcal{I}, \mathcal{D}^\beta, \mathcal{Z}(f^\beta)\rangle \ \text{ with } \ \mathcal{D}^\beta := (2-\beta)t\partial_t + x\partial_x + (3-\beta)y\partial_y, \ \mathcal{Z}(f^\beta) := f^\beta\partial_u,$$

where the parameter function $f^\beta = f^\beta(t,x,y)$ runs through the solution set of this

equation, and $\beta \in (-\infty, 5/2]$ *modulo the* $G^{\sim}_{\mathcal{F}'}$*-equivalence. The maximal Lie invariance algebra* $\mathfrak{g}_\beta$ *of the equation* $\mathcal{F}_\beta$ *coincides with* $\mathfrak{g}^{\rm gen}_\beta$ *if and only if* $\beta \in \mathbb{R}\setminus\{0,2,3,5\}$*. A complete list of* $G^{\sim}_{\mathcal{F}'}$*-inequivalent essential Lie symmetry extensions in the class* $\mathcal{F}'$ *is exhausted by the following cases:*

$$
\begin{aligned}
&\beta = 2\colon \quad \mathfrak{g}_2 = \mathfrak{g}^{\rm gen}_2 \dot{+} \langle \mathcal{K}_2\rangle \quad \text{with} \quad \mathcal{K}_2 = 2xy\partial_x + y^2\partial_y - xu\partial_u,\\
&\beta = 0\colon \quad \mathfrak{g}_0 = \mathfrak{g}^{\rm gen}_0 \dot{+} \langle \mathcal{K}_0, \mathcal{P}^3, \mathcal{P}^2, \mathcal{P}^1\rangle \quad \text{with}\\
&\qquad \mathcal{K}_0 = t^2\partial_t + (tx+3y)\partial_x + 3ty\partial_y - (x^2+2t)u\partial_u,\\
&\qquad \mathcal{P}^3 = 3t^2\partial_x + t^3\partial_y + 3(y-tx)u\partial_u, \quad \mathcal{P}^2 = 2t\partial_x + t^2\partial_y - xu\partial_u,\\
&\qquad \mathcal{P}^1 = \partial_x + t\partial_y.
\end{aligned}
$$

Proof. Splitting the system of determining equations (3.15) with respect to the arbitrary element β, we find the kernel Lie invariance algebra $\mathfrak{g}^{\cap}_{\mathcal{F}'}$ of the equations from the class $\mathcal{F}'$.

To find the algebra $\mathfrak{g}_\beta$, we start with solving the system of the equations (3.15a) with the fourth equation from (3.15b), obtaining

$$
\begin{aligned}
&\tau = \tau(t,y), \quad \xi^x = -x^2\tau_y + x\xi^y_y - x\tau_t + \xi^y_t, \quad \xi^y = \xi^y(t,y),\\
&\eta = \eta^1(t,x,y)u + \eta^0(t,x,y),
\end{aligned}
$$

where τ, ξ^y, η^0 and η^1 are smooth functions of their arguments. Substituting this solution into the third equation from (3.15b) and splitting the result with respect to x lead to the system

$$
(\beta-5)\tau_y = 0, \quad (\beta-3)\tau_t - (\beta-2)\xi^y_y = 0, \quad \beta\xi^y_t = 0,
$$

which is of maximal rank as a system of differential equations with the independent variables (t,y) and the dependent variables (τ,ξ^y) if and only if $\beta \in \mathbb{R}\setminus\{0,2,3,5\}$. In this case, the general solution of the system is given by

$$
\tau = c_1(2-\beta)t + c_2, \quad \xi^y = c_1(3-\beta)y + c_3.
$$

Substituting this solution into the rest of the system (3.15b) gives the algebra $\mathfrak{g}_\beta$ for $\beta \in \mathbb{R}\setminus\{0,2,3,5\}$, or, equivalently, for $\beta \in (-\infty,5/2]\setminus\{0,2\}$ under the gauge $\beta \leqslant 5/2$.

The equivalence transformation (3.4) reduces the cases $\beta = 5$ and $\beta = 3$ to the cases $\beta = 0$ and $\beta = 2$, respectively. Solving the determining equations for the latter two cases results in the algebras $\mathfrak{g}_0$ and $\mathfrak{g}_2$. □

For any β, the vector fields $\mathcal{Z}(f^\beta)$ constitute the infinite-dimensional abelian ideal $\mathfrak{g}^{\rm lin}_\beta$ of $\mathfrak{g}_\beta$ that is associated with the linear superposition of solutions of $\mathcal{F}'_\beta$, $\mathfrak{g}^{\rm lin}_\beta := \{\mathcal{Z}(f^\beta)\}$. The algebra $\mathfrak{g}_\beta$ splits over this ideal, $\mathfrak{g}_\beta = \mathfrak{g}^{\rm ess}_\beta \in \mathfrak{g}^{\rm lin}_\beta$, where $\mathfrak{g}^{\rm ess}_\beta$ is a (finite-dimensional) subalgebra of $\mathfrak{g}_\beta$, which is called the essential Lie invariance algebra of $\mathcal{F}'_\beta$. In the above notation,

$$\begin{aligned}
&\mathfrak{g}^{\rm ess}_\beta := \langle \mathcal{P}^t,\, \mathcal{P}^y,\, \mathcal{I},\, \mathcal{D}^\beta\rangle, \quad \beta \in \mathbb{R}\setminus\{0,2,3,5\},\\
&\mathfrak{g}^{\rm ess}_2 := \langle \mathcal{P}^y, \mathcal{D}_2, \mathcal{K}, \mathcal{P}^t, \mathcal{I}\rangle,\\
&\mathfrak{g}^{\rm ess}_0 := \langle \mathcal{P}^t, \mathcal{D}_0, \mathcal{K}, \mathcal{P}^3, \mathcal{P}^2, \mathcal{P}^1, \mathcal{P}^0, \mathcal{I}\rangle.
\end{aligned}$$

Similar splitting occurs for the corresponding Lie symmetry pseudogroups G_β for each β. The set of point symmetry transformations $G^{\rm lin}_\beta = \{\mathcal{Z}(f)\colon \tilde t = t, \tilde x = x, \tilde y = y, \tilde u = f^\beta(t,x,y)\}$, where f^β is an arbitrary solution of the equation $\mathcal{F}'_\beta$, constitute a normal pseudosubgroup in G_β. Moreover, the pseudogroup G_β splits over $G^{\rm lin}_\beta$, $G_\beta = G^{\rm ess}_\beta \ltimes G^{\rm lin}_\beta$. Here $G^{\rm ess}_\beta$ is a pseudosubgroup in G_β called the essential Lie invariance pseudogroup.

Theorems 3.4 and 3.16 and Corollary 3.5 jointly imply the solution of the group classification problem for the class $\mathcal{F}$.

Corollary 3.17. *The kernel Lie invariance algebra $\mathfrak{g}^\cap_{\mathcal{F}}$ of the equations from the class $\mathcal{F}$ coincides with that for the class $\mathcal{F}'$, $\mathfrak{g}^\cap_{\mathcal{F}} = \mathfrak{g}^\cap_{\mathcal{F}'}$. Any equation $\mathcal{F}_{\alpha\beta}$ from $\mathcal{F}$ is invariant with respect to the algebra*

$$\mathfrak{g}^{\rm gen}_{\alpha\beta} = \big\langle \mathcal{P}^t, \mathcal{P}^y, \mathcal{I}, \mathcal{D}^{\alpha\beta}, \mathcal{Z}(f^{\alpha\beta})\big\rangle$$

with $\mathcal{D}^{\alpha\beta} := (2-\beta)t\partial_t + (x-\alpha)\partial_x + \big((3-\beta)y - \alpha t\big)\partial_y$ and $\mathcal{Z}(f^{\alpha\beta}) := f^{\alpha\beta}\partial_u$, and the parameter function $f^{\alpha\beta} = f^{\alpha\beta}(t,x,y)$ running through the solution set of this equation. Modulo the $\mathcal{G}^\sim_{\mathcal{F}}$-equivalence, we can assume $\beta \in (-\infty, 5/2]$, and a complete list of $\mathcal{G}^\sim_{\mathcal{F}}$-inequivalent essential Lie symmetry extensions in the class $\mathcal{F}$ is exhausted by the counterparts of those in the class $\mathcal{F}'$, $\mathcal{F}_{00}$ and $\mathcal{F}_{02}$. An analogous list up to the $G^\sim_{\mathcal{F}}$-equivalence consists of the equations $\mathcal{F}_{00}$, $\mathcal{F}_{02}$, $\mathcal{F}_{03}$ and $\mathcal{F}_{05}$.

The maximal Lie invariance algebras of the equations $\mathcal{F}_{03}\sim\mathcal{F}'_3$ and $\mathcal{F}_{05}\sim\mathcal{F}'_5$ are presented in Propositions 3.24 and 3.20, respectively.

3.6 Remarkable Fokker–Planck equation and its counterpart

The equation $\mathcal{F}'_0$ is of the simplest form among the equations from the classes $\mathcal{F}'$, $\mathcal{F}$ and even $\bar{\mathcal{F}}$,

$$\mathcal{F}'_0\colon\quad u_t+xu_y=u_{xx},\tag{3.16}$$

and its essential Lie invariance algebra is of the greatest dimension, which is equal to eight. Moreover, up to the equivalence with respect to point transformations, this is a unique equation with the above property within each of the above classes. This is why in [70] we called it the remarkable (1+2)-dimensional Fokker–Planck equation. We briefly review the results of [70, 120] on this equation and then extend them to the equation (3.5) with $\beta=5$ using the point transformation (3.6).

The maximal Lie invariance algebra of the equation (3.16) is (see, e.g., [76])

$$\mathfrak{g}_0:=\langle\mathcal{P}^t,\,\mathcal{D},\,\mathcal{K},\,\mathcal{P}^3,\,\mathcal{P}^2,\,\mathcal{P}^1,\,\mathcal{P}^0,\,\mathcal{I},\mathcal{Z}(f)\rangle,$$

where

$$\begin{aligned}&\mathcal{P}^t=\partial_t,\quad \mathcal{D}=2t\partial_t+x\partial_x+3y\partial_y-2u\partial_u,\\&\mathcal{K}=t^2\partial_t+(tx+3y)\partial_x+3ty\partial_y-(x^2+2t)u\partial_u,\\&\mathcal{P}^3=3t^2\partial_x+t^3\partial_y+3(y-tx)u\partial_u,\quad \mathcal{P}^2=2t\partial_x+t^2\partial_y-xu\partial_u,\\&\mathcal{P}^1=\partial_x+t\partial_y,\quad \mathcal{P}^0=\partial_y,\quad \mathcal{I}=u\partial_u,\quad \mathcal{Z}(f)=f(t,x,y)\partial_u.\end{aligned}$$

Here the parameter function f runs through the solution set of the equation (3.16). The essential Lie invariance algebra $\mathfrak{g}_0^{\rm ess}$ splits over its radical $\mathfrak{r}_0=\langle\mathcal{P}^3,\mathcal{P}^2,\mathcal{P}^1,\mathcal{P}^0,\mathcal{I}\rangle$, which is isomorphic to the real rank-two Heisenberg algebra $\mathrm{h}(2,\mathbb{R})$, $\mathfrak{g}_0^{\rm ess}=\mathfrak{f}_0\in\mathfrak{r}_0$, where $\mathfrak{f}_0=\langle\mathcal{P}^t,\mathcal{D},\mathcal{K}\rangle$ is a Levi factor of $\mathfrak{g}_0^{\rm ess}$ and it is isomorphic to the real order-two special linear Lie algebra $\mathrm{sl}(2,\mathbb{R})$.

Theorem 3.18 ([70]). *The complete point symmetry pseudogroup G_0 of the remarkable Fokker–Planck equation (3.16) consists of the transformations of the form*

$$\begin{aligned}&\tilde t=\frac{\alpha t+\beta}{\gamma t+\delta},\quad \tilde x=\frac{\hat x}{\gamma t+\delta}-\frac{3\gamma\hat y}{(\gamma t+\delta)^2},\quad \tilde y=\frac{\hat y}{(\gamma t+\delta)^3},\\&\tilde u=\sigma(\gamma t+\delta)^2\exp\left(\frac{\gamma\hat x^2}{\gamma t+\delta}-\frac{3\gamma^2\hat x\hat y}{(\gamma t+\delta)^2}+\frac{3\gamma^3\hat y^2}{(\gamma t+\delta)^3}\right)\\&\quad\times\exp\big(3\lambda_3(y-tx)-\lambda_2x-(3\lambda_3^2t^3+3\lambda_3\lambda_2t^2+\lambda_2^2t)\big)\big(u+f(t,x,y)\big),\end{aligned}\tag{3.17}$$

where $\hat x:=x+3\lambda_3t^2+2\lambda_2t+\lambda_1$, $\hat y:=y+\lambda_3t^3+\lambda_2t^2+\lambda_1t+\lambda_0$; α, β, γ and δ are arbitrary constants with $\alpha\delta-\beta\gamma=1$; λ_0, …, λ_3 and σ are arbitrary constants with $\sigma\neq0$, and f is an arbitrary solution of (3.16).

Pulling back an arbitrary solution $u=h(t,x,y)$ of (3.16) by an arbitrary point symmetry transformation of the form (3.17), we obtain, in the notation of Theorem 3.18, the formula of generating new solutions of (3.16) from known ones under the action of elements of G_0,

$$\begin{aligned}u=&\frac{\mathrm e^{\lambda_2x-3\lambda_3(y-tx)+3\lambda_3^2t^3+3\lambda_3\lambda_2t^2+\lambda_2^2t}}{\sigma(\gamma t+\delta)^2}\exp\left(-\frac{\gamma\hat x^2}{\gamma t+\delta}+\frac{3\gamma^2\hat x\hat y}{(\gamma t+\delta)^2}-\frac{3\gamma^3\hat y^2}{(\gamma t+\delta)^3}\right)\\&\times h\left(\frac{\alpha t+\beta}{\gamma t+\delta},\frac{\hat x}{\gamma t+\delta}-\frac{3\gamma\hat y}{(\gamma t+\delta)^2},\frac{\hat y}{(\gamma t+\delta)^3}\right)-f(t,x,y).\end{aligned}\tag{3.18}$$

The following families of G_0-inequivalent solutions of the equation (3.16) that arise from its codimension-one Lie reductions were constructed in [70]:

- $\beta=0\colon\quad u=|x|^{-\frac14}\vartheta^\mu\left(\frac94\tilde\varepsilon y,|x|^{\frac32}\right)\quad$ with $\quad\mu=\frac{5}{36},\quad\tilde\varepsilon:=\operatorname{sgn}x,$ (3.19)
- $\beta=0\colon\quad u=|t|^{-\frac12}\mathrm e^{-\frac{x^2}{4t}}\vartheta^0\left(\frac13t^3+2\varepsilon t-t^{-1},2y-(t+\varepsilon t^{-1})x\right)$ with $\varepsilon\in\{-1,1\}$, (3.20)
- $\beta=0\colon\quad u=\vartheta^0\left(\frac13t^3,y-tx\right),$ (3.21)
- $\beta=0\colon\quad u=\vartheta^0(t,x),$ (3.22)

where $\vartheta^\mu=\vartheta^\mu(z_1,z_2)$ is an arbitrary solution of the equation $\vartheta^\mu_1=\vartheta^\mu_{22}+\mu z_2^{-2}\vartheta^\mu$.[4]

[4] A complete collection of inequivalent Lie invariant solutions of equations of such form with generic $\mu\neq0$ is presented in [70, Section A], see also [44, 45]. For $\mu=0$, this equation is just

The algebra of generalized symmetries of the equation (3.16) was recently computed in [120]. The properties of the associative algebra of differential recursion operators of (3.16) played the key role in that computation. Moreover, these operators were applied therein to generating new solutions of $\mathcal{F}_0$ from known ones. In particular, using the solution families (3.19)–(3.22) of the Lie-invariant solutions as seed ones, large families of essentially new solutions of (3.16) were generated.

More specifically, consider the operators in total derivatives that are associated with the Lie-symmetry vector fields $-\mathcal{P}^3$, $-\mathcal{P}^2$, $-\mathcal{P}^1$, $-\mathcal{P}^0$ and $-\mathcal{P}^t$, $-\mathcal{D}$, $-\mathcal{K}$ of (3.16),

$$\begin{aligned}
&\mathrm{P}^3 := 3t^2\mathrm{D}_x + t^3\mathrm{D}_y - 3(y-tx), \quad \mathrm{P}^2 := 2t\mathrm{D}_x + t^2\mathrm{D}_y + x,\\
&\mathrm{P}^1 := \mathrm{D}_x + t\mathrm{D}_y, \quad \mathrm{P}^0 := \mathrm{D}_y,\\
&\mathrm{P}^t := \mathrm{D}_t, \quad \mathrm{D} := 2t\mathrm{D}_t + x\mathrm{D}_x + 3y\mathrm{D}_y + 2,\\
&\mathrm{K} := t^2\mathrm{D}_t + (tx+3y)\mathrm{D}_x + 3ty\mathrm{D}_y + x^2 + 2t.
\end{aligned}$$

These operators are recursion operators of the equation (3.16) [103, Proposition 5.22], which we call Lie-symmetry operators. By $\Upsilon_{\mathfrak{r}_0}$ we denote the associative algebra generated by the operators P^3, P^2, P^1 and P^0. On solutions of the equation (3.16), the operators P^t, D and K are equivalent to the elements

$$\begin{aligned}
&\hat{\mathrm{P}}^t := (\mathrm{P}^1)^2 - \mathrm{P}^2\mathrm{P}^0 = \mathrm{D}_x^2 - x\mathrm{D}_y,\\
&\hat{\mathrm{D}} := \mathrm{P}^2\mathrm{P}^1 - \mathrm{P}^3\mathrm{P}^0 + 2 = 2t\mathrm{D}_x^2 + x\mathrm{D}_x + (3y-2tx)\mathrm{D}_y + 2,\\
&\hat{\mathrm{K}} := (\mathrm{P}^2)^2 - \mathrm{P}^3\mathrm{P}^1 = t^2\mathrm{D}_x^2 + (3y+tx)\mathrm{D}_x + t(3y-tx)\mathrm{D}_y + x^2 + 2t
\end{aligned}$$

of the associative algebra $\Upsilon_{\mathfrak{r}_0}$, respectively, and thus they are inessential in the course of generating solutions. The only nontrivial relations between the generators of the algebra $\Upsilon_{\mathfrak{r}_0}$ are given by the commutator relations $[\mathrm{P}^1,\mathrm{P}^2]=1$, $[\mathrm{P}^0,\mathrm{P}^3]=-3$. Using Bergman's diamond lemma [11], which extends the Gröbner bases technique to noncommutative rings, we explicitly construct a basis of the algebra $\Upsilon_{\mathfrak{r}_0}$ as follows. For any (ordered) basis $(\mathrm{Q}^1,\mathrm{Q}^2,\mathrm{Q}^3,\mathrm{Q}^4)$ of the span $\langle \mathrm{P}^3,\mathrm{P}^2,\mathrm{P}^1,\mathrm{P}^0\rangle$, the monomials of the form $(\mathrm{Q}^1)^{i_1}(\mathrm{Q}^2)^{i_2}(\mathrm{Q}^3)^{i_3}(\mathrm{Q}^4)^{i_4}$ with $(i_1,i_2,i_3,i_4)\in\mathbb{N}_0^4$ constitute a basis of $\Upsilon_{\mathfrak{r}_0}$.

---

the (1+1)-dimensional linear heat equation. An enhanced complete collection of inequivalent Lie invariant solutions of this equation was presented in [147, Section A], following Examples 3.3 and 3.17 in [103].

Theorem 3.19 ([120]). *The algebra of canonical representatives of generalized symmetries of the remarkable Fokker–Planck equation* (3.16) *is* $\Sigma_0=\Lambda_0\Subset\Sigma_0^{-\infty}$, *where*

$$\Lambda_0=\big\langle\big((\mathrm{P}^3)^{i_3}(\mathrm{P}^2)^{i_2}(\mathrm{P}^1)^{i_1}(\mathrm{P}^0)^{i_0}u\big)\partial_u\mid i_0,i_1,i_2,i_3\in\mathbb{N}_0\big\rangle,\quad \Sigma_0^{-\infty}:=\big\{\mathcal{Z}(f)\big\}.$$

Here the parameter function f *runs through the solution set of* (3.16).

Acting on an arbitrary solution $u=h(t,x,y)$ of the equation (3.16) by an element

$$Q=\sum_{(i_1,i_2,i_3,i_4)\in\mathbb{N}_0^4}c_{i_1i_2i_3i_4}(\mathrm{Q}^1)^{i_1}(\mathrm{Q}^2)^{i_2}(\mathrm{Q}^3)^{i_3}(\mathrm{Q}^4)^{i_4}$$

of the algebra $\Upsilon_{\mathfrak{r}_0}$, where only finitely many real constants $c_{i_1i_2i_3i_4}$ are nonzero, we obtain a solution Qh of (3.16). In fact, applying this procedure to a known solution, one may obtain a solution that is known as well. It is shown in [120, Section 5.3] that for the solution families (3.19), (3.20), (3.21) and (3.22), only the action by the monomials $(\mathrm{P}^1)^k$, $(\mathrm{P}^1)^k$, $(\mathrm{P}^2)^k$ and $(\mathrm{P}^3)^k$, $k\in\mathbb{N}$, respectively, in general leads to essentially new solutions of (3.16),

- $\beta=0\colon\quad u=(\mathrm{P}^1)^k\Big(|x|^{-\frac14}\vartheta^\mu\big(\frac94\tilde\varepsilon y,|x|^{\frac32}\big)\Big)$ with $\mu=\frac{5}{36},\quad \tilde\varepsilon:=\operatorname{sgn}x,$
- $\beta=0\colon\quad u=(\mathrm{P}^1)^k\Big(|t|^{-\frac12}\mathrm{e}^{-\frac{x^2}{4t}}\vartheta^0\big(\frac13t^3+2\varepsilon t-t^{-1},2y-(t+\varepsilon t^{-1})x\big)\Big),\ \varepsilon=\pm1,$
- $\beta=0\colon\quad u=(\mathrm{P}^2)^k\vartheta^0\big(\frac13t^3,y-tx\big),$
- $\beta=0\colon\quad u=(\mathrm{P}^3)^k\vartheta^0(t,x).$

Note that for the above cases of solution generation, the generated solutions with $k=0,\dots,n-1$, $n\in\mathbb{N}$, span the spaces of solutions of the equation (3.16) that are respectively invariant with respect to generalized symmetries $\big((\mathrm{P}^3)^nu\big)\partial_u$, $\big((\mathrm{P}^2+\varepsilon\mathrm{P}^0)^nu\big)\partial_u$, $\big((\mathrm{P}^1)^nu\big)\partial_u$ and $\big((\mathrm{P}^0)^nu\big)\partial_u$, see [70, Section 8] and [120, Section 5.3].

Pushing forward the above results for the equation (3.16) by the transformation (3.4), we derive the results for the equation $\mathcal{F}_5'$. Following Remark 3.1, instead of the latter equation, we consider its counterpart without the absolute value sign,

$$u_t+xu_y=x^5u_{xx},\tag{3.23}$$

and replace the transformation (3.4) by the simpler transformation (3.6).

Proposition 3.20. *The maximal Lie invariance algebra* $\mathfrak{g}_5$ *of the equation* (3.23) *is spanned by the vector fields*

$$\begin{aligned}
&\mathcal{P}^y=\partial_y,\quad \mathcal{D}=3t\partial_t-x\partial_x+2y\partial_y-3u\partial_u,\\
&\mathcal{K}=3ty\partial_t-x(3tx+y)\partial_x+y^2\partial_y-(3tx+3y+x^{-2})u\partial_u,\\
&\mathcal{P}^3=y^3\partial_t-3x^2y^2\partial_x-3(xy^2-t+x^{-1}y)u\partial_u,\\
&\mathcal{P}^2=y^2\partial_t-2x^2y\partial_x-(2yx+x^{-1})u\partial_u,\\
&\mathcal{P}^1=y\partial_t-x^2\partial_x-xu\partial_u,\quad \mathcal{P}^0=\partial_t,\\
&\mathcal{I}=u\partial_u,\quad \mathcal{Z}(f)=f(t,x,y)\partial_u,
\end{aligned}$$

where the parameter function f *runs through the solution set of the equation* (3.23).

The algebra $\mathfrak{g}_5^{\rm ess}$ admits a Levi decomposition $\mathfrak{g}_5^{\rm ess}=\mathfrak{f}_5\in\mathfrak{r}_5$, where $\mathfrak{f}_5$ and $\mathfrak{r}_5$ are the images of $\mathfrak{f}_0$ and $\mathfrak{r}_0$ under the pushforward of (3.4).

Theorem 3.21. *The complete point symmetry pseudogroup* G_5 *of the equation* (3.23) *consists of the transformations of the form*

$$\begin{aligned}
&\tilde t=\frac{\hat t}{(\gamma y+\delta)^3},\quad \tilde x=\frac{x(\gamma y+\delta)^2}{\hat x(\gamma y+\delta)-3\gamma\hat t x},\quad \tilde y=\frac{\alpha y+\beta}{\gamma y+\delta},\\
&\tilde u=\frac{\sigma(\gamma y+\delta)^4}{\hat x(\gamma y+\delta)-3\gamma\hat t x}\exp\left(\frac{\gamma\hat x^2}{x^2(\gamma y+\delta)}-\frac{3\gamma^2\hat x\hat t}{x(\gamma y+\delta)^2}+\frac{3\gamma^3\hat t^2}{(\gamma y+\delta)^3}\right)\\
&\qquad\times\exp\big(3\lambda_3(t-y/x)-\lambda_2/x-(3\lambda_3^2y^3+3\lambda_3\lambda_2y^2+\lambda_2^2y)\big)\big(u+f(t,x,y)\big),
\end{aligned}$$

where $\hat x:=1+3\lambda_3xy^2+2\lambda_2xy+\lambda_1x$, $\hat t:=t+\lambda_3y^3+\lambda_2y^2+\lambda_1y+\lambda_0$; α, β, γ *and* δ *are arbitrary constants with* $\alpha\delta-\beta\gamma=1$; $\lambda_0,\dots,\lambda_3$ *and* σ *are arbitrary constants with* $\sigma\neq0$, *and* f *is an arbitrary solution of* (3.23).

The counterpart of (3.18) for generating new solutions of the equation (3.23) using elements of G_5 takes, in the notation of Theorem 3.21, the following form:

$$\begin{aligned}
u=&\frac{\mathrm{e}^{\lambda_2/x-3\lambda_3(t-y/x)+3\lambda_3^2y^3+3\lambda_3\lambda_2y^2+\lambda_2^2y}}{\sigma(\gamma y+\delta)^4}\exp\left(\frac{-\gamma\hat x^2}{x^2(\gamma y+\delta)}+\frac{3\gamma^2\hat x\hat t}{x(\gamma y+\delta)^2}-\frac{3\gamma^3\hat t^2}{(\gamma y+\delta)^3}\right)\\
&\times\big(\hat x(\gamma y+\delta)-3\gamma\hat t x\big)h\left(\frac{\hat t}{(\gamma y+\delta)^3},\frac{x(\gamma y+\delta)^2}{\hat x(\gamma y+\delta)-3\gamma\hat t x},\frac{\alpha y+\beta}{\gamma y+\delta}\right)-f(t,x,y).
\end{aligned}$$

The essential Lie-invariant solutions of the equation (3.23) are exhausted by the families

- $\beta=5\colon\quad u=|x|^{\frac54}\vartheta^\mu\big(\frac94\tilde\varepsilon t,|x|^{-\frac32}\big),\quad \mu=\frac{5}{36},\ \tilde\varepsilon:=\operatorname{sgn}x,$
- $\beta=5\colon\quad u=x|y|^{-\frac12}\mathrm{e}^{-\frac{1}{4x^2y}}\vartheta^0\Big(\frac13y^3+2\varepsilon y-y^{-1},2t-(y+\varepsilon y^{-1})x^{-1}\Big),\quad \varepsilon=\pm1,$
- $\beta=5\colon\quad u=x\vartheta^0\Big(\frac13y^3,t-yx^{-1}\Big),$
- $\beta=5\colon\quad u=x\vartheta^0\big(y,x^{-1}\big),$

where $\vartheta^\mu=\vartheta^\mu(z_1,z_2)$ is an arbitrary solution of the equation $\vartheta^\mu_1=\vartheta^\mu_{22}+\mu z_2^{-2}\vartheta^\mu$, see footnote 4.

The counterpart $\Upsilon_{\mathfrak{r}_5}$ of the algebra $\Upsilon_{\mathfrak{r}_0}$ is generated by the differential operators

$$\begin{aligned}&\mathrm{P}^3:=y^3\mathrm{D}_t-3x^2y^2\mathrm{D}_x+3(xy^2-t+x^{-1}y),\\&\mathrm{P}^2:=y^2\mathrm{D}_t-2x^2y\mathrm{D}_x+(2yx+x^{-1}),\\&\mathrm{P}^1:=y\mathrm{D}_t-x^2\mathrm{D}_x+x,\quad \mathrm{P}^0:=\mathrm{D}_t,\end{aligned}$$

which is associated with the Lie-symmetry vector fields $-\mathcal{P}^3$, $-\mathcal{P}^2$,$-\mathcal{P}^1$ and $-\mathcal{P}^0$ of the equation (3.23). The following assertion is just a particular case of a more general result [120, Corollary 20].

Corollary 3.22. *The algebra of canonical representatives of generalized symmetries of the equation* (3.23) *is* $\Sigma_5=\Lambda_5\in\Sigma_5^{-\infty}$, *where*

$$\Lambda_5=\big\langle\big((\mathrm{P}^3)^{i_3}(\mathrm{P}^2)^{i_2}(\mathrm{P}^1)^{i_1}(\mathrm{P}^0)^{i_0}u\big)\partial_u\mid i_0,i_1,i_2,i_3\in\mathbb{N}_0\big\rangle,\quad \Sigma_5^{-\infty}:=\big\{\mathcal{Z}(f)\big\}.$$

Here the parameter function f *runs through the solution set of* (3.23).

To switch from the algebra Σ_5 to the algebra of standard canonical representatives of generalized symmetries of the equation (3.23) as an evolution equation, we need to replace, in view of this equation, the operator D_t in $\mathrm{P}^3,\dots,\mathrm{P}^0$ by the operator $x^5\mathrm{D}_x^2-x\mathrm{D}_y$, which complicates the expanded form of the representatives and increases their order.

The above solutions of the equation (3.16) generated by its recursion operators

are mapped to the following solutions of (3.23):

- $\beta=5\colon\ u=(\mathrm{P}^1)^k\Big(|x|^{\frac54}\vartheta^\mu\big(\frac94\tilde\varepsilon t,|x|^{-\frac32}\big)\Big)\quad$ with $\quad\mu=\frac{5}{36},\quad\tilde\varepsilon:=\operatorname{sgn}x,$
- $\beta=5\colon\ u=(\mathrm{P}^1)^k\Big(x|y|^{-\frac12}\mathrm{e}^{-\frac{1}{4yx^2}}\vartheta^0\big(\frac{y^3}{3}+2\varepsilon y-y^{-1},2t-\frac{y+\varepsilon y^{-1}}{x}\big)\Big),\ \varepsilon\in\{-1,1\},$
- $\beta=5\colon\ u=(\mathrm{P}^2)^k\Big(x\vartheta^0\big(\frac13y^3,t-x^{-1}y\big)\Big),$
- $\beta=5\colon\ u=(\mathrm{P}^3)^k\big(x\vartheta^0(y,x^{-1})\big).$

3.7 Fine Kolmogorov backward equation and its counterpart

Another distinguished equation within the classes $\mathcal{F}'$, $\mathcal{F}$ and $\bar{\mathcal{F}}$ is the Kolmogorov backward equation $\mathcal{F}'_2$,

$$\mathcal{F}'_2\colon\quad u_t+xu_y=x^2u_{xx}. \tag{3.24}$$

Its essential Lie invariance algebra is five-dimensional and nonsolvable, and modulo the point equivalence, it is the only equation from the above classes with this property. This is why in [75], we refer to $\mathcal{F}'_2$ as to the *fine Kolmogorov backward equation*. Some of the results from [75] are reviewed below.

The maximal Lie invariance $\mathfrak{g}_2$ algebra of (3.24) is spanned by the vector fields

$$\mathcal{P}^y=\partial_y,\quad \mathcal{D}=x\partial_x+y\partial_y,\quad \mathcal{K}=2xy\partial_x+y^2\partial_y-ux\partial_u,$$
$$\mathcal{P}^t=\partial_t,\quad \mathcal{I}=u\partial_u,\quad \mathcal{Z}(f)=f(t,x,y)\partial_u,$$

where the parameter function f ranges through the solution set of (3.24).

Theorem 3.23 ([75]). *The point symmetry pseudogroup G_2 of the fine Kolmogorov backward equation (3.24) consists of the point transformations of the form*

$$\tilde t=t+\lambda,\quad \tilde x=\frac{\alpha\delta-\beta\gamma}{(\gamma y+\delta)^2}x,\quad \tilde y=\frac{\alpha y+\beta}{\gamma y+\delta},$$
$$\tilde u=\sigma\exp\left(-\frac{\gamma x}{\gamma y+\delta}\right)\big(u+f(t,x,y)\big)$$

where α, β, γ and δ are arbitrary constants with $\alpha\delta-\beta\gamma=\pm1$ that are defined up to simultaneously alternating their signs; λ and σ are arbitrary constants with $\sigma\neq0$, and f is an arbitrary solution of the equation (3.24).

In the above notation, the most general form of the G_2-equivalent counterparts of an arbitrary seed solution $u=h(t,x,y)$ of (3.24) is

$$\tilde u=\sigma\exp\left(-\frac{\gamma x}{\alpha-\gamma y}\right)h\left(t-\lambda,\frac{\alpha\delta-\beta\gamma}{(\alpha-\gamma y)^2}x,\frac{\delta y-\beta}{\alpha-\gamma y}\right)+f(t,x,y).$$

Essential $G_2^{\rm ess}$-inequivalent Lie-invariant solutions of (3.24) in closed form were constructed in [75] only by codimension-one Lie reductions,

- $\beta=2$: $u=\mathrm{e}^{\mu t}|x|^{1/4}\vartheta^{4\mu+\frac34}(\varepsilon y,2\sqrt{|x|})$ with $\varepsilon:=\operatorname{sgn}x$,
- $\beta=2$: $u=\mathrm{e}^{-\frac14 t}|x|^{1/2}\vartheta^0(t,\ln|x|)$,

where $\vartheta^\mu=\vartheta^\mu(z_1,z_2)$ is an arbitrary solution of the linear (1+1)-dimensional heat equation with the inverse square potential $\vartheta^\mu_1=\vartheta^\mu_{22}+\mu z_2^{-2}\vartheta^\mu$, see footnote 4.

For each of the values $\mu_n=\frac14n(n+1)-\frac{3}{16}$, $n\in\mathbb{N}$, the first of the above solution families can be expressed using Darboux transformations in terms of the general solution ϑ^0 of the linear (1+1)-dimensional heat equation, see Section 3.B,

- $\beta=2$: $u=\mathrm{e}^{\mu_n t}|x|^{\frac{2n+1}{4}}\left(\partial_x-\dfrac{1}{2x}\right)^n\vartheta^0(\varepsilon y,2\sqrt{|x|})$ with $\mu_n:=\dfrac{n(n+1)}{4}-\dfrac{3}{16}$.

In [75] we also constructed two families of solutions of (3.24) that are invariant with respect to generalized symmetries of (3.24),

- $\beta=2$: $u=\mathrm{e}^{-\frac14 t}|x|^{n-\frac12}\displaystyle\sum_{s=0}^{n-1}\frac{y^sx^{-s}}{s!}\left(\prod_{k=n-s}^{n-1}(2kx\partial_x+k^2)\right)\vartheta^0(t,\ln|x|)$,
- $\beta=2$: $u=|x|^{1/4}\Big(t\vartheta_2^{3/4}(\varepsilon y,2\sqrt{|x|})-(\frac14t-1)|x|^{-1/2}\vartheta^{3/4}(\varepsilon y,2\sqrt{|x|})\Big)$,

where ϑ^μ_2 denotes the derivative of ϑ^μ with respect to its second argument. According to [75, Theorem 14], for any $n\in\mathbb{N}$, the solutions from the former family, which are invariant with respect to the generalized symmetries $(\mathrm{D}_y^nu)\partial_u$, can be expressed in terms of the iterative action by the Lie-symmetry operator $\mathrm{K}:=$

$2xy\mathrm{D}_x+y^2\mathrm{D}_y+x$ on Lie invariant solutions of (3.24) from the second family of such solutions, $u=\mathrm{K}^{n-1}\big(\mathrm{e}^{-\frac14 t}|x|^{1/2}\theta(t,\ln|x|)\big)$. Moreover, this is the only nontrivial case of generating new solutions of (3.24) by acting Lie-symmetry operators.

Following Remark 3.1, instead of $\mathcal{F}'_3$ and the equivalence transformation (3.4), we can consider the equation

$$u_t+xu_y=x^3u_{xx} \tag{3.25}$$

as the counterpart of $\mathcal{F}'_2$ with respect to the simpler transformation (3.6) and push forward the above results for $\mathcal{F}'_2$ by the latter transformation.

Proposition 3.24. *The maximal Lie invariance algebra $\mathfrak{g}_3$ of the equation (3.25) is spanned by the vector fields*

$$\begin{aligned}&\mathcal{P}^t=\partial_t,\quad \mathcal{D}=t\partial_t-x\partial_x-u\partial_u,\quad \mathcal{K}=t^2\partial_t-2tx\partial_x-(2t+x^{-1})u\partial_u,\\ &\mathcal{P}^y=\partial_y,\quad \mathcal{I}=u\partial_u,\quad \mathcal{Z}(f)=f(t,x,y)\partial_u,\end{aligned}$$

where the parameter function f runs through the solution set of the equation (3.25).

Theorem 3.25. *The point symmetry pseudogroup G_3 of the equation (3.25) consists of the point transformations of the form*

$$\begin{aligned}&\tilde t=\frac{\alpha t+\beta}{\gamma t+\delta},\quad \tilde x=\frac{(\gamma t+\delta)^2}{\alpha\delta-\beta\gamma}x,\quad \tilde y=y+\lambda,\\ &\tilde u=\sigma\frac{(\gamma t+\delta)^2}{\alpha\delta-\beta\gamma}\exp\left(-\frac{\gamma x^{-1}}{\gamma t+\delta}\right)\big(u+f(t,x,y)\big),\end{aligned} \tag{3.26}$$

where α, β, γ and δ are arbitrary constants with $\alpha\delta-\beta\gamma=\pm1$ that are defined up to simultaneously alternating their signs; λ and σ are arbitrary constants with $\sigma\neq0$, and f is an arbitrary solution of the equation (3.25).

Thus, under the action of an element of G_3, which is of the form (3.26), a given solution $u=h(t,x,y)$ of (3.25) is mapped to the solution

$$\tilde u=\sigma\frac{\alpha\delta-\beta\gamma}{(\alpha-\gamma t)^2}\exp\left(-\frac{\gamma x^{-1}}{\alpha-\gamma t}\right)h\left(\frac{\delta t-\beta}{\alpha-\gamma t},\frac{(\alpha-\gamma t)^2}{\alpha\delta-\beta\gamma}x,y-\lambda\right)+f(t,x,y).$$

In the above notation, the constructed $G_2^{\rm ess}$-inequivalent closed-form Lie invariant

solutions of $\mathcal{F}'_3$ are exhausted by the following:

- $\beta=3$: $\quad u=\mathrm{e}^{\mu y}|x|^{3/4}\vartheta^{4\mu+\frac{3}{4}}(\varepsilon t,2|x|^{-1/2})\quad$ with $\quad\varepsilon:=\operatorname{sgn}x$,
- $\beta=3$: $\quad u=\mathrm{e}^{-\frac{1}{4}y}|x|^{1/2}\vartheta^0(y,-\ln|x|)$.

In the similar way for each of the values $\mu_n=\frac{1}{4}n(n+1)-\frac{3}{16}$, $n\in\mathbb{N}$, the first of the above solution families can be expressed using Darboux transformations in terms of the general solution ϑ^0 of the linear (1+1)-dimensional heat equation,

- $\beta=3$: $u=\mathrm{e}^{\mu_n y}x|x|^{-\frac{2n+1}{4}}\left(x^2\partial_x+\dfrac{x}{2}\right)^n\vartheta^0(\varepsilon t,2|x|^{-1/2})$ with $\mu_n:=\dfrac{n^2+n}{4}-\dfrac{3}{16}$.

The above solutions of the equation (3.24) that are invariant with respect to generalized symmetries are mapped to the following solutions of (3.25):

- $\beta=3$: $\quad u=\mathrm{e}^{-\frac{1}{4}y}x|x|^{\frac{1}{2}-n}\displaystyle\sum_{s=0}^{n-1}\frac{t^sx^s}{s!}\left(\prod_{k=n-s}^{n-1}(-2kx\partial_x+k^2)\right)\vartheta^0(y,-\ln|x|)$,
- $\beta=3$: $\quad u=x|x|^{-1/4}\Big(xy\vartheta_2^{3/4}(\varepsilon t,2|x|^{-1/2})-(\frac{1}{4}y-1)|x|^{1/2}\vartheta^{3/4}(\varepsilon t,2|x|^{-1/2})\Big)$.

3.8 Generic case

As the generic case, we consider the case $\beta\neq0,2,3,5$. In view of Remark 3.12, up to the $G^{\sim}_{\mathcal{F}'}$-equivalence, it suffices to consider only the equations

$$\mathcal{F}'_\beta\colon\quad u_t+xu_y=|x|^\beta u_{xx},\quad \beta\in(-\infty,5/2]\setminus\{0,2\}.$$

According to Section 3.5, the essential Lie invariance algebra $\mathfrak{g}^{\rm ess}_\beta$ of the equation $\mathcal{F}'_\beta$ with such a value of β is spanned by the vector fields

$$\mathcal{P}^t:=\partial_t,\quad \mathcal{P}^y:=\partial_y,\quad \mathcal{D}^\beta:=(2-\beta)t\partial_t+x\partial_x+(3-\beta)y\partial_y,\quad \mathcal{I}:=u\partial_u.$$

Up to the skew-symmetry of the Lie bracket of vector fields, the only nonzero commutation relations among the basis elements of $\mathfrak{g}^{\rm ess}_\beta$ are the following:

$$[\mathcal{P}^t,\mathcal{D}^\beta]=(2-\beta)\mathcal{P}^t,\quad [\mathcal{P}^y,\mathcal{D}^\beta]=(3-\beta)\mathcal{P}^y.$$

The algebra $\mathfrak{g}^{\rm ess}_\beta$ is isomorphic to the algebra $A^a_{3.4}\oplus A_1$ with $a=(2-\beta)/(3-\beta)$, see [125] for notation, which is consistent with Mubarakzyanov's algebra numeration [99].

3.8.1 Point symmetry pseudogroup

Theorem 3.26. *(i) For $\beta\in\mathbb{R}\setminus\{0,2,5/2,3,5\}$, the point symmetry pseudogroup G_β of the equation $\mathcal{F}'_\beta$ consists of the point transformations*

$$\tilde t=|\alpha|^{2-\beta}t+\lambda_0,\quad \tilde x=\alpha x,\quad \tilde y=\alpha|\alpha|^{2-\beta}y+\lambda_1,\quad \tilde u=\sigma u+f(t,x,y),\qquad (3.27)$$

where α, λ_0, λ_1 and σ are arbitrary constants with $\alpha\sigma\neq0$, and f is an arbitrary solution of $\mathcal{F}'_\beta$.

(ii) In comparison with the generic case of β, the point symmetry pseudogroup $G_{5/2}$ of the equation $\mathcal{F}'_{5/2}$ is extended by the point transformations

$$\begin{aligned}&\tilde t=\operatorname{sgn}(x)\alpha|\alpha|^{2-\beta}y+\lambda_1,\quad \tilde x=\frac{\alpha}{x},\quad \tilde y=\operatorname{sgn}(x)|\alpha|^{2-\beta}t+\lambda_0,\\ &\tilde u=\sigma\frac{u}{x}+f(t,x,y).\end{aligned}\qquad (3.28)$$

Proof. The standard algebraic method and its known modifications for computing point symmetry (pseudo)groups of systems of differential equations, see, e.g., [16, 26, 37, 48–51, 89, 108] and references therein, do not work properly in the case of linear well-determined systems of partial differential equations. This is why we combine the algebraic part of this method with the advanced direct method in an original way. More specifically, Theorem 3.2 implies that for any fixed β, any point symmetry transformation Φ of the equation $\mathcal{F}'_\beta$ is of the form (3.7a), where U^0/U^1 is a solution of $\mathcal{F}'_\beta$ in view of the last equation in (3.7c) and the homogeneousness of $\mathcal{F}'_\beta$ (i.e., $C=0$ and $\tilde C=0$). Thus, up to composing with transformations of linear superposition of solutions, we can assume that $U^0=0$. Then the transformation Φ preserves the algebra $\mathfrak{g}^{\rm ess}_\beta$ and thus induces an automorphism of this algebra.

Fixing the basis $(\mathcal{P}^t,\mathcal{P}^y,\mathcal{D}^\beta,\mathcal{I})$, we identify each automorphism of $\mathfrak{g}^{\rm ess}_\beta$ with its matrix in the this basis. The automorphisms of the algebra $A^a_{3.4}\oplus A_1$, which is isomorphic to $\mathfrak{g}^{\rm ess}_\beta$, were first computed in [31, Table 1] in a consistent basis using a different algebra notation ($A_{3.4}\oplus A_1$ instead of $A^{-1}_{3.4}\oplus A_1$ and $A^\alpha_{3.5}\oplus A_1$ instead of $A^\alpha_{3.4}\oplus A_1$ for $-1<\alpha<1$); see also [125, Section A.2] for a more convenient exposition.

In the case $\beta \neq 5/2$, the automorphism group $\mathrm{Aut}(\mathfrak{g}^{\mathrm{ess}}_{\beta})$ of $\mathfrak{g}^{\mathrm{ess}}_{\beta}$ is constituted by the matrices

$$\begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ 0 & a_{22} & a_{23} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix}, \tag{3.29}$$

where all the involved parameters a_{ij} are arbitrary real constants with $a_{11}a_{22}a_{44} \neq 0$. If $\beta = 5/2$, then the group $\mathrm{Aut}(\mathfrak{g}^{\mathrm{ess}}_{\beta})$ is extended, in comparison with the generic case, by the matrices

$$\begin{pmatrix} 0 & a_{12} & a_{13} & 0 \\ a_{21} & 0 & a_{23} & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix},$$

where all the involved parameters a_{ij} are arbitrary real constants with $a_{12}a_{21}a_{44} \neq 0$. The singularity of the value $\beta = 5/2$ is related to the fact that the equivalence transformation $\mathcal{J}'$ preserves this value of β and, therefore, it induces an additional point symmetry transformation of $\mathcal{F}'_{5/2}$. Factoring out the induced transformation from $G_{5/2}$, we can consider only automorphisms of the form (3.29) for each relevant value β.

Pushing forward the basis elements of $\mathfrak{g}^{\mathrm{ess}}_{\beta}$ by Φ, we obtain the equalities

$$\Phi_*(\mathcal{P}^t) = a_{11}\mathcal{P}^t, \quad \Phi_*(\mathcal{P}^y) = a_{22}\mathcal{P}^y, \quad \Phi_*(\mathcal{I}) = a_{44}\mathcal{I},$$
$$\Phi_*(\mathcal{D}^\beta) = \mathcal{D}^\beta + a_{13}\mathcal{P}^t + a_{23}\mathcal{P}^y + a_{43}\mathcal{I}.$$

for some values the involved parameters a_{ij}. Splitting these equalities componentwise results in the following system of determining equations for the components of Φ:

$$T_t = a_{11}, \quad X_t = Y_t = U^1_t = 0,$$
$$Y_y = a_{22}, \quad T_y = X_y = U^1_y = 0,$$
$$U^1 = a_{44}, \quad xX_x = X,$$

which integrates to

$$T = a_{11}t + b_1, \quad X = cx, \quad Y = a_{22}y + b_2, \quad U = a_{44}u,$$

where b_1, b_2 and c are real constants with $c \neq 0$. Proceeding with the direct method, we take the equation $\mathcal{F}'_\beta$ in the variables with tildes, expand, using the chain rule, all the involved derivatives in terms of the variables without tildes, substitute the expression for u_t in view of the equation $\mathcal{F}'_\beta$ into the expanded equation and split the result with respect to all the remaining jet variables. This gives two equations for the constant parameters a_{11}, a_{22} and c,

$$a_{11} = |c|^{2-\beta}, \quad a_{22} = c|c|^{2-\beta}.$$

The final step is to redenote the remaining constant parameters b_1, b_2, c and a_{44}. □

Remark 3.27. The essential point symmetry group $G^{\rm ess}_\beta$ coincides with the kernel point symmetry group $G^{\cap}_{\mathcal{F}'}$ of the class $\mathcal{F}'$ if $\beta \in \mathbb{R} \setminus \{0, 2, 5/2, 3, 5\}$ and with the group $\pi_* G^{\sim}_{\mathcal{F}'}$ if $\beta \neq 5/2$, where π is the natural projection $\pi\colon \mathbb{R}^5_{t,x,y,u,\beta} \to \mathbb{R}^4_{t,x,y,u}$, cf. Remark 3.11. In other words, the group $G^{\rm ess}_\beta$ is extended for value $\beta = 5/2$ by the element $\pi_*\mathcal{J}'$.

Remark 3.28. A complete list of independent discrete point symmetries of the equation $\mathcal{F}'_\beta$ is exhausted by the transformations $\pi_*\mathcal{I}'_u$ and $\pi_*\mathcal{I}'_{\rm s}$ if $\beta \neq 5/2$ and by the transformations $\pi_*\mathcal{I}'_u$ and $\pi_*(\mathcal{I}'_{\rm s} \circ \mathcal{J}')$ if $\beta \neq 5/2$, cf. Remark 3.11 and Corollary 3.15.

3.8.2 Classification of subalgebras

The established isomorphism between $\mathfrak{g}^{\rm ess}_\beta$ and $A^a_{3.4} \oplus A_1$ with $a = (2-\beta)/(3-\beta)$ and results of [114] (in a different algebra notation with $A_{3.4} \oplus A_1$ instead of $A^{-1}_{3.4} \oplus A_1$ and $A^\alpha_{3.5} \oplus A_1$ instead of $A^\alpha_{3.4} \oplus A_1$ for $-1 < \alpha < 1$) straightforwardly give the classification of subalgebras of $\mathfrak{g}^{\rm ess}_\beta$.

Lemma 3.29. *A complete list of inequivalent (with respect to inner automorphisms) subalgebras of the algebra $\mathfrak{g}^{\rm ess}_\beta$ with $\beta \in \mathbb{R} \setminus \{0, 2, 3, 5\}$ is exhausted by the following subalgebras:*

$$\text{1D}\colon\ \mathfrak{s}^\kappa_{1.1} := \langle \mathcal{D}^\beta + \kappa\mathcal{I}\rangle, \quad \mathfrak{s}^{\varepsilon,\kappa}_{1.2} := \langle \mathcal{P}^y + \varepsilon\mathcal{P}^t + \kappa\mathcal{I}\rangle, \quad \mathfrak{s}^\delta_{1.3} := \langle \mathcal{P}^y + \delta\mathcal{I}\rangle,$$

$$
\begin{aligned}
&\mathfrak{s}^{\delta}_{1.4} := \langle \mathcal{P}^t + \delta\mathcal{I}\rangle, \quad \mathfrak{s}_{1.5} := \langle \mathcal{I}\rangle,\\
\text{2D:}\ &\mathfrak{s}^{\delta}_{2.1} := \langle \mathcal{P}^y, \mathcal{P}^t + \delta\mathcal{I}\rangle,\ \mathfrak{s}^{\varepsilon,\kappa}_{2.2} := \langle \mathcal{P}^y + \varepsilon\mathcal{I}, \mathcal{P}^t + \kappa\mathcal{I}\rangle,\ \mathfrak{s}^{\kappa}_{2.3} := \langle \mathcal{D}^\beta + \kappa\mathcal{I}, \mathcal{P}^y\rangle,\\
&\mathfrak{s}^{\kappa}_{2.4} := \langle \mathcal{D}^\beta + \kappa\mathcal{I}, \mathcal{P}^t\rangle, \quad \mathfrak{s}_{2.5} := \langle \mathcal{D}^\beta, \mathcal{I}\rangle, \quad \mathfrak{s}^{\delta}_{2.6} := \langle \mathcal{P}^y + \delta\mathcal{P}^t, \mathcal{I}\rangle,\\
&\mathfrak{s}_{2.7} := \langle \mathcal{P}^t, \mathcal{I}\rangle,\\
\text{3D:}\ &\mathfrak{s}^{\kappa}_{3.1} := \langle \mathcal{D}^\beta + \kappa\mathcal{I}, \mathcal{P}^t, \mathcal{P}^y\rangle, \quad \mathfrak{s}_{3.2} := \langle \mathcal{D}^\beta, \mathcal{P}^y, \mathcal{I}\rangle, \quad \mathfrak{s}_{3.3} := \langle \mathcal{D}^\beta, \mathcal{P}^t, \mathcal{I}\rangle,
\end{aligned}
$$

where $\delta \in \{-1, 0, 1\}$, $\varepsilon \in \{-1, 1\}$ *and* κ *is an arbitrary constant.*

Corollary 3.30. *(i) For the generic case of* $\beta \in \mathbb{R} \setminus \{0, 2, 5/2, 3, 5\}$*, a complete list of* $G^{\rm ess}_\beta$*-inequivalent subalgebras of the algebra* $\mathfrak{g}^{\rm ess}_\beta$ *is exhausted by the subalgebras*

$$
\begin{aligned}
\text{1D:}\ &\mathfrak{s}^{\kappa}_{1.1}, \quad \mathfrak{s}^{1,\kappa}_{1.2}, \quad \mathfrak{s}^{\delta'}_{1.3}, \quad \mathfrak{s}^{\delta}_{1.4}, \quad \mathfrak{s}_{1.5},\\
\text{2D:}\ &\mathfrak{s}^{\delta}_{2.1}, \quad \mathfrak{s}^{1,\kappa}_{2.2}, \quad \mathfrak{s}^{\kappa}_{2.3}, \quad \mathfrak{s}^{\kappa}_{2.4}, \quad \mathfrak{s}_{2.5}, \quad \mathfrak{s}^{\delta'}_{2.6}, \quad \mathfrak{s}_{2.7},\\
\text{3D:}\ &\mathfrak{s}^{\kappa}_{3.1}, \quad \mathfrak{s}_{3.2}, \quad \mathfrak{s}_{3.3},
\end{aligned}
$$

where $\delta' \in \{0, 1\}$ *and* κ *is an arbitrary constant.*

(ii) An analogous list for $\beta = 5/2$ *is given by the subagebras*

$$
\begin{aligned}
\text{1D:}\ &\mathfrak{s}^{\kappa}_{1.1}, \quad \mathfrak{s}^{1,\kappa}_{1.2}, \quad \mathfrak{s}^{\delta'}_{1.3}, \quad \mathfrak{s}_{1.5},\\
\text{2D:}\ &\mathfrak{s}^{\delta}_{2.1}, \quad \mathfrak{s}^{1,\kappa}_{2.2}, \quad \mathfrak{s}^{\kappa}_{2.3}, \quad \mathfrak{s}_{2.5}, \quad \mathfrak{s}^{\delta'}_{2.6}, \quad \mathfrak{s}_{2.7},\\
\text{3D:}\ &\mathfrak{s}^{\kappa}_{3.1}, \quad \mathfrak{s}_{3.2}.
\end{aligned}
$$

Among the subalgebras listed in Lemma 3.29, only the following satisfy the transversality condition and are thus appropriate for Lie reduction:

$$
\begin{aligned}
\text{1D:}\ &\mathfrak{s}^{\kappa}_{1.1}, \quad \mathfrak{s}^{\varepsilon,\kappa}_{1.2}, \quad \mathfrak{s}^{\delta}_{1.3}, \quad \mathfrak{s}^{\delta}_{1.4},\\
\text{2D:}\ &\mathfrak{s}^{\delta}_{2.1}, \quad \mathfrak{s}^{\varepsilon,\kappa}_{2.2}, \quad \mathfrak{s}^{\kappa}_{2.3}, \quad \mathfrak{s}^{\kappa}_{2.4},\\
\text{3D:}\ &\mathfrak{s}^{\kappa}_{3.1}.
\end{aligned}
$$

Instead of the classification of Lie reductions of a single equation $\mathcal{F}'_\beta$ modulo its essential point-symmetry (pseudo)group $G^{\rm ess}_\beta$, it is convenient to classify Lie reductions within the class $\mathcal{F}'$ up to the $G^{\sim}_{\mathcal{F}'}$-equivalence. More specifically, pushing forward the subalgebras $\mathfrak{s}^{\delta}_{1.3}$ and $\mathfrak{s}^{\kappa}_{2.3}$ of the algebra $\mathfrak{g}^{\rm ess}_\beta$ by the equivalence transformation (3.4),

we obtain the subalgebras $\mathfrak{s}^{\delta}_{1.4}$ and $\mathfrak{s}^{\kappa}_{2.4}$ of the algebra $\mathfrak{g}^{\rm ess}_{5-\beta}$, respectively. In view of this, we can without loss of generality omit the Lie reductions with respect to the subalgebras $\mathfrak{s}^{\delta}_{1.4}$ and $\mathfrak{s}^{\kappa}_{2.4}$. However, this requires us to relax the restriction on the parameter β, from $\beta \in (-\infty, 5/2] \setminus \{0, 2\}$ to $\beta \in \mathbb{R} \setminus \{0, 2, 3, 5\}$. The suggested adjustment simplifies the Lie reduction procedure for the equations $\mathcal{F}'_{\beta}$ with $\beta \in \mathbb{R} \setminus \{0, 2, 3, 5\}$ by reducing the number of cases to be considered.

3.8.3 Lie reductions of codimension one

We use the subalgebras $\mathfrak{s}^{\kappa}_{1.1}$, $\mathfrak{s}^{\varepsilon,\kappa}_{1.2}$ and $\mathfrak{s}^{\delta}_{1.3}$ for carrying out codimension-one Lie reductions. For each of these subalgebras, we construct an ansatz for u with new unknown function $w = w(z_1, z_2)$ of two new independent variables (z_1, z_2) and the corresponding reduced partial differential equation for w, which are listed below. In what follows, the subscripts in w_1 and w_2 indicate the derivatives of w with respect to z_1 and z_2, respectively.

1.1. $\mathfrak{s}^{\kappa}_{1.1} = \langle \mathcal{D}^{\beta} + \kappa \mathcal{I} \rangle$ with $\kappa \in \mathbb{R}$.

$$z_1 := y|t|^{-(\beta-3)/(\beta-2)}, \quad z_2 := x|t|^{1/(\beta-2)}, \quad u = w(z_1, z_2)|t|^{\kappa(\beta-3)/(\beta-2)};$$
$$(\epsilon(\beta-2)z_2 + (3-\beta)z_1)w_1 = \epsilon(\beta-2)|z_2|^{\beta} w_{22} - z_2 w_2 - \kappa(\beta-3)w, \quad \epsilon = \operatorname{sgn} t.$$

For each value of (κ, ϵ), the maximal Lie invariance algebra of reduced equation $1.1^{\kappa\epsilon}$ is trivial as for a linear homogeneous equation since it is spanned by the vector fields $w\partial_w$ and $f(z_1, z_2)\partial_w$, where the parameter function f runs through the solution set of reduced equation $1.1^{\kappa\epsilon}$.[5] Moreover, this entire algebra is induced by the normalizer $\mathrm{N}_{\mathfrak{g}_{\beta}}(\mathfrak{s}^{\kappa}_{1.1})$ of the subalgebra $\mathfrak{s}^{\kappa}_{1.1}$ in the algebra $\mathfrak{g}_{\beta}$,

$$\mathrm{N}_{\mathfrak{g}_{\beta}}(\mathfrak{s}^{\kappa}_{1.1}) = \langle \mathcal{D}^{\beta}, \mathcal{I} \rangle \notin \langle \hat{f}(t, x, y)\partial_u \rangle,$$

[5] The construction of the maximal Lie invariance algebras of reduced equations $1.1^{\kappa\epsilon}$ and $1.2^{\kappa\epsilon}$ is too complicated. The corresponding system of determining equations cannot be solved by standard symbolic computation programs for studying overdetermined systems of differential equations. The main reason for this is the parameterization of the determining equations by β and κ in a cumbersome way. To overcome the computational obstacles, we combine general techniques from the theory of compatibility of systems of differential equations and certain specific tools with the classical results by Lie on group classification of (1+1)-dimensional linear evolution equations [86]; see modern treatment of these results in [111].

where the parameter function $\hat{f}$ runs through the set of $\mathfrak{s}^{\kappa}_{1.1}$-invariant solutions of $\mathcal{F}'_{\beta}$.

1.2. $\mathfrak{s}^{\varepsilon,\kappa}_{1.2} = \langle \mathcal{P}^y + \varepsilon\mathcal{P}^t + \kappa\mathcal{I}\rangle$ with $\varepsilon = \pm 1$ and $\kappa \in \mathbb{R}$.

$$z_1 := y - \varepsilon t, \quad z_2 := x, \quad u = \mathrm{e}^{\varepsilon\kappa t} w(z_1, z_2);$$
$$(z_2 - \varepsilon) w_1 = |z_2|^{\beta} w_{22} - \varepsilon\kappa w.$$

For each value of (ε, κ), the maximal Lie invariance algebra of the reduced equation $1.2^{\varepsilon\kappa}$ coincides with the span $\langle \partial_1, w\partial_w, f(z_1, z_2)\partial_w\rangle$, where the parameter function f runs through the solution set of this equation, see again footnote 5. This entire algebra is induced by the normalizer $\mathrm{N}_{\mathfrak{g}_\beta}(\mathfrak{s}^{\varepsilon,\kappa}_{1.2})$ of the subalgebra $\mathfrak{s}^{\varepsilon,\kappa}_{1.2}$ in the algebra $\mathfrak{g}_\beta$,

$$\mathrm{N}_{\mathfrak{g}_\beta}(\mathfrak{s}^{\varepsilon,\kappa}_{1.2}) = \langle \mathcal{P}^t, \mathcal{P}^y, \mathcal{I}\rangle \in \langle \hat{f}(t, x, y)\partial_u\rangle,$$

where the parameter function $\hat{f}$ runs through the set of $\mathfrak{s}^{\varepsilon,\kappa}_{1.2}$-invariant solutions of $\mathcal{F}'_{\beta}$.

1.3. $\mathfrak{s}^{\delta}_{1.3} = \langle \mathcal{P}^y + \delta\mathcal{I}\rangle$ with $\delta \in \{-1, 0, 1\}$.

$$z_1 := t, \quad z_2 := x, \quad u = \mathrm{e}^{\delta y} w(z_1, z_2);$$
$$w_1 = |z_2|^{\beta} w_{22} - \delta z_2 w.$$

Using the point transformation $\tilde{z}_1 = z_1$, $\tilde{z}_2 = 2(\beta - 2)^{-1} z_2 |z_2|^{-\beta/2}$ and $\tilde{w} = |z_2|^{-\beta/4} w$, we further map the reduced equation to its canonical form within the class of (1+1)-dimensional linear second-order evolution equations,

$$\tilde{w}_1 = \tilde{w}_{22} + \left(\frac{\beta(\beta - 4)}{4\tilde{z}_2^2(\beta - 2)^2} - \epsilon\delta \left(\frac{(\beta - 2)^2}{4} \tilde{z}_2^2 \right)^{\frac{1}{2-\beta}} \right) \tilde{w}, \tag{3.30}$$

where $\epsilon = \operatorname{sgn} z_2$.

For any β, the normalizer $\mathrm{N}_{\mathfrak{g}_\beta}(\mathfrak{s}^{\delta}_{1.3})$ of the subalgebra $\mathfrak{s}^{\delta}_{1.3}$ in the algebra $\mathfrak{g}_\beta$ is

$$\mathrm{N}_{\mathfrak{g}_\beta}(\mathfrak{s}^{\delta}_{1.3}) = \mathfrak{g}^{\mathrm{ess}}_{\beta} \in \langle \hat{f}(t, x, y)\partial_u\rangle \quad \text{if} \quad \delta = 0,$$
$$\mathrm{N}_{\mathfrak{g}_\beta}(\mathfrak{s}^{\delta}_{1.3}) = \langle \mathcal{P}^t, \mathcal{P}^y, \mathcal{I}\rangle \in \langle \hat{f}(t, x, y)\partial_u\rangle \quad \text{if} \quad \delta = \pm 1,$$

where the parameter function $\hat{f}$ runs through the set of $\mathfrak{s}^{\delta}_{1.3}$-invariant solutions of the equation $\mathcal{F}'_{\beta}$.

The equation (3.30) with $\delta = 0$ and $\beta = 4$ coincides with the linear (1+1)-dimensional heat equation $\tilde{w}_1 = \tilde{w}_{22}$. Denoting its general solution by $\tilde{w} = \vartheta^0(\tilde{z}_1, \tilde{z}_2)$, cf. footnote 4, results in the solution family

- $\beta = 4$: $u = x\vartheta^0(t, x^{-1})$.

The maximal Lie invariance algebra of the linear (1+1)-dimensional heat equation $\tilde{w}_1 = \tilde{w}_{22}$ is spanned by the vector fields

$$\partial_1, \quad \mathcal{D} := 2z_1\partial_1 + z_2\partial_2 - \tfrac{1}{2}\tilde{w}\partial_{\tilde{w}}, \quad \mathcal{K} := z_1^2\partial_1 + z_1z_2\partial_2 - \tfrac{1}{4}(z_2^2 + 2z_1)\tilde{w}\partial_w,$$
$$z_1\partial_2 - \tfrac{1}{2}z_2\tilde{w}\partial_{\tilde{w}}, \quad \partial_2, \quad \tilde{w}\partial_{\tilde{w}}, \quad f(z_1, z_2)\partial_{\tilde{w}},$$

where the parameter function f runs through the solution set of this equation.

The subalgebra $\langle \partial_1, \mathcal{D}, \tilde{w}\partial_{\tilde{w}}, f(z_1, z_2)\partial_{\tilde{w}}\rangle$ is induced by the normalizer $\mathrm{N}_{\mathfrak{g}_4}(\mathfrak{s}^0_{1.3})$. Thus, the vector fields $\mathcal{K}$, $z_1\partial_2 - \frac{1}{2}z_2\tilde{w}\partial_{\tilde{w}}$ and ∂_2 exhaust a complete list of independent genuine hidden symmetries of the equation $\mathcal{F}'_4$.

If $\delta = 0$ and $\beta \neq 4$, then the equation (3.30) is the linear (1+1)-dimensional heat equation with the inverse square potential,

$$\tilde{w}_1 = \tilde{w}_{22} + \frac{\beta(\beta-4)}{4\tilde{z}_2^2(\beta-2)^2}\tilde{w},$$

and thus its general solution can be denoted as $\tilde{w} = \vartheta^\mu(\tilde{z}_1, \tilde{z}_2)$ with $\mu = \beta(\beta-4)(\beta-2)^{-2}/4$. The maximal Lie invariance algebra of this equation is spanned by the vector fields

$$\partial_{\tilde{z}_1}, \quad \tilde{\mathcal{D}} := \tilde{z}_1\partial_{\tilde{z}_1} + \tfrac{1}{2}\tilde{z}_2\partial_{\tilde{z}_2} - \tfrac{1}{4}\tilde{w}\partial_{\tilde{w}}, \quad \tilde{\mathcal{K}} := \tilde{z}_1^2\partial_{\tilde{z}_1} + \tilde{z}_1\tilde{z}_2\partial_{\tilde{z}_2} - \tfrac{1}{4}(\tilde{z}_2^2 - 2\tilde{z}_2)\tilde{w}\partial_{\tilde{w}},$$
$$\tilde{w}\partial_{\tilde{w}}, \quad f(\tilde{z}_1, \tilde{z}_2)\partial_{\tilde{w}},$$

where $f(\tilde{z}_1, \tilde{z}_2)$ is an arbitrary solution of it. The subalgebra $\langle \partial_{\tilde{z}_1}, \tilde{\mathcal{D}}, \tilde{w}\partial_{\tilde{w}}, f(\tilde{z}_1, \tilde{z}_2)\partial_{\tilde{w}}\rangle$ is induced by the normalizer $\mathrm{N}_{\mathfrak{g}_\beta}(\mathfrak{s}^0_{1.3})$. Thus, the vector field $\tilde{\mathcal{K}}$ is a single independent genuine hidden Lie-symmetry of the equation $\mathcal{F}'_\beta$ with $\beta \neq 4$.

Pulling back the general solution $\tilde{w} = \vartheta^\mu(\tilde{z}_1, \tilde{z}_2)$ with $\mu = \beta(\beta-4)(\beta-2)^{-2}/4$ by the above transformation and substituting the resulting expression into the ansatz,

we obtain the following solution family of the equation $\mathcal{F}'_\beta$:

- $\beta\in\mathbb{R}\setminus\{0,2,3,5\}\colon\ u=|x|^{\beta/4}\vartheta^\mu(t,2(\beta-2)^{-1}x|x|^{-\beta/2})$. (3.31)

Let $\delta=\pm1$ and $\beta\neq1$. The maximal Lie invariance algebra of the reduced equation 1.3^δ is equal to the span $\langle\partial_1,w\partial_w,f(z_1,z_2)\partial_w\rangle$ with f running through the solution set of this equation, which is entirely induced by the normalizer $\mathrm{N}_{\mathfrak{g}_\beta}(\mathfrak{s}^\delta_{1.3})$.

A specific value of β for $\delta=\pm1$ is $\beta=1$. In this case, we can alternate the sign of $\tilde z_2$ and set $\tilde z_2=2|z_2|^{1/2}$. The corresponding modified reduced equations (3.30) with $\epsilon\delta=-1$ and $\epsilon\delta=1$, $\tilde w_1=\tilde w_{22}-\left(\frac34\tilde z_2^{-2}+\frac14\epsilon\delta\tilde z_2^2\right)\tilde w$, are respectively mapped by the point transformations

$$\hat z_1=\frac12\tan\tilde z_1,\ \hat z_2=\frac{\sqrt2}{2}\frac{\tilde z_2}{\cos\tilde z_1},\ \hat w=\sqrt{|\cos\tilde z_1|}\mathrm{e}^{-\frac14\tilde z_2^2\tan\tilde z_1}\tilde w,$$
$$\hat z_1=\frac14\mathrm{e}^{2\tilde z_1},\ \hat z_2=\frac{\sqrt2}{2}\mathrm{e}^{\tilde z_1}\tilde z_2,\ \hat w=\mathrm{e}^{-\frac14\tilde z_2^2-\frac12\tilde z_1}\tilde w$$

to the equation of the same form with $\delta=0$ in terms of the variables with hats. As a result, instead of the single solution family (3.31) as in the generic case of β, we construct three families of solutions that are parameterized by arbitrary solutions of the linear (1+1)-dimensional heat equation with the inverse square potential with $\mu=-\frac34$,

- $\beta=1\colon\ u=|x|^{1/4}\vartheta^\mu(t,2\sqrt{|x|})$,
- $\beta=1\colon\ u=|x|^{1/4}\mathrm{e}^{\delta y}\dfrac{\mathrm{e}^{|x|\tan t}}{\sqrt{|\cos t|}}\vartheta^\mu\left(\dfrac12\tan t,\dfrac{\sqrt{2|x|}}{\cos t}\right)$, (3.32)
- $\beta=1\colon\ u=|x|^{1/4}\mathrm{e}^{x+\frac12t+\delta y}\vartheta^\mu\left(\dfrac14\mathrm{e}^{2t},\sqrt{2|x|}\mathrm{e}^t\right)$. (3.33)

The maximal Lie invariance algebras of the modified reduced equations (3.30) with $\epsilon\delta=-1$ and $\epsilon\delta=1$ are respectively spanned by the vector fields

$$\begin{aligned}&2\cos(2\tilde z_1)\partial_{\tilde z_1}-2\sin(2\tilde z_1)\tilde z_2\partial_{\tilde z_2}+(\cos(2z_1)\tilde z_2^2+\sin(2z_1))\tilde w\partial_{\tilde w},\\&2\sin(2\tilde z_1)\partial_{\tilde z_1}+2\cos(2z_1)\tilde z_2\partial_{\tilde z_2}+(\sin(2\tilde z_1)\tilde z_2^2-\cos(2z_1))\tilde w\partial_{\tilde w},\\&\partial_{\tilde z_1},\quad\tilde w\partial_{\tilde w},\quad f(\tilde z_1,\tilde z_2)\partial_{\tilde w}\end{aligned}\tag{3.34}$$

and

$$\begin{aligned}&\mathrm{e}^{-2\tilde z_1}\big(2\partial_{\tilde z_1}-2\tilde z_2\partial_{\tilde z_2}+(1-\tilde z_2^2)\tilde w\partial_{\tilde w}\big),\ \mathrm{e}^{2\tilde z_1}\big(2\partial_{\tilde z_1}+2\tilde z_2\partial_{\tilde z_2}-(1+\tilde z_2^2)\tilde w\partial_{\tilde w}\big),\\&\partial_{\tilde z_1},\quad \tilde w\partial_{\tilde w},\quad f(\tilde z_1,\tilde z_2)\partial_{\tilde w},\end{aligned}\tag{3.35}$$

and the normalizer $\mathrm{N}_{\mathfrak{g}_1}(\mathfrak{s}^{\delta}_{1.3})$ induces the subalgebras $\langle\partial_{\tilde z_1},\tilde w\partial_{\tilde w},f(\tilde z_1,\tilde z_2)\partial_{\tilde w}\rangle$ in each of these algebras. Here the parameter function f runs through the solution set of the respective equation. Therefore, a complete list of independent genuine hidden symmetries of the equation $\mathcal{F}'_1$ related to each of the reductions under consideration is exhausted by the first two vector fields presented in (3.34) and (3.35).

Remark 3.31. Mapping the obtained solutions using the discrete equivalence transformation (3.4) or its counterpart (3.6) for equations (3.5), we obtain the solutions of $\mathcal{F}'_\beta$ invariant with respect to the subalgebra $\mathfrak{s}^{\delta}_{1.4}=\langle\mathcal{P}^t+\delta\mathcal{I}\rangle$,

- $\beta=1$: $u=\vartheta^0(y,x)$,
- $\beta\in\mathbb{R}\setminus\{0,2,3,5\}$: $u=|x|^{(1-\beta)/4}\vartheta^{\mu}\big(y\operatorname{sgn}x,2(3-\beta)^{-1}|x|^{(3-\beta)/2}\operatorname{sgn}x\big)$,
- $\beta=4$: $u=|x|^{3/4}\vartheta^{-3/4}\big(y,2|x|^{-1/2}\big)$,
- $\beta=4$: $u=|x|^{3/4}\mathrm{e}^{\delta t}\dfrac{\mathrm{e}^{|x|^{-1}\tan y}}{\sqrt{|\cos y|}}\vartheta^{-3/4}\left(\dfrac12\tan y,\dfrac{\sqrt{2|x|^{-1}}}{\cos y}\right)$, (3.36)
- $\beta=4$: $u=|x|^{3/4}\mathrm{e}^{x^{-1}+\frac12 y+\delta t}\vartheta^{-3/4}\left(\dfrac14\mathrm{e}^{2y},\sqrt{2|x|^{-1}}\mathrm{e}^{y}\right)$. (3.37)

3.8.4 Lie reductions of codimensions two and three

The Lie reductions with respect to the subalgebras $\mathfrak{s}^{\delta}_{2.1}$, $\mathfrak{s}^{\varepsilon,0}_{2.2}$, $\mathfrak{s}^{\kappa}_{2.3}$, $\mathfrak{s}^{\kappa}_{2.4}$ and $\mathfrak{s}^{\kappa}_{3.1}$ do not give new solutions in comparison with the two-step Lie reductions, where the first step is associated with the subalgebra $\mathfrak{s}^{0}_{1.3}$ and leads to (1+1)-dimensional linear heat equations with zero or inverse square potentials as reduced equations, and such equations are well studied within the framework of symmetry analysis of differential equations.

This is why the only essential codimension-two Lie reductions of the equations from the class $\mathcal{F}'$ with $\beta\notin\{0,2,3,5\}$ are those associated with the subalgebras $\mathfrak{s}^{\varepsilon,\kappa}_{2.2}:=\langle\mathcal{P}^y+\varepsilon\mathcal{I},\mathcal{P}^t+\kappa\mathcal{I}\rangle$, where $\varepsilon\in\{-1,1\}$ and $\kappa\neq0$. A corresponding ansatz is

$u = e^{\varepsilon y+\kappa t}\varphi(\omega)$ with $\omega = x$, and the reduced equation for φ is

$$|\omega|^{\beta}\varphi_{\omega\omega} = (\varepsilon\omega + \kappa)\varphi.$$

In some particular cases of parameters, we can construct closed-form solutions of this equation.

For example, in the cases $\beta = 1$ and $\beta = 4$, the counterparts of this equation for the corresponding equations of the form (3.5) coincide with the equations [60, Chapter C, Section 2.273, Eq. (12)] with $a = -\kappa$, $b^2 = \varepsilon$, $c = 1$ and $a = -\varepsilon$, $b^2 = \kappa$, $c = -1$, respectively. At the same time, in view of results of Section 3.8.3 on $\mathfrak{s}_{1.3}^{\delta}$-invariant solutions with $\delta \neq 0$, the consideration of the Lie reduction with respect to the subalgebras $\mathfrak{s}_{2.2}^{\varepsilon,\kappa}$ makes sense only for $\beta \neq 1, 4$.

For the case $\beta = -1$, the change of variable $\tilde{\omega} = \omega + \frac{1}{2}\varepsilon\kappa$ results in the equation [60, Chapter C, Section 2.273, Eq. (11)] with $a = 0$, $b = -\frac{1}{4}\varepsilon\kappa^2$ and $c^2 = -\varepsilon$, whose general solution is

$$\varphi = |\tilde{\omega}|^{-1/2}\operatorname{Re}\Big((C_1 + \mathrm{i}C_2)W_{\frac{\mathrm{i}\kappa^2}{16},\frac{1}{4}}(\mathrm{i}\tilde{\omega})\Big) \quad \text{if} \quad \varepsilon = 1,$$
$$\varphi = |\tilde{\omega}|^{-1/2}\Big(C_1 M_{\frac{\kappa^2}{16},\frac{1}{4}}(\tilde{\omega}) + C_2 W_{\frac{\kappa^2}{16},\frac{1}{4}}(\tilde{\omega})\Big) \quad \text{if} \quad \varepsilon = -1.$$

Here $M_{a,b}(z)$ and $W_{a,b}(z)$ are the Whittaker functions, which constitute the fundamental solution set of the Whittaker equation

$$\varphi_{zz} + \left(-\frac{1}{4} + \frac{a}{z} + \frac{1/4 - b^2}{z^2}\right)\varphi = 0$$

with constant parameters a and b. The corresponding solutions of the equation $\mathcal{F}'_{-1}$ are

- $\beta = -1\colon\ u = \left|x + \dfrac{\kappa}{2}\right|^{-1/2}\mathrm{e}^{y+\kappa t}\operatorname{Re}\Big((C_1 + \mathrm{i}C_2)W_{\frac{\mathrm{i}\kappa^2}{16},\frac{1}{4}}\big(\mathrm{i}(x + \frac{1}{2}\kappa)\big)\Big),$ (3.38)
- $\beta = -1\colon\ u = \left|x - \dfrac{\kappa}{2}\right|^{-1/2}\mathrm{e}^{\kappa t-y}\Big(C_1 M_{\frac{\kappa^2}{16},\frac{1}{4}}\big(x - \frac{1}{2}\kappa\big) + C_2 W_{\frac{\kappa^2}{16},\frac{1}{4}}\big(x - \frac{1}{2}\kappa\big)\Big).$ (3.39)

Using the transformation (3.6), we can map these solutions to solution of the

equation $\mathcal{F}'_6$,

- $\beta=6\colon\ u=x\Big|x^{-1}+\dfrac{\kappa}{2}\Big|^{-1/2}\mathrm{e}^{t+\kappa y}\,\mathrm{Re}\Big((C_1+\mathrm{i}C_2)W_{\frac{\mathrm{i}\kappa^2}{16},\frac14}\big(\mathrm{i}(x^{-1}+\frac12\kappa)\big)\Big)$,
- $\beta=6\colon\ u=x\Big|x^{-1}-\dfrac{\kappa}{2}\Big|^{-1/2}\mathrm{e}^{\kappa y-t}\Big(C_1M_{\frac{\kappa^2}{16},\frac14}\big(x^{-1}-\frac12\kappa\big)+C_2W_{\frac{\kappa^2}{16},\frac14}\big(x^{-1}-\frac12\kappa\big)\Big)$.

3.8.5 Generating solutions

The simplest way to generate solutions for an equation $\mathcal{F}'_\beta$ with $\beta\in\mathbb{R}\setminus\{0,2,3,5\}$ is to use its point symmetry pseudogroup from Theorem 3.26. More specifically, given a solution $u=h(t,x,y)$ of $\mathcal{F}'_\beta$, the action of a transformation of the form (3.27) leads to the solution

$$u=\sigma h\big(|\alpha|^{\beta-2}t-\lambda_0,\alpha^{-1}x,\alpha^{-1}|\alpha|^{\beta-2}y-\lambda_1\big)+f(t,x,y),$$

of the same equation. In the case $\beta=5/2$, we can in addition use a transformation of the form (3.28) and derive the solution

$$u=\sigma h\big(\operatorname{sgn}(x)|\alpha|^{\beta-2}y-\lambda_0,\alpha x^{-1},\operatorname{sgn}(x)\alpha^{-1}|\alpha|^{\beta-2}t-\lambda_1\big)+f(t,x,y).$$

Since each equation $\mathcal{F}'_\beta$ is linear, we can also apply its generalized symmetries to the solution generation following [75]. The differential operators in total derivatives that are associated with the Lie-symmetry vector fields $-\mathcal{P}^t$, $-\mathcal{P}^y$, $-\mathcal{D}^\beta$ and $\mathcal{I}$ are

$$\mathrm{P}^t:=\mathrm{D}_t,\quad \mathrm{P}^y:=\mathrm{D}_y,\quad \mathrm{D}^\beta:=(2-\beta)t\mathrm{D}_t+x\mathrm{D}_x+(3-\beta)y\mathrm{D}_y$$

and the identity operator, respectively. These differential operators generate the associative algebra $\Upsilon_{\mathfrak{s}}$ associated with the subalgebra $\mathfrak{s}=\langle\mathcal{P}^t,\mathcal{P}^y,\mathcal{D}^\beta\rangle\simeq A^a_{3.4}$ of the algebra $\mathfrak{g}_\beta$, where $a:=(\beta-2)/(\beta-3)$. The operators P^t and D^β on the solutions of $\mathcal{F}'_\beta$ respectively coincide with the operators

$$\hat{\mathrm{P}}^t:=-x\mathrm{D}_y+|x|^\beta\mathrm{D}_x^2\quad\text{and}\quad \hat{\mathrm{D}}^\beta:=(2-\beta)t\hat{\mathrm{P}}^t+x\mathrm{D}_x+(3-\beta)y\mathrm{D}_y.$$

Consider the associative algebra $\bar{\Upsilon}_{\mathfrak{s}}$ generated by the operators $\hat{\mathrm{P}}^t$, P^y and $\hat{\mathrm{D}}^\beta$. In

other words, the algebra $\bar{\Upsilon}_{\mathfrak{s}}$ is subjected to the presentation

$$\begin{aligned}\bar{\Upsilon}_{\mathfrak{s}} =\big\langle \hat{\mathrm{P}}^t, \mathrm{P}^y, \hat{\mathrm{D}}^\beta \mid \\ [\hat{\mathrm{P}}^t, \hat{\mathrm{D}}^\beta] = (2-\beta)\hat{\mathrm{P}}^t,\ [\mathrm{P}^y, \hat{\mathrm{D}}^\beta] = (3-\beta)\mathrm{P}^y,\ [\hat{\mathrm{P}}^t, \mathrm{P}^y] = 0\big\rangle.\end{aligned} \tag{3.40}$$

Lemma 3.32. *The monomials* $\mathbf{Q}^\alpha := (\mathrm{Q}^1)^{\alpha_1}(\mathrm{Q}^2)^{\alpha_2}(\mathrm{Q}^3)^{\alpha_3}$, *where* $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$ *and* $(\mathrm{Q}^1, \mathrm{Q}^2, \mathrm{Q}^3)$ *is any fixed ordering of* $\hat{\mathrm{P}}^t$, P^y *and* $\hat{\mathrm{D}}^\beta$, *constitute a basis of the algebra* $\bar{\Upsilon}_{\mathfrak{s}}$.

Proof. We again make use of Bergman's diamond lemma [11]. Without loss of generality, we fix the ordering

$$\hat{\mathrm{D}}^\beta < \mathrm{P}^y < \hat{\mathrm{P}}^t$$

and extend it to the degree lexicographic order on words in the alphabet $(\hat{\mathrm{D}}^\beta, \mathrm{P}^y, \hat{\mathrm{P}}^t)$. According to the chosen ordering, the relations in the presentation (3.40) of the algebra $\bar{\Upsilon}_{\mathfrak{s}}$ can be written as the following reduction system:

$$\hat{\mathrm{P}}^t\hat{\mathrm{D}}^\beta = \hat{\mathrm{D}}^\beta\hat{\mathrm{P}}^t + (2-\beta)\hat{\mathrm{P}}^t, \quad \mathrm{P}^y\hat{\mathrm{D}}^\beta = \hat{\mathrm{D}}^\beta\mathrm{P}^y + (3-\beta)\mathrm{P}^y, \quad \hat{\mathrm{P}}^t\mathrm{P}^y = \mathrm{P}^y\hat{\mathrm{P}}^t.$$

It has exactly one overlap ambiguity, $\hat{\mathrm{P}}^t\mathrm{P}^y\hat{\mathrm{D}}^\beta$, which is resolvable since

$$\begin{aligned}\hat{\mathrm{P}}^t\mathrm{P}^y\hat{\mathrm{D}}^\beta &= \mathrm{P}^y\hat{\mathrm{P}}^t\hat{\mathrm{D}}^\beta = \mathrm{P}^y\hat{\mathrm{D}}^\beta\hat{\mathrm{P}}^t + (2-\beta)\mathrm{P}^y\hat{\mathrm{P}}^t \\ &= \hat{\mathrm{D}}^\beta\mathrm{P}^y\hat{\mathrm{P}}^t + (3-\beta)\mathrm{P}^y\hat{\mathrm{P}}^t + (2-\beta)\mathrm{P}^y\hat{\mathrm{P}}^t, \\ \hat{\mathrm{P}}^t\mathrm{P}^y\hat{\mathrm{D}}^\beta &= \hat{\mathrm{P}}^t\hat{\mathrm{D}}^\beta\mathrm{P}^y + (3-\beta)\hat{\mathrm{P}}^t\mathrm{P}^y = \hat{\mathrm{D}}^\beta\hat{\mathrm{P}}^t\mathrm{P}^y + (2-\beta)\hat{\mathrm{P}}^t\mathrm{P}^y + (3-\beta)\hat{\mathrm{P}}^t\mathrm{P}^y \\ &= \hat{\mathrm{D}}^\beta\mathrm{P}^y\hat{\mathrm{P}}^t + (2-\beta)\mathrm{P}^y\hat{\mathrm{P}}^t + (3-\beta)\mathrm{P}^y\hat{\mathrm{P}}^t.\end{aligned}$$

Thus, the diamond lemma implies that the monomials of the form $(\hat{\mathrm{P}}^t)^{\alpha_1}(\mathrm{P}^y)^{\alpha_2}(\hat{\mathrm{D}}^\beta)^{\alpha_3}$ constitute an $\mathbb{R}$-basis of the algebra $\bar{\Upsilon}_{\mathfrak{s}}$. Similar arguments work for all the other orderings of $\hat{\mathrm{P}}^t$, P^y and $\hat{\mathrm{D}}^\beta$. □

Corollary 3.33. *The algebra* $\bar{\Upsilon}_{\mathfrak{s}}$ *is isomorphic to the universal enveloping algebra* $\mathfrak{U}(A_{3.4}^a)$ *of the Lie algebra* $A_{3.4}^a = \langle e_1, e_2, e_3\rangle$, $[e_1, e_3] = e_1$, $[e_2, e_3] = ae_2$, *where* $a = (2-\beta)/(3-\beta)$.

Proof. The correspondence $-\mathcal{P}^t \mapsto \hat{\mathrm{P}}^t$, $-\mathcal{P}^y \mapsto \mathrm{P}^y$ and $-\mathcal{D}^\beta \mapsto \hat{\mathrm{D}}^\beta$ by linearity

extends to the Lie algebra homomorphism φ from $\mathfrak{s}=\langle \mathcal{P}^t,\mathcal{P}^y,\mathcal{D}^\beta\rangle$ to the Lie algebra $\bar{\Upsilon}_{\mathfrak{s}}^{(-)}$ associated with the associative algebra $\bar{\Upsilon}_{\mathfrak{s}}$, $\varphi\colon \mathfrak{s}\to\bar{\Upsilon}_{\mathfrak{s}}^{(-)}$. By the universal property of the universal enveloping algebra $\mathfrak{U}(\mathfrak{s})$, the Lie algebra homomorphism φ extends to the (unital) associative algebra homomorphism $\hat{\varphi}\colon \mathfrak{U}(\mathfrak{s})\to\bar{\Upsilon}_{\mathfrak{s}}$, i.e., $\varphi=\hat{\varphi}\circ\iota$ as homomorphisms of vector spaces, where $\iota\colon \mathfrak{s}\to\mathfrak{U}(\mathfrak{s})$ is the canonical embedding of $\mathfrak{s}$ in $\mathfrak{U}(\mathfrak{s})$. Since the algebra $\Upsilon_{\mathfrak{s}}$ is generated by $\varphi(\mathfrak{s})$, the homomorphism $\hat{\varphi}$ is surjective. Fixed an ordering of the basis elements of $\mathfrak{s}$, in view of the Poincaré–Birkhoff–Witt theorem, the homomorphism $\hat{\varphi}$ maps the corresponding Poincaré–Birkhoff–Witt basis of $\mathfrak{U}(\mathfrak{s})$ to a basis of $\bar{\Upsilon}_{\mathfrak{s}}$. Therefore, $\hat{\varphi}$ is an isomorphism. The isomorphism $\mathfrak{s}\simeq A^a_{3.4}$ implies the isomorphism $\mathfrak{U}(\mathfrak{s})\simeq\mathfrak{U}(A^a_{3.4})$. □

Corollary 3.34. *The algebra $\bar{\Upsilon}_{\mathfrak{s}}$ is isomorphic to the algebra $\big\langle(\mathbf{Q}^\alpha u)\partial_u \mid \alpha\in\mathbb{N}_0^3\big\rangle$ of canonical representatives for a subalgebra of the quotient algebra of generalized symmetries of an equation $\mathcal{F}'_\beta$ with $\beta\in\mathbb{R}\setminus\{0,2,3,5\}$ with respect to the equivalence of generalized symmetries.*

Similarly to what has been indicated in Sections 3.6 and 3.7, we use elements of the algebra $\bar{\Upsilon}_{\mathfrak{s}}$ to generate solutions of the equation $\mathcal{F}'_\beta$. More specifically, acting on a solution $u=h(t,x,y)$ of this equation by an arbitrary element

$$Q=\sum_{(\alpha_1,\alpha_2,\alpha_3)\in\mathbb{N}_0^3} c_{\alpha_1\alpha_2\alpha_3}(\mathrm{Q}^1)^{\alpha_1}(\mathrm{Q}^2)^{\alpha_2}(\mathrm{Q}^3)^{\alpha_3}$$

of the algebra $\bar{\Upsilon}_{\mathfrak{s}}$, gives the solution Qh of the same equation. Here all but finitely many (real) constants $c_{\alpha_1\alpha_2\alpha_3}$ are equal to zero, and any ordering in $\{\hat{\mathrm{D}}^\beta,\mathrm{P}^y,\hat{\mathrm{P}}^t\}$ can be fixed. In view of Corollary 3.34, the latter generation is consistent with the generation of solutions using generalized symmetries as indicated in [75, Section 6], however, since the algebra of generalized symmetries of $\mathcal{F}'_\beta$ is yet not described, we use its known subalgebra.

Nevertheless, depending on h and $c_{\alpha_1\alpha_2\alpha_3}$, this formula can result in a known or even the zero solution. We analyze which solutions generated according to this from Lie invariant solutions of $\mathcal{F}'_\beta$ may be of interest. It suffices to consider only the solutions that are invariant with respect to the subalgebras $\mathfrak{s}^\kappa_{1.1}=\langle\mathcal{D}^\beta+\kappa\mathcal{I}\rangle$, $\mathfrak{s}^{\varepsilon,\kappa}_{1.2}=\langle\mathcal{P}^y+\varepsilon\mathcal{P}^t+\kappa\mathcal{I}\rangle$, $\mathfrak{s}^\delta_{1.3}=\langle\mathcal{P}^y+\delta\mathcal{I}\rangle$, $\mathfrak{s}^{\varepsilon,\kappa}_{2.2}=\langle\mathcal{P}^y+\varepsilon\mathcal{I},\mathcal{P}^t+\kappa\mathcal{I}\rangle$, with $\varepsilon=\pm1$, $\delta\in\{-1,0,1\}$, $\kappa\in\mathbb{R}$ and, for the last subalgebra family, $\kappa\neq0$, see Section 3.8.2. These solutions were discussed in Sections 3.8.3 and 3.8.4. In view of the structure

of the algebra $\mathfrak{g}_\beta$ with $\beta \in \mathbb{R} \setminus \{0,2,3,5\}$, the analysis reduces to that for single monomials $(\hat{\mathrm{D}}^\beta)^{\alpha_1}(\mathrm{P}^y)^{\alpha_2}(\hat{\mathrm{P}}^t)^{\alpha_3}$.

1.1. The vector field $\mathcal{D}^\beta + \kappa\mathcal{I}$ corresponds to the differential operator $\hat{\mathrm{D}}^\beta - \kappa$. Since $\hat{\mathrm{P}}^t\hat{\mathrm{D}}^\beta = (\hat{\mathrm{D}}^\beta + 2 - \beta)\hat{\mathrm{P}}^t$ and $\mathrm{P}^y\hat{\mathrm{D}}^\beta = (\hat{\mathrm{D}}^\beta + 3 - \beta)\mathrm{P}^y$, we have

$$\begin{aligned}&(\hat{\mathrm{D}}^\beta)^{\alpha_3}(\hat{\mathrm{P}}^t)^{\alpha_1}(\mathrm{P}^y)^{\alpha_2}(\hat{\mathrm{D}}^\beta - \kappa)\\ &\quad = (\hat{\mathrm{D}}^\beta + \alpha_1(2-\beta) + \alpha_2(3-\beta) - \kappa)(\hat{\mathrm{D}}^\beta)^{\alpha_3}(\hat{\mathrm{P}}^t)^{\alpha_1}(\mathrm{P}^y)^{\alpha_2}.\end{aligned}$$

In other words, the monomial $(\hat{\mathrm{P}}^t)^{\alpha_1}(\mathrm{P}^y)^{\alpha_2}(\hat{\mathrm{D}}^\beta)^{\alpha_3}$ maps any $\mathfrak{s}^\kappa_{1.1}$-invariant solution to $\mathfrak{s}^{\tilde\kappa}_{1.1}$-invariant solution with $\tilde\kappa = \kappa - \alpha_1(2-\beta) - \alpha_2(3-\beta)$. Thus, any element of $\bar\Upsilon_{\mathfrak{s}}$ maps the space spanned by $\mathfrak{s}^\kappa_{1.1}$-invariant solutions with κ running through $\mathbb{R}$ into itself.

1.2. It is clear that any polynomial in $\hat{\mathrm{P}}^t$ and P^y commutes with $Q := \mathrm{P}^y + \varepsilon\hat{\mathrm{P}}^t - \kappa$. Therefore, its action maps the set of $\mathfrak{s}^\kappa_{1.2}$-invariant solutions into itself. At the same time, the operator $\hat{\mathrm{D}}^\beta$ does not commute with Q, $Q\hat{\mathrm{D}}^\beta = (\hat{\mathrm{D}}^\beta + 2 - \beta)Q + \mathrm{P}^y + (2-\beta)\kappa$. Thus, for an $\mathfrak{s}^{\varepsilon,\kappa}_{1.2}$-invariant solution h, the solution $\hat{\mathrm{D}}^\beta h$ is $\mathfrak{s}^{\varepsilon,\kappa}_{1.2}$-invariant if and only if the solution h is in addition $\langle \mathcal{P}^y + (\beta-2)\kappa\mathcal{I}\rangle$-invariant. This is why, to generate new solutions of $\mathcal{F}'_\beta$ from $\mathfrak{s}^{\varepsilon,\kappa}_{1.2}$- invariant solutions within the framework discussed, up to linearly combining solutions, it suffices to iteratively act by the operator $\hat{\mathrm{D}}^\beta$ on other $\mathfrak{s}^{\varepsilon,\kappa}_{1.2}$-invariant solutions if they are known.

1.3. It is clear that acting by a polynomial in P^y or $\hat{\mathrm{P}}^t$ on $\mathfrak{s}^\delta_{1.3}$-invariant solutions does not result in new solutions since it commutes with the operator $\mathrm{P}^y - \delta$. To describe the action of powers of $\hat{\mathrm{D}}^\beta$, we split the consideration into two cases.

Assume that $\delta = 0$. Despite the fact that P^y and $\hat{\mathrm{D}}^\beta$ do not commute, the action by $\hat{\mathrm{D}}^\beta$ does not give us new solutions since $\mathrm{P}^y\hat{\mathrm{D}}^\beta = (\hat{\mathrm{D}}^\beta + 3 - \beta)\mathrm{P}^y$. Therefore, any element of $\bar\Upsilon_{\mathfrak{s}}$ maps the space of $\mathfrak{s}^0_{1.3}$-invariant solutions of $\mathcal{F}'_\beta$ into itself.

In the case $\delta \neq 0$, the equality $(\mathrm{P}^y + \delta)\hat{\mathrm{D}}^\beta = (\hat{\mathrm{D}}^\beta + 3 - \beta)(\mathrm{P}^y + \delta) - \delta(3-\beta)$ implies that for any $\mathfrak{s}^\delta_{1.3}$-invariant solution h, the solution $\hat{\mathrm{D}}^\beta h$ is not $\mathfrak{s}^\delta_{1.3}$-invariant. This is why to generate new solutions of $\mathcal{F}'_\beta$ from $\mathfrak{s}^\delta_{1.3}$-invariant solutions within the framework discussed, up to linearly combining solutions, it again suffices to iteratively act by the operator $\hat{\mathrm{D}}^\beta$. Since for the specific case of $\beta = 1$, we constructed the families (3.32) and (3.33) of $\mathfrak{s}^\delta_{1.3}$-invariant solutions with $\delta \neq 0$, we can extend them to more new

wide families of solutions of the equation $\mathcal{F}'_\beta$,

- $\beta=1\colon\ u=(\hat{\mathrm{D}}^1)^k\left(|x|^{1/4}\mathrm{e}^{\delta y}\dfrac{\mathrm{e}^{|x|\tan t}}{\sqrt{|\cos t|}}\vartheta^{-3/4}\left(\dfrac{\tan t}{2},\dfrac{\sqrt{2|x|}}{\cos t}\right)\right)$,
- $\beta=1\colon\ u=(\hat{\mathrm{D}}^1)^k\left(|x|^{1/4}\mathrm{e}^{x+\frac12 t+\delta y}\vartheta^{-3/4}\left(\dfrac14\mathrm{e}^{2t},\sqrt{2|x|}\mathrm{e}^{t}\right)\right)$,

where $\hat{\mathrm{D}}^1:=x(t\mathrm{D}_x+1)\mathrm{D}_x+(2y-tx)\mathrm{D}_y$. These solution families can be mapped by the transformation (3.6) to new solution families of the equation $\mathcal{F}'_4$, which can be rearranged up to linearly combining. An equivalent but simpler way is to act on solutions of the form (3.36) and (3.37) by the operator $\hat{\mathrm{D}}^4:=-2tx^4\mathrm{D}_x^2+x\mathrm{D}_x+(2tx-y)\mathrm{D}_y$, which is associated with the Lie-symmetry vector field $-\mathcal{D}^4$ of $\mathcal{F}'_4$ and thus is the counterpart of $\hat{\mathrm{D}}^1$,

- $\beta=4\colon\ u=(\hat{\mathrm{D}}^4)^k\left(|x|^{3/4}\mathrm{e}^{\delta t}\dfrac{\mathrm{e}^{|x|^{-1}\tan y}}{\sqrt{|\cos y|}}\vartheta^{-3/4}\left(\dfrac{\tan y}{2},\dfrac{\sqrt{2|x|^{-1}}}{\cos y}\right)\right)$,
- $\beta=4\colon\ u=(\hat{\mathrm{D}}^4)^k\left(|x|^{3/4}\mathrm{e}^{x^{-1}+\frac12 y+\delta t}\vartheta^{-3/4}\left(\dfrac14\mathrm{e}^{2y},\sqrt{2|x|^{-1}}\mathrm{e}^{y}\right)\right)$,

Note that $\mathcal{J}'_*\hat{\mathrm{D}}^1=-\hat{\mathrm{D}}^4+1$.

2.2. Similarly to case 1.3, the operators $\mathrm{P}^y-\varepsilon$ and $\hat{\mathrm{P}}^t-\kappa$ commute with the operators $\hat{\mathrm{P}}^t$ and P^y and permute with the operator $\hat{\mathrm{D}}^\beta$ in the following way:

$$(\mathrm{P}^y-\varepsilon)\hat{\mathrm{D}}^\beta=(\hat{\mathrm{D}}^\beta+3-\beta)(\mathrm{P}^y-\varepsilon)+(3-\beta)\varepsilon,$$
$$(\hat{\mathrm{P}}^t-\kappa)\hat{\mathrm{D}}^\beta=(\hat{\mathrm{D}}^\beta+2-\beta)(\hat{\mathrm{P}}^t-\kappa)+(2-\beta)\kappa,$$

therefore, for any $\mathfrak{s}_{2.2}^{\varepsilon,\kappa}$-invariant solution h, the solution $\hat{\mathrm{D}}^\beta h$ is not $\mathfrak{s}_{2.2}^{\varepsilon,\kappa}$-invariant. Therefore, to construct essentially new solutions of $\mathcal{F}'_\beta$, it suffices, up to linearly combining solutions, to act by the powers of the operator $\hat{\mathrm{D}}^\beta$.

In Section 3.8.4, we have constructed the solution families (3.38) and (3.39) for the specific case $\beta=-1$. Analogously to generating solutions in case 1.3 with $\delta\neq0$, we extend these solutions to more new solution families of the equation $\mathcal{F}'_{-1}$ and contract their counterparts for the equation $\mathcal{F}'_6$,

- $\beta=-1\colon\ u=(\hat{\mathrm{D}}^{-1})^k\,\mathrm{Re}\left(\left|x+\dfrac{\kappa}{2}\right|^{-1/2}\mathrm{e}^{y+\kappa t}(C_1+\mathrm{i}C_2)W_{\frac{\mathrm{i}\kappa^2}{16},\frac14}\left(\mathrm{i}\left(x+\tfrac{\kappa}{2}\right)\right)\right)$,

- $\beta=-1\colon\ u=(\hat{\mathrm{D}}^{-1})^k\left(\left|x-\frac{\kappa}{2}\right|^{-1/2}\mathrm{e}^{\kappa t-y}\left(C_1M_{\frac{\kappa^2}{16},\frac{1}{4}}\left(x-\frac{\kappa}{2}\right)+C_2W_{\frac{\kappa^2}{16},\frac{1}{4}}\left(x-\frac{\kappa}{2}\right)\right)\right)$,
- $\beta=6\colon\ u=(\hat{\mathrm{D}}^{6})^k\,\mathrm{Re}\left(x\left|x^{-1}+\frac{\kappa}{2}\right|^{-1/2}\mathrm{e}^{t+\kappa y}(C_1+\mathrm{i}C_2)W_{\frac{\mathrm{i}\kappa^2}{16},\frac{1}{4}}\left(\mathrm{i}\left(x^{-1}+\frac{\kappa}{2}\right)\right)\right)$,
- $\beta=6\colon\ u=(\hat{\mathrm{D}}^{6})^k\left(x\left|x^{-1}-\frac{\kappa}{2}\right|^{-1/2}\mathrm{e}^{\kappa y-t}\left(C_1M_{\frac{\kappa^2}{16},\frac{1}{4}}\left(x^{-1}-\frac{\kappa}{2}\right)+C_2W_{\frac{\kappa^2}{16},\frac{1}{4}}\left(x^{-1}-\frac{\kappa}{2}\right)\right)\right)$,

where $\hat{\mathrm{D}}^{-1}:=3tx^{-1}\mathrm{D}_x^2+x\mathrm{D}_x+(4y-3tx)\mathrm{D}_y$ and $\hat{\mathrm{D}}^{6}:=-4tx^{6}\mathrm{D}_x^2+x\mathrm{D}_x+(4tx-3y)\mathrm{D}_y$.

3.9 Conclusion

The class of ultraparabolic (1+2)-dimensional Kolmogorov backward equations with power diffusivity $\mathcal{F}$ and its properly gauged counterpart $\mathcal{F}'$ have many outstanding properties from the perspective of symmetry analysis of differential equations. As expected, the generic elements of the class $\mathcal{F}'$ are related to one of the inequivalent cases of Lie symmetry extension within the superclass $\bar{\mathcal{F}}$ of the ultraparabolic linear (1+2)-dimensional second-order partial differential equations, which is a subcase of a case with at least four-dimensional decomposable solvable essential Lie invariance algebras. Moreover, specific exponents 0, 2, 3 and 5 of power diffusivity correspond to additional Lie symmetry extensions.

At the same time, the number of surprising properties of the classes $\mathcal{F}$ and $\mathcal{F}'$ is essentially greater than those we expected.

The class $\mathcal{F}'$ admits the nontrivial discrete equivalence transformation (3.4), which is the only essential among elements of the equivalence group $G^{\sim}_{\mathcal{F}'}$ in the course of group classification of this class. Under passing to the class $\mathcal{F}$, this transformation induces elements from the complement of the action groupoid of the group $G^{\sim}_{\mathcal{F}}$ in the groupoid $\mathcal{G}^{\sim}_{\mathcal{F}}$, and thus it is associated with no element of $G^{\sim}_{\mathcal{F}}$. This shows the naturality and, moreover, the necessity of the gauge $\alpha=0$.

There are only two $G^{\sim}_{\mathcal{F}'}$-inequivalent cases of essential Lie symmetry extensions within the class $\mathcal{F}'$. They are associated with the remarkable Fokker–Planck equation $\mathcal{F}'_0$ and the fine Kolmogorov backward equation $\mathcal{F}'_2$. Furthermore, these are equations that are singular with respect to their symmetry properties in the entire

superclass $\bar{\mathcal{F}}$, representing the cases with maximum eight-dimensional and nonsolvable five-dimensional essential Lie symmetry extensions, respectively. As a result, the study of $\mathcal{F}'_0$ and $\mathcal{F}'_2$ in [70, 75, 120] provides a significant part of symmetry analysis of elements of not only the classes $\mathcal{F}$ and $\mathcal{F}'$ but also their superclass $\bar{\mathcal{F}}$.

The equations $\mathcal{F}'_0$ and $\mathcal{F}'_2$ are also distinguished by other symmetry properties. They admit a number of nontrivial hidden Lie symmetries that are associated with their codimension-one Lie reductions to the (1+1)-dimensional linear heat equations with inverse square or zero potentials. Moreover, all the closed-form Lie invariant solutions that can be constructed for $\mathcal{F}'_0$ and $\mathcal{F}'_2$ are related to such reductions. These solutions constitute wide families and, furthermore, can be used as seeds for generating new solutions via acting by recursion operators related to Lie symmetries, and such generation indeed efficiently works for $\mathcal{F}'_0$ and $\mathcal{F}'_2$.

In a practical sense, these exceptional values correspond to more applied models. For instance, the case $\beta = 0$ is used in modeling cell populations, whereas $\beta = 2$ arises in modeling stock options (see discussions in Section 3.1). In contrast, the physical scope and applicability of the equation for generic values of β remain less clear. We emphasize that the physical relevance of the cases $\beta = 0, 2$ follows only from the cited modeling contexts, not from the larger symmetry algebra itself. Although symmetry extensions make these cases mathematically distinguished, facilitating the construction of exact solutions via Lie reductions, the symmetry extension alone does not establish physical realism.

Even more surprising is that, among the other equations in the class $\mathcal{F}'$ up to the $G^{\sim}_{\mathcal{F}'}$-equivalence, only the equation $\mathcal{F}'_1$ possesses all the above properties of $\mathcal{F}'_0$ and $\mathcal{F}'_2$, except the extension of essential Lie invariance algebras. This is why the singularity of $\mathcal{F}'_1$ within the class $\mathcal{F}'$ can be found out only after the comprehensive study of Lie reductions of the equations from this class.

Another but less singular equation is $\mathcal{F}'_{-1}$. Its singularity becomes apparent only in the course of carrying out its codimension-two Lie reductions to ordinary differential equations that integrate, in contrast to their analogues for regular elements of the class $\mathcal{F}'$, in terms of Whittaker functions. The solutions of $\mathcal{F}'_{-1}$ constructed in this way can also be used as seeds for generating new solutions via acting by a recursion operator arising from a Lie symmetry.

We have found hidden Lie symmetries for the generic equations from the class $\mathcal{F}'$,

but their number is essentially less than for the singular equations from this class with $\beta \in \{0, 1, 2, 3, 4, 5\}$. All of these symmetries are associated with codimension-one Lie reductions.

In total, we have singled out eight values of the arbitrary element β, including $G^{\sim}_{\mathcal{F}'}$-equivalent counterparts, -1, 0, 1, 2, 3, 4, 5 and 6. An open question is whether there exist other properties that distinguish the listed values. It is also not clear whether the study of more complicated structures like generalized symmetries or reduction modules can allow one to reveal the singularity of some values of β that are regular from the point of view of Lie and hidden Lie symmetries, the construction of families of Lie-invariant solutions and their extension by recursion operators.

The results of the present paper have several novel features.

We not only have completely solved the properly posed group classification problems for the classes $\mathcal{F}'$ and $\mathcal{F}$ with respect to equivalences of two kinds but have also computed the point symmetry pseudogroups of all equations from these classes. The point symmetry pseudogroups of the equations $\mathcal{F}'_\beta$ with $\beta = 0$, $\beta = 5$, $\beta = 2$, $\beta = 3$ and $\beta \in \mathbb{R} \setminus \{0, 2, 3, 5\}$ are given in Theorems 3.18, 3.21, 3.23, 3.25 and 3.26, respectively, which provides the exhaustive descriptions of the fundamental groupoids $\mathcal{G}^{\rm f}_{\mathcal{F}'}$ and, hence, $\mathcal{G}^{\rm f}_{\mathcal{F}}$. Combining the above result on $\mathcal{G}^{\rm f}_{\mathcal{F}'}$ with Theorem 3.4(iv), we obtain the complete description of the entire equivalence groupoid $\mathcal{G}^{\sim}_{\mathcal{F}'}$.

Theorem 3.35. *The class $\mathcal{F}'$ is semi-normalized. Its equivalence groupoid $\mathcal{G}^{\sim}_{\mathcal{F}'}$ is the Frobenius product of the action groupoid of the subgroup generated by the discrete equivalence transformation $\mathcal{J}'$ and the fundamental groupoid $\mathcal{G}^{\rm f}_{\mathcal{F}'} = \{(\beta, \Phi, \beta) \mid \Phi \in G_\beta,\ \beta \in \mathbb{R}\}$, where G_β is the point-symmetry (pseudo)group of the equation $\mathcal{F}'_\beta$, presented in Theorems 3.18, 3.21, 3.23, 3.25 and 3.26 for $\beta = 0$, $\beta = 5$, $\beta = 2$, $\beta = 3$ and $\beta \in \mathbb{R} \setminus \{0, 2, 3, 5\}$, respectively.*

Corollary 3.5 combined with Theorem 3.35 completes the description of $\mathcal{G}^{\sim}_{\mathcal{F}}$.

Theorem 3.36. *The equivalence groupoid $\mathcal{G}^{\sim}_{\mathcal{F}}$ is constituted by the admissible transformations*

$$\big((\alpha, \beta), \pi_*\mathcal{S}(\tilde\alpha) \circ \Phi \circ \pi_*\mathcal{S}(-\alpha), (\tilde\alpha, \tilde\beta)\big),$$

where $(\beta, \Phi, \tilde\beta) \in \mathcal{G}^{\sim}_{\mathcal{F}'}$.

We have first applied Hydon's automorphism-based version of the algebraic method [48–51] to compute equivalence groups of classes of differential equations, which are the groups $G^{\sim}_{\mathcal{F}}$ and $G^{\sim}_{\mathcal{F}'}$. At the same time, the required automorphism groups of the associated equivalence algebras $\mathfrak{g}^{\sim}_{\mathcal{F}}$ and $\mathfrak{g}^{\sim}_{\mathcal{F}'}$ cannot be found in the straightforward "brute-force" way. This is why we needed first to construct a sufficient number of megaideals of these algebras and then to take into account the derived constraints on entries of the automorphism matrices in the course of the regular computation of the automorphism groups. It was unexpected that the megaideals preliminarily constructed using their basic properties exhaust the entire sets of megaideals of the corresponding algebras.

The groups $G^{\sim}_{\mathcal{F}}$ and $G^{\sim}_{\mathcal{F}'}$ have also been constructed using the original advanced version of the direct method. There are two outputs of the double computation. It provides a cross-check of the correctness of the obtained groups, which is important since knowing them underlies all aspects of symmetry analysis of equations from the classes $\mathcal{F}$ and $\mathcal{F}'$. Due to the double computation, we can also compare the effectiveness of the applied method. We did not expect that the advanced version of the direct method turns out to be more efficient than the algebraic method.

In general, the study of the particular equivalence groupoids and equivalence groups in the present paper gives a justified insight into further development of computational tools for such studies.

To find the point symmetry pseudogroup G_β of a generic equation $\mathcal{F}'_\beta$ in the class $\mathcal{F}'$, we have used an original combination of the above methods, see the proof of Theorem 3.26. We have first applied the advanced version of the direct method to prove the factorization of the pseudogroup G_β and the maximal Lie invariance algebra $\mathfrak{g}_\beta$ of $\mathcal{F}'_\beta$ into their essential parts and those associated with the linear superposition of solutions. The latter parts are known and can be neglected. The fact that the adjoint action of the group $G^{\rm ess}_\beta$ preserves the algebra $\mathfrak{g}^{\rm ess}_\beta$ has allowed us to use techniques of Hydon's algebraic method for obtaining a restrictive form for elements of the group $G^{\rm ess}_\beta$, which is parameterized by six constants. Then we complete the computation of this group by the direct method, deriving two constraints for these parameters.

The classes $\mathcal{F}$ and $\mathcal{F}'$ are constituted by linear differential equations. This is why it was natural and instructive to generate new wide solution families of equations

from these classes by acting with recursion operators associated with their essential Lie symmetries. We have singled out, up to the $G^{\sim}_{\mathcal{F}}$- and $G^{\sim}_{\mathcal{F}'}$-equivalences within the respective classes and up to the G_β-equivalence for each particular value of β, all nontrivial generations among the above ones and then carried out them. This provided the first example in the literature on exhaustive study of such generations for a class of linear differential equations. The first similar examples for single differential linear equations were presented in [75] and [120] for the fine Kolmogorov backward equation $\mathcal{F}'_2$ and the remarkable Fokker–Planck equation $\mathcal{F}'_0$, respectively.

Despite the diverse results obtained for the classes $\mathcal{F}$ and $\mathcal{F}'$ and their particular elements in [70, 75, 120] and in the present paper, there are still many open problems on these objects, not to mention the superclass $\bar{\mathcal{F}}$.

We have comprehensively studied the generation of solutions of equations from $\mathcal{F}'$ by their recursion operators that are related to their Lie symmetries. General linear recursion operators, which are associated with linear genuinely generalized symmetries, are also relevant for such generation but their classification is much more complicated than the classification of recursion operators stemming from Lie symmetries. We proved in [120] that the entire algebra of generalized symmetries of the remarkable Fokker–Planck equation $\mathcal{F}'_0$ is generated with successively acting by its Lie-symmetry operators on the trivial Lie symmetry $u\partial_u$. We also reasonably conjecture the analogous claim for the fine Kolmogorov backward equation $\mathcal{F}'_2$ in [75, Section 6]. At the same time, we have no conjecture on generalized symmetries of equations $\mathcal{F}'_\beta$ with $\beta \in \mathbb{R} \setminus \{0, 2, 3, 5\}$.

In the context of the theory of singular reduction modules [25] and in view of the ultraparabolicity of equations from the classes $\mathcal{F}$ and $\mathcal{F}'$, it will be instructive to study reduction modules of these equations. In comparison with Lie symmetries, the main obstacle for finding non-Lie reduction modules is that it reduces to solving nonlinear determining equations even for linear differential equations, which makes the related computations highly nontrivial and cumbersome. For example, even for the simplest class constituted by linear second-order partial differential equations with two independent variables, an analogous study was carried out only for the subclass of evolution equations [123].

Local conservation laws of equations from the class $\mathcal{F}$ were described in [151], which

can be viewed as a preliminary step for the further consideration of nonlocal structures associated with these equations, e.g., potential conservation laws and potential symmetries. With this objective in mind, one should remember the principal difference between the two- and multi-dimensional cases in this aspect [3, Theorem 2.7]. More specifically, for the existence of nontrivial potential structures in the latter case, one should gauge potentials arising from the conservation laws, and the selection of suitable and natural gauges can still be carried out only in a heuristic way.

All the above problems can be extended to equations from the superclass $\bar{\mathcal{F}}$. This class has been extensively studied for the past couple of decades. Its partial preliminary group classification was carried out in [35]. Some subclasses of $\bar{\mathcal{F}}$ were considered within the Lie-symmetry framework in [45, 76, 138–140, 151]. Despite the number of papers on this subject, there are still many open problems in the symmetry analysis of the entire class $\bar{\mathcal{F}}$, its subclasses and even particular equations from this class, as can be seen from the above discussion on the classes $\mathcal{F}$ and $\mathcal{F}'$ and on the equations $\mathcal{F}'_0$ and $\mathcal{F}'_2$. We are close to completely solve the group classification problem for the class $\bar{\mathcal{F}}$ using the algebraic approach, and this involves almost all recently developed advanced techniques of group classification and the famous description of finite-dimensional Lie algebras of vector fields on the real plane, see [87, pp. 122–178] for the first presentation and [43] for the modern and enhanced version of this result.

Appendix 3.A Algebraic method for finding equivalence groups

The straightforward construction of the equivalence group of a class of differential equations by the direct method in general leads to cumbersome and complicated computations. We simplify computing the equivalence groups $G^{\sim}_{\mathcal{F}}$ and $G^{\sim}_{\mathcal{F}'}$ due to using an advanced version of the direct method and knowing the equivalence groupoid of the superclass $\bar{\mathcal{F}}$, which is described in Theorem 3.2. Nevertheless, it is still instructive to find the groups $G^{\sim}_{\mathcal{F}}$ and $G^{\sim}_{\mathcal{F}'}$ by Hydon's *algebraic method* [48–51] extended to equivalence groups in [13], which in addition leads to the double-check of Theorems 3.6 and 3.10.

The algebraic method is based on knowing the corresponding equivalence algebra,

which should be computed independently using the infinitesimal approach. In the course of this computation, we can assume that the constant arbitrary elements α and β of the class $\mathcal{F}$ (resp. β of the class $\mathcal{F}'$) play the role of additional dependent variables satisfying the equations $\alpha_t = \alpha_x = \alpha_y = 0$ and $\beta_t = \beta_x = \beta_y = 0$ (resp. $\beta_t = \beta_x = \beta_y = 0$). The corresponding infinitesimal invariance condition should be split with respect to both the parametric jet variables and the arbitrary elements α and β (resp. the arbitrary element β).

Since the equivalence algebras $\mathfrak{g}^{\sim}_{\mathcal{F}}$ and $\mathfrak{g}^{\sim}_{\mathcal{F}'}$ are finite-dimensional, we proceed with the original automorphism-based version of the algebraic method by Hydon [48–51]. In fact, this is the first example of such a computation in the literature, see [13, Section 4] for the first application of the megaideal-based version of the same method to finding the equivalence group of a class of differential equations.

3.A.1 The original class

Despite the fact that the dimension of the algebra $\mathfrak{g} := \mathfrak{g}^{\sim}_{\mathcal{F}}$ is not high, see Corollary 3.8, the direct construction of the automorphism group $\mathrm{Aut}(\mathfrak{g})$ using symbolic computation programs seems expensive, if at all possible. To make the construction feasible, we first find a sufficient number of *megaideals* [125] of $\mathfrak{g}$, i.e., linear subspaces of $\mathfrak{g}$ that are stable with respect to the action of $\mathrm{Aut}(\mathfrak{g})$; another name for megaideals is *fully characteristic ideals* [47, Exercise 14.1.1]. The choice of a basis of $\mathfrak{g}$ agreed with the megaideal hierarchy $\mathfrak{g}$ guarantees the presence of an essential number of zero entries in the matrix of a general automorphism of $\mathfrak{g}$. The more megaideals we construct, the simpler computations become.

In view of elementary properties of megaideals, all elements of the derived series, the upper and the lower central series of $\mathfrak{g}$, including the center $\mathfrak{z}(\mathfrak{g})$ and the derivative $\mathfrak{g}'$ of $\mathfrak{g}$, are megaideals of $\mathfrak{g}$. Thus,

$$\mathfrak{m}_1 := \mathfrak{g}'' = \langle \partial_u \rangle,$$
$$\mathfrak{m}_2 := \mathfrak{z}(\mathfrak{g}') = \langle \partial_u, x\partial_u \rangle,$$
$$\mathfrak{m}_4 := \mathfrak{g}^3_{\mathcal{F}} = [\mathfrak{g}, \mathfrak{g}'_{\mathcal{F}}] = \langle \partial_u, x\partial_u, (tx - y)\partial_u \rangle$$

are megaideals of $\mathfrak{g}$. A useful tool for constructing megaideals of Lie algebras is [37, Proposition 1], which states that if $\mathfrak{i}_0$, $\mathfrak{i}_1$ and $\mathfrak{i}_2$ are the megaideals of $\mathfrak{g}$, then the set $\mathfrak{s}$

of elements from $\mathfrak{i}_0$ whose commutators with arbitrary elements from $\mathfrak{i}_1$ belong to $\mathfrak{i}_2$ is also a megaideal of $\mathfrak{g}$. By considering $(\mathfrak{i}_0,\mathfrak{i}_1,\mathfrak{i}_2)=(\mathfrak{g},\mathfrak{g},\mathfrak{m}_1)$ and $(\mathfrak{i}_0,\mathfrak{i}_1,\mathfrak{i}_2)=(\mathfrak{g},\mathfrak{g},\mathfrak{m}_3)$ we respectively obtain the megaideals

$$\mathfrak{m}_3:=\langle\partial_u,\partial_y\rangle,$$
$$\mathfrak{m}_5:=\langle\partial_u,\partial_y,\partial_x+t\partial_y+\partial_\alpha\rangle.$$

The sum of megaideals is a megaideal. Taking $(\mathfrak{i}_0,\mathfrak{i}_1,\mathfrak{i}_2)=(\mathfrak{m},\mathfrak{g},\{0\})$, we derive that the centralizer $\mathrm{C}_{\mathfrak{g}}(\mathfrak{m})$ of a megaideal $\mathfrak{m}$ is a megaideal. By this, we obtain

$$\mathfrak{m}_6:=\mathrm{C}_{\mathfrak{g}}(\mathfrak{m}_2+\mathfrak{m}_3)=\langle\partial_u,x\partial_u,\partial_y,\partial_t\rangle.$$

Applying the technique from [26, Lemma 10], we construct one more megaideal in $\mathfrak{g}$.

Proposition 3.37. *The span* $\mathfrak{m}_7:=\langle\partial_u,x\partial_u,(tx-y)\partial_u,u\partial_u\rangle$ *is a megaideal.*

Proof. Consider the megaideal $\mathfrak{s}=\langle\partial_u,x\partial_u,(tx-y)\partial_u,u\partial_u,\partial_y\rangle$ of $\mathfrak{g}$ obtained using [37, Proposition 1] for $(\mathfrak{i}_0,\mathfrak{i}_1,\mathfrak{i}_2)=(\mathfrak{g},\mathfrak{g},\mathfrak{m}_4)$. The megaideals $\mathfrak{m}_2$, $\mathfrak{m}_4$ and $\mathfrak{m}_3+\mathfrak{m}_4$ of $\mathfrak{g}$ are contained in $\mathfrak{s}$. Fix an arbitrary element Φ of $\mathrm{Aut}(\mathfrak{g})$. Since $\Phi(\mathfrak{m}_2)=\mathfrak{m}_2$, $\Phi(\mathfrak{m}_4)=\mathfrak{m}_4$, $\Phi(\mathfrak{m}_3+\mathfrak{m}_4)=\mathfrak{m}_3+\mathfrak{m}_4$, $\Phi(\mathfrak{s})=\mathfrak{s}$, $(tx-y)\partial_u\in\mathfrak{m}_4\setminus\mathfrak{m}_2$ and $u\partial_u\in\mathfrak{s}\setminus(\mathfrak{m}_3+\mathfrak{m}_4)$, we have that

$$\Phi((tx-y)\partial_u)=\hat{c}_1\partial_u+\hat{c}_2x\partial_u+\hat{c}_3(tx-y)\partial_u,$$
$$\Phi(u\partial_u)=c_1\partial_u+c_2x\partial_u+c_3(tx-y)\partial_u+c_4u\partial_u+c_5\partial_y$$

with $c_4\hat{c}_3\neq0$. Therefore,

$$\Phi([(tx-y)\partial_u,u\partial_u])=\Phi((tx-y)\partial_u)=\hat{c}_1\partial_u+\hat{c}_2x\partial_u+\hat{c}_3(tx-y)\partial_u$$
$$=\big[\Phi((tx-y)\partial_u),\Phi(u\partial_u)\big]=c_4(\hat{c}_1\partial_u+\hat{c}_2x\partial_u+\hat{c}_3(tx-y)\partial_u)+c_5\hat{c}_3\partial_u,$$

which gives $c_4\hat{c}_3=\hat{c}_3$ and $\hat{c}_1=c_4\hat{c}_1+c_5\hat{c}_3$, i.e., that $c_4=1$ and $c_5=0$. Hence $\Phi(u\partial_u)\in\langle u\partial_u\rangle+\mathfrak{m}_2+\mathfrak{m}_3=\mathfrak{m}_7$. Thus, the span $\mathfrak{m}_7$ is stable under $\Phi\in\mathrm{Aut}(\mathfrak{g})$. □

Fixing a basis of a Lie algebra, we identify automorphisms of this algebra with their matrices. In the algebra $\mathfrak{g}$, we fix the basis $(\partial_u,x\partial_u,(tx-y)\partial_u,\partial_y,u\partial_u,\partial_t,\partial_x+t\partial_y+\partial_\alpha)$. In view of the presence of the megaideals $\mathfrak{m}_1,\dots,\mathfrak{m}_7$, any automorphism

$\Phi = (a_{ij})_{i,j=1,\dots,7}$ of $\mathfrak{g}$ satisfies the following constraints on its entries:

$$a_{ij} = 0, \quad 1 \leqslant j < i \leqslant 7,$$
$$a_{24} = a_{34} = a_{45} = a_{36} = a_{56} = a_{27} = a_{37} = a_{57} = a_{67} = 0.$$

We take into account these constraints and proceed with the direct construction of the group $\mathrm{Aut}(\mathfrak{g})$ using a symbolic computation program. As a result, we obtain that the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of $\mathfrak{g}$ consists of the matrices

$$\begin{pmatrix}
a_{33}a_{66}a_{77} & -a_{33}a_{46} & a_{13} & a_{35}a_{66}a_{77} & a_{15} & a_{35}a_{46} & a_{35}a_{47} - a_{25}a_{77} \\
0 & a_{33}a_{66} & a_{33}a_{47}a_{77}^{-1} & 0 & a_{25} & -a_{35}a_{66} & 0 \\
0 & 0 & a_{33} & 0 & a_{35} & 0 & 0 \\
0 & 0 & 0 & a_{66}a_{77} & 0 & a_{46} & a_{47} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_{66} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a_{77}
\end{pmatrix},$$

where the remaining parameters a_{ij} are arbitrary real constants with $a_{33}a_{66}a_{77} \neq 0$. It is clear from the form of automorphisms that a complete list of essential megaideals of the algebra $\mathfrak{g}$, which are not the sums of other megaideals, is exhausted by that constructed above:

$$\mathfrak{m}_1, \quad \mathfrak{m}_2, \quad \mathfrak{m}_3, \quad \mathfrak{m}_4, \quad \mathfrak{m}_5, \quad \mathfrak{m}_6, \quad \mathfrak{m}_7.$$

The remaining nonzero megaideals of the algebra $\mathfrak{g}$ are $\mathfrak{m}_2 + \mathfrak{m}_3$, $\mathfrak{m}_3 + \mathfrak{m}_4$, $\mathfrak{m}_2 + \mathfrak{m}_5$, $\mathfrak{m}_3+\mathfrak{m}_7$, $\mathfrak{m}_4+\mathfrak{m}_6$, $\mathfrak{m}_4+\mathfrak{m}_5$, $\mathfrak{m}_5+\mathfrak{m}_6$, $\mathfrak{m}_5+\mathfrak{m}_7$, $\mathfrak{m}_6+\mathfrak{m}_7$, $\mathfrak{m}_4+\mathfrak{m}_5+\mathfrak{m}_6$, $\mathfrak{m}_5+\mathfrak{m}_6+\mathfrak{m}_7 = \mathfrak{g}$.

Lemma 3.38. *A complete list of independent discrete equivalence transformations of the class $\mathcal{F}$ of Kolmogorov backward equations with power diffusivity (3.1) is exhausted by two involutions, one alternating the sign of the dependent variable and one simultaneously alternating the signs of x, y and α, $(t, x, y, u, \alpha, \beta) \mapsto (t, x, y, -u, \alpha, \beta)$ and $(t, x, y, u, \alpha, \beta) \mapsto (t, -x, -y, u, -\alpha, \beta)$.*

Proof. The inner automorphism group $\mathrm{Inn}(\mathfrak{g})$ is constituted by the matrices of the

form

$$\begin{pmatrix} \mathrm{e}^{\delta_5} & \delta_7\mathrm{e}^{\delta_5} & -\delta_4\mathrm{e}^{\delta_5} & \delta_3 & \delta_1 & -\delta_3\delta_7 & \delta_3\delta_6-\delta_2 \\ 0 & \mathrm{e}^{\delta_5} & \delta_6\mathrm{e}^{\delta_5} & 0 & \delta_2 & -\delta_3 & 0 \\ 0 & 0 & \mathrm{e}^{\delta_5} & 0 & \delta_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\delta_7 & \delta_6 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where the parameters $\delta_1, \dots, \delta_7$ are arbitrary real constants. Consider the subgroup $D \subset \mathrm{Aut}(\mathfrak{g})$ consisting of the diagonal matrices

$$\mathrm{diag}(\varepsilon b_1 b_2, \varepsilon b_1, \varepsilon, b_1 b_2, 1, b_1, b_2),$$

where b_1 and b_2 are arbitrary real constants with $b_1 b_2 \neq 0$, and $\varepsilon = \pm 1$. The group of outer automorphisms $\mathrm{Out}(\mathfrak{g}) := \mathrm{Aut}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})$ is canonically isomorphic to D. Moreover, the group $\mathrm{Aut}(\mathfrak{g})$ splits over its normal subgroup $\mathrm{Inn}(\mathfrak{g})$, $\mathrm{Aut}(\mathfrak{g}) = D \ltimes \mathrm{Inn}(\mathfrak{g})$.

The general form of a fiber-preserving point transformation $\mathcal{T}$ acting on the foliated space $\mathbb{R}^4_{t,x,y,u} \times \mathbb{R}^2_{\alpha,\beta}$ is

$$\mathcal{T}\colon \quad (\tilde t, \tilde x, \tilde y, \tilde u) = (T, X, Y, U)(t, x, y, u), \quad (\tilde\alpha, \tilde\beta) = (A, B)(t, x, y, u, \alpha, \beta).$$

If $\mathcal{T}$ is an equivalence transformation of the class $\mathcal{F}$, then the pushforward $\mathcal{T}_*$ of vector fields by $\mathcal{T}$ is an automorphism of the algebra $\mathfrak{g}$. Up to factoring out continuous equivalence transformations, which are known, we can assume that $\mathcal{T}_* \in D$. Therefore,

$$\begin{aligned} &\mathcal{T}_*(\partial_u) = \varepsilon b_1 b_2 \partial_{\tilde u}, \quad \mathcal{T}_*(x\partial_u) = \varepsilon b_1 \tilde x \partial_{\tilde u}, \quad \mathcal{T}_*((tx-y)\partial_u) = \varepsilon(\tilde t\tilde x - \tilde y)\partial_{\tilde u}, \\ &\mathcal{T}_*(\partial_y) = b_1 b_2 \partial_{\tilde y}, \quad \mathcal{T}_*(u\partial_u) = \tilde u \partial_{\tilde u}, \quad \mathcal{T}_*(\partial_t) = b_1 \partial_{\tilde t}, \\ &\mathcal{T}_*(\partial_x + t\partial_y + \partial_\alpha) = b_2(\partial_{\tilde x} + \tilde t\partial_{\tilde y} + \partial_{\tilde\alpha}). \end{aligned}$$

This results in a system of determining equations for the components of $\mathcal{T}$,

$$\begin{aligned} &X_t = Y_t = U_t = A_t = B_t = 0, \quad T_t = b_1, \\ &T_x = U_x = B_x + B_\alpha = 0, \quad X_x = A_x + A_\alpha = b_2, \quad Y_x + tY_y = b_2 T, \end{aligned}$$

$$T_y = X_y = U_y = A_y = B_y = 0, \quad Y_y = b_1 b_2,$$
$$T_u = X_u = Y_u = A_u = B_u = 0, \quad U_u = \varepsilon b_1 b_2,$$
$$xU_u = \varepsilon b_1 X, \quad (tx - y)U_u = \varepsilon(TX - Y), \quad uU_u = U,$$

whose general solution is

$$T = b_1 t, \quad X = b_2 x, \quad Y = b_1 b_2 y, \quad U = \varepsilon b_1 b_2 u,$$
$$A = b_2 \alpha + A^0(\omega, \beta), \quad B = B(\omega, \beta), \quad \omega := x - \alpha.$$

Using the chain rule, we express all the required transformed derivatives in terms of the initial ones, $\tilde{u}_{\tilde{t}} = \varepsilon b_2 u_t$, $\tilde{u}_{\tilde{y}} = \varepsilon u_y$, $\tilde{u}_{\tilde{x}\tilde{x}} = \varepsilon b_1 b_2^{-1} u_{xx}$, $\tilde{\alpha}_{\tilde{x}} = \alpha_x + b_2^{-1}\big(A^0_\omega(1-\alpha_x) + A^0_\beta \beta_x\big)$ and $\tilde{\beta}_{\tilde{x}} = b_2^{-1}\big(B_\omega(1-\alpha_x) + B_\beta \beta_x\big)$. Then substitute the obtained expressions into the transformed counterpart of (3.1). The resulted equation should be satisfied identically for all the solutions of (3.1), which results in the constraints $b_1 = 1$, $b_2^2 = 1$, $A^0 = 0$ and $B = \beta$. Therefore, the class $\mathcal{F}$ possesses only two independent discrete point equivalence transformations, which are the involutions $\mathcal{I}_u$ and $\mathcal{I}_s$ associated with the values $\varepsilon = -1$ and $b_2 = -1$, respectively, where the other parameters take the values as for the identity transformation, see Corollary 3.9. □

3.A.2 The gauged class

Similarly to what has been done in Section 3.A.1, to find the automorphism group $\mathrm{Aut}(\hat{\mathfrak{g}})$ of the algebra $\hat{\mathfrak{g}} := \mathfrak{g}^{\sim}_{\mathcal{F}'}$, we start with constructing megaideals of the algebra $\hat{\mathfrak{g}}$. Since the algebra is not abelian, we straightforwardly obtain the megaideal

$$\hat{\mathfrak{m}}_2 := \hat{\mathfrak{g}}' = \langle \partial_u, x\partial_u, (tx - y)\partial_u \rangle.$$

Recall that a nilradical of a Lie algebra and its derivatives are megaideals of the algebra. The nilradical of the algebra $\hat{\mathfrak{g}}$ and its derivative respectively are

$$\hat{\mathfrak{n}} = \langle \partial_u, x\partial_u, (tx - y)\partial_u, \partial_t, \partial_y \rangle \quad \text{and}$$
$$\hat{\mathfrak{m}}_1 := \hat{\mathfrak{n}}' = \langle \partial_u, x\partial_u \rangle.$$

Applying [37, Proposition 1] to the triple $(\mathfrak{i}_0, \mathfrak{i}_1, \mathfrak{i}_2) = (\hat{\mathfrak{n}}, \hat{\mathfrak{g}}, \hat{\mathfrak{m}}_1)$, we obtain the megaideal

$$\hat{\mathfrak{m}}_4 = \langle \partial_u, x\partial_u, \partial_t, \partial_y \rangle.$$

Lemma 3.39. *The span* $\hat{\mathfrak{m}}_3 = \langle \partial_u, x\partial_u, (tx-y)\partial_u, u\partial_u \rangle$ *is a megaideal of the algebra* $\hat{\mathfrak{g}}$.

Proof. Since $u\partial_u \in \hat{\mathfrak{g}} \setminus \hat{\mathfrak{n}}$ and $(tx-y)\partial_u \in \hat{\mathfrak{m}}_2 \setminus \hat{\mathfrak{m}}_1$, for any $\Phi \in \operatorname{Aut}(\hat{\mathfrak{g}})$ we have

$$\Phi(u\partial_u) = c_1\partial_u + c_2 x\partial_u + c_3(tx-y)\partial_u + c_4 u\partial_u + c_5\partial_y + c_6\partial_t \quad \text{and}$$
$$\Phi((tx-y)\partial_u) = \hat{c}_1\partial_u + \hat{c}_2 x\partial_u + \hat{c}_3(tx-y)\partial_u,$$

where $c_1, \ldots, c_6$, $\hat{c}_1$,$\hat{c}_2$ and $\hat{c}_3$ are real constants with $\hat{c}_3 c_4 \neq 0$. Therefore,

$$\begin{aligned}\Phi([(tx-y)\partial_u, u\partial_u]) &= \Phi((tx-y)\partial_u) = \hat{c}_1\partial_u + \hat{c}_2 x\partial_u + \hat{c}_3(tx-y)\partial_u \\ &= \big[\Phi((tx-y)\partial_u), \Phi(u\partial_u)\big] \\ &= c_4(\hat{c}_1\partial_u + \hat{c}_2 x\partial_u + \hat{c}_3(tx-y)\partial_u) + \hat{c}_3 c_5\partial_u - \hat{c}_3 c_6 x\partial_u,\end{aligned}$$

which results in the equations $\hat{c}_1 = \hat{c}_1 c_4 + \hat{c}_3 c_5$, $\hat{c}_2 = \hat{c}_2 c_4 - \hat{c}_3 c_6$ and $\hat{c}_3 = \hat{c}_3 c_4$, i.e., $c_4 = 1$ and $c_5 = c_6 = 0$. Hence,

$$\Phi(u\partial_u) \in \langle u\partial_u \rangle + \hat{\mathfrak{m}}_2.$$

Thus, we get that $\hat{\mathfrak{m}}_4$ is stable under the action of the group $\operatorname{Aut}(\hat{\mathfrak{g}})$. □

Fixing the basis $(\partial_u, x\partial_u, (tx-y)\partial_u, u\partial_u, \partial_t, \partial_y)$ of the algebra $\hat{\mathfrak{g}}$ and using the knowledge of the proper megaideals $\hat{\mathfrak{m}}_1, \ldots, \hat{\mathfrak{m}}_4$, we can impose the following constraints on the matrix $(a_{ij})_{i,j=1,\ldots,6}$ of an arbitrary automorphism of $\hat{\mathfrak{g}}$ in this basis:

$$a_{i1} = a_{i2} = a_{j3} = a_{j4} = a_{k5} = a_{k6} = 0, \quad i = 3,4,5,6, \quad j = 5,6, \quad k = 3,4,$$
$$a_{65} = 0.$$

Under the derived constraints, we complete the computation of $\operatorname{Aut}(\hat{\mathfrak{g}})$ in Maple. We

obtain that the group $\mathrm{Aut}(\hat{\mathfrak{g}})$ consists of the matrices

$$\begin{pmatrix} a_{44}a_{55} & -a_{43}a_{55} & a_{43}a_{56} & a_{44}a_{56} & a_{15} & a_{16} \\ -a_{34}a_{55} & a_{33}a_{55} & -a_{33}a_{56} & -a_{34}a_{56} & a_{25} & a_{26} \\ 0 & 0 & a_{33} & a_{34} & 0 & 0 \\ 0 & 0 & a_{43} & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & a_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where the remaining parameters a_{ij} are arbitrary real constants with $a_{33}(a_{55}a_{66} - a_{56}a_{65}) \neq 0$. It is then clear that the complete list of the essential megaideals of $\hat{\mathfrak{g}}$, which are not the sums of other megaideals, is exhausted by $\hat{\mathfrak{m}}_1$, $\hat{\mathfrak{m}}_2$, $\hat{\mathfrak{m}}_3$ and $\hat{\mathfrak{m}}_4$. The only other nonzero megaideals are $\hat{\mathfrak{n}} = \hat{\mathfrak{m}}_2 + \hat{\mathfrak{m}}_4$ and $\hat{\mathfrak{g}} = \hat{\mathfrak{m}}_3 + \hat{\mathfrak{m}}_4$. This is why the nilradical $\hat{\mathfrak{n}}$ and the entire algebra $\hat{\mathfrak{g}}$ are not essential in the course of computing the automorphisms of the algebra $\hat{\mathfrak{g}}$ and the equivalence group $G^{\sim}_{\mathcal{F}'}$ by the algebraic method.

The inner automorphism group $\mathrm{Inn}(\hat{\mathfrak{g}})$ is constituted by the matrices

$$\begin{pmatrix} \mathrm{e}^{\delta_6} & 0 & 0 & \delta_5 & -\delta_4\mathrm{e}^{\delta_6} & -\delta_4\delta_5 + \delta_1 \\ 0 & \mathrm{e}^{\delta_6} & -\delta_5 & 0 & \delta_3\mathrm{e}^{\delta_6} & \delta_3\delta_5 + \delta_2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{e}^{\delta_6} & \delta_5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where the parameters $\delta_1, \dots, \delta_6$ are arbitrary real constants. The group $\mathrm{Aut}(\hat{\mathfrak{g}})$ splits over its normal subgroup $\mathrm{Inn}(\hat{\mathfrak{g}})$, $\mathrm{Aut}(\hat{\mathfrak{g}}) = H \ltimes \mathrm{Inn}(\hat{\mathfrak{g}})$. The subgroup $H \subset \mathrm{Aut}(\hat{\mathfrak{g}})$ consists of the matrices

$$\begin{pmatrix} \varepsilon c_{22} & -\varepsilon c_{21} \\ -\varepsilon c_{12} & \varepsilon c_{11} \end{pmatrix} \oplus \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \oplus \mathrm{diag}(\varepsilon, 1),$$

where the parameters c_{kl}, $k, l = 1, 2$, are arbitrary real constants with $c_{11}c_{22} - c_{21}c_{12} \neq 0$, and $\varepsilon = \pm 1$. The general form of a point fiber-preserving transformation $\mathcal{T}$ in the foliated space $\mathbb{R}^4_{t,x,y,u} \times \mathbb{R}_\beta$ is

$$\mathcal{T}: \quad (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u}) = (T, X, Y, U)(t, x, y, u), \quad \tilde{\beta} = B(t, x, y, u, \beta).$$

If $\mathcal{T}$ is an equivalence transformation of the class $\mathcal{F}'$, then the pushforward $\mathcal{T}_*$ of vector fields by $\mathcal{T}$ is an automorphism of the algebra $\hat{\mathfrak{g}}$. Moreover, up to composing $\mathcal{T}$ with elements of the identity component of $G^{\sim}_{\mathcal{F}'}$, which is known if $\hat{\mathfrak{g}}$ is, we can assume that $\mathcal{T}_* \in H$. Therefore,

$$\begin{aligned}
&\mathcal{T}_*(\partial_u) = \varepsilon c_{22}\partial_u - \varepsilon c_{12}x\partial_u, \quad \mathcal{T}_*(x\partial_u) = -\varepsilon c_{21}\partial_u + \varepsilon c_{11}x\partial_u,\\
&\mathcal{T}_*(\partial_t) = c_{11}\partial_t + c_{21}\partial_y, \quad \mathcal{T}_*(\partial_y) = c_{12}\partial_t + c_{22}\partial_y,\\
&\mathcal{T}_*((tx-y)\partial_u) = \varepsilon(tx-y)\partial_u, \quad \mathcal{T}_*(u\partial_u) = u\partial_u.
\end{aligned}$$

This leads to a system of linear differential equations on the components of $\mathcal{T}$,

$$\begin{aligned}
&X_t = U_t = B_t = 0, \quad T_t = c_{11}, \quad Y_t = c_{21},\\
&X_y = U_y = B_y = 0, \quad T_y = c_{12}, \quad Y_y = c_{22},\\
&T_u = X_u = Y_u = B_u = 0, \quad U_u = \varepsilon(c_{22} - c_{12}X),\\
&xU_u = \varepsilon(c_{11}X - c_{21}), \quad uU_u = U, \quad (tx-y)U_u = \varepsilon(TX - Y),
\end{aligned}$$

which integrates to

$$\begin{aligned}
&T = c_{11}t + c_{12}y, \quad X = \frac{c_{22}x + c_{21}}{c_{12}x + c_{11}}, \quad Y = c_{21}t + c_{22}y, \quad U = \varepsilon\frac{c_{11}c_{22} - c_{12}c_{21}}{c_{12}x + c_{11}}u,\\
&B = B(x, \beta).
\end{aligned}$$

This is the most restricted form of candidates for equivalence transformations of $\mathcal{F}'$ that can be obtained within the framework of the algebraic method. We complete the computation of $G^{\sim}_{\mathcal{F}'}$, applying the direct method. Using the chain rule, we find the transformation components for involved derivatives,

$$\begin{aligned}
&\tilde{u}_{\tilde{t}} = \varepsilon\frac{c_{22}u_t - c_{21}u_y}{c_{12}x + c_{11}}, \quad \tilde{u}_{\tilde{y}} = \varepsilon\frac{c_{11}u_y - c_{12}u_t}{c_{12}x + c_{11}}, \quad \tilde{u}_{\tilde{x}\tilde{x}} = \frac{\varepsilon}{\Delta}(c_{12}x + c_{11})^3 u_{xx},\\
&\tilde{\beta}_{\tilde{x}} = \frac{(c_{12}x + c_{11})^2}{\Delta}(B_x + B_\beta\beta_x),
\end{aligned}$$

where $\Delta := c_{11}c_{22} - c_{12}c_{21}$. The transformed counterpart $\tilde{u}_{\tilde{t}} + \tilde{x}\tilde{u}_{\tilde{y}} = |\tilde{x}|^{\tilde{\beta}}\tilde{u}_{\tilde{x}\tilde{x}}$ of the equation (3.2) in the initial coordinates takes the form

$$u_t + xu_y = \left|\frac{c_{22}x + c_{21}}{c_{12}x + c_{11}}\right|^{\tilde{\beta}} \frac{(c_{12}x + c_{11})^5}{\Delta^2}u_{xx},$$

which should coincide with (3.2). This results in the equation

$$|c_{22}x+c_{21}|^{\tilde\beta}=\Delta^2\operatorname{sgn}(c_{12}x+c_{11})|x|^{\beta}|c_{12}x+c_{11}|^{\tilde\beta-5}.$$

Hence $\operatorname{sgn}(c_{12}x+c_{11})=1$. In view of Lemma 3.3 and the inequality $c_{11}c_{22}-c_{21}c_{12}\neq 0$, in the case $c_{22}=0$, we successively derive $c_{12}c_{21}\neq 0$, $c_{11}=0$, $c_{12}=\operatorname{sgn}x$, $c_{21}=\pm\operatorname{sgn}x$ and $\beta=5-\tilde\beta$. Analogously, for $c_{22}\neq 0$ we obtain $c_{12}=0$, $c_{11}\neq 0$, $c_{21}=0$, $c_{11}=1$, $\tilde\beta=\beta$, $|c_{22}|^{\beta-2}=1$ and thus $c_{22}=\pm1$. Hence, there are only three independent discrete equivalence transformations in $G^{\sim}_{\mathcal{F}'}$, $\mathcal{I}'_u$, $\mathcal{I}'_{\rm s}$ and $\mathcal{J}'$, which are respectively associated with the values $\varepsilon=-1$, $(\varepsilon,c_{22})=(-1,1)$ (if the other parameters in these two cases take the values as for the identity transformation) and $(c_{11},c_{12},c_{21},c_{22},\varepsilon)=(0,\operatorname{sgn}x,\operatorname{sgn}x,0,-\operatorname{sgn}x)$, see Corollary 3.14.

Appendix 3.B Darboux transformations of linear heat equations with potentials

Consider a (1+1)-dimensional linear heat equations with an arbitrary potential $V=V(t,x)$,

$$u_t-u_{xx}+V(t,x)u=0. \tag{3.41}$$

By $\mathrm{W}(f^1,\dots,f^k)$ we denote the Wronskian of sufficiently smooth functions f^1, …, f^k of (t,x) with respect to x, $\mathrm{W}(f^1,\dots,f^k)=\det\big(\partial_x^{l'-1}f^l\big)_{l,l'=1,\dots,k}$. The Darboux transformation

$$\tilde u=\mathrm{DT}[f^1,\dots,f^k]u:=\frac{\mathrm{W}(f^1,\dots,f^k,u)}{\mathrm{W}(f^1,\dots,f^k)},$$

with a fixed tuple $(f^1,\dots,f^k)$ of solutions of (3.41) maps the solution set of (3.41) onto the solution set of the equation $\tilde u_t-\tilde u_{xx}+\tilde V(t,x)\tilde u=0$, where

$$\tilde V=V-2\left(\frac{\big(\mathrm{W}(f^1,\dots,f^k)\big)_x}{\mathrm{W}(f^1,\dots,f^k)}\right)_x,$$

see [91, 127]. An evident relation is established by Darboux transformations between (1+1)-dimensional linear heat equations with inverse square potentials, $V=-\mu x^{-2}$, where μ is an arbitrary constant with $\mu\leqslant\frac{1}{4}$. The function $u=|x|^{\alpha}$ with $\alpha=\frac{1}{2}\pm(\frac{1}{4}-\mu)^{1/2}$ is a solution of the equation

$$u_t=u_{xx}+\mu x^{-2}u, \tag{3.42}$$

and the corresponding Darboux transformation $\mathrm{DT}[|x|^{\alpha}]$ maps this equation to the equation $\tilde{u}_t=\tilde{u}_{xx}+\tilde{\mu}x^{-2}\tilde{u}$ with $\tilde{\mu}=\mu-2\alpha$.

The most interesting is the case when solutions of (1+1)-dimensional linear heat equations with inverse square potentials are expressed in terms of the general solution of the (1+1)-dimensional linear heat equation. Since the Darboux transformations $\mathrm{DT}[x^n]$ maps the equation of the form (3.42) with $\mu=n(n-1)$ to the equation of the same form with $\mu=n(n+1)$. As a result, beginning from $n=1$ we derive the representation

$$\begin{aligned} u&=\left(\partial_x-\frac{n}{x}\right)\left(\partial_x-\frac{n-1}{x}\right)\cdots\left(\partial_x-\frac{1}{x}\right)\vartheta^0(t,x)\\ &=\mathrm{DT}[P_1,P_3,\dots,P_{2n-1}]\vartheta^0(t,x) \end{aligned} \tag{3.43}$$

for an arbitrary solution of the equation of the form (3.42) with $\mu=n(n+1)$, where ϑ^0 is an arbitrary solution of the (1+1)-dimensional linear heat equation, for which $\mu=0$. Here P_k is the canonical heat polynomial of degree $k\in\mathbb{N}\cup\{0\}$, which is, for odd $k=2m-1$ with $m\in\mathbb{N}$, of the form

$$\begin{aligned} P_{2m-1}(t,x)=\frac{x^{2m-1}}{(2m-1)!}+\frac{t}{1!}\frac{x^{2m-3}}{(2m-3)!}+\frac{t^2}{2!}\frac{x^{2m-5}}{(2m-5)!}+\cdots\\ +\frac{t^{m-2}}{(m-2)!}\frac{x^3}{3!}+\frac{t^{m-1}}{(m-1)!}\frac{x}{1!}. \end{aligned}$$

To check the second equality in (3.43), it suffices to prove that

$$R_mR_{m-1}\cdots R_1P_{2m-1}=0,\quad \text{where}\quad R_l:=\partial_x-\frac{l}{x},\quad l\in\mathbb{N},$$

which we carry out by induction. The induction base $m=1$ is clear since $P_1(t,x)=x$

and $(\partial_x - x^{-1})P_1 = 0$. The induction step follows from the following facts:

$$P_k(t,x) = \frac{2^k}{k!}\mathrm{G}^k 1, \quad \text{where} \quad \mathrm{G} := t\partial_x + \frac{x}{2}, \quad \partial_x P_k = P_{k-1},$$
$$R_m R_{m-1} \cdots R_1 \mathrm{G} = \left(tR_m + \frac{x}{2}\right) R_{m-1} \cdots R_1 \partial_x.$$

Chapter 4

Symmetry analysis and exact solutions of multi-layer quasi-geostrophic problem

4.1 Introduction

In ocean or atmosphere modeling, the amount of literature devoted to numerical investigation of dynamics of fluid motion greatly outweighs that focused on finding explicit exact closed-form solutions. This trend is certainly not surprising, given that the underlying systems of partial differential equations are nonlinear and strongly coupled. However, even within numerical analysis, exact solutions remain important as they provide a reference point for developing and verifying finite difference schemes or other numerical techniques. Among various mathematical approaches, Lie group analysis stands as one of the most powerful frameworks for constructing exact solutions of systems of nonlinear partial differential equations.

Based on analysis of horizontal and vertical characteristic length scales, the complete set of governing hydro-thermodynamical equations for meteorology and oceanology can be reduced to a hierarchy of simplified models. Each model from this hierarchy accounts only for selected phenomena in dynamics of atmosphere or ocean.

The most elementary model is the famous Lorenz system, which is a system of three ordinary differential equations that captures nonlinear chaos but sacrifices spatial

resolution [88]. The next, more complicated model is the barotropic vorticity equation, which simplifies the fluid to a single constant-density layer, while still allowing for spatial dynamics. Results on Lie symmetries and exact solutions of the barotropic vorticity equation were collected in [53, Section 9.6]; this equation was called the equation of geopotential forecast therein. Its point symmetry group was computed in [16], see also [37] for similar results on the vorticity equation on the rotating sphere.

Analogously to the barotropic vorticity equation, the shallow water equations model a single constant-density fluid layer, but take into account depth variations and gravity wave dynamics for cases where the horizontal scale is significantly greater than the vertical one. The shallow water equations with flat bottom topography coincide with a particular case of the equations of motion of polytropic gas with the parameter $\gamma = 2$; see [52] for Lie symmetries, some exact solutions and some conservation laws of these equations. Lie symmetries and zeroth-order conservation laws of the shallow water equations with variable bottom topography were classified in [14] and in [18], respectively.

Increasing the precision of the approximation to the multi-layer quasi-geostrophic model, one considers the fluid as stacked layers of constant densities allowing for simulating baroclinic instability and synoptic-scale weather patterns. The classical symmetry analysis for the two-layer model was carried out in [15], while its Lax representation and its conservation laws were constructed in [94].

The primitive equations consist of three main sets of balance equations: the continuity, momentum and thermal energy equations. This is a standard model for global forecasting. The partial classical symmetry analysis of the primitive equations in [38] included computing their maximal Lie invariance algebra, point-symmetry pseudogroup and some families of Lie invariant solutions. The most complicated and complete model for meteorology and oceanology is given by the full nonhydrostatic Euler equations, which resolves vertical accelerations and acoustic waves to account for small-scale, high-energy phenomena like deep convection and turbulence.

For other relevant results on application of Lie group analysis in fluid dynamics, see [53, Chapters 9–11], [8, 30, 54, 62, 84, 121, 142] and references therein.

The purpose of the present paper is to carry out extended Lie symmetry analysis of the quasi-geostrophic model with an arbitrary number of layers, essentially generalizing the study of the case of two layers initiated in [15] and continued in [94].

This includes the construction of conservation laws and a Hamiltonian structure of the model, the computation of its maximal Lie invariance algebra $\mathfrak{g}$ and its complete pseudogroup G of point symmetries, the classification of one- and two-dimensional subalgebras of the algebra $\mathfrak{g}$, the exhaustive study of codimension-one, -two and -three Lie reductions and hidden Lie symmetries and finding exact invariant solutions of the model. Beyond its clear physical relevance to large-scale oceanic and atmospheric dynamics, the model possesses distinguishing symmetry features that have never been represented in the literature on Lie group analysis of differential equations. The multi-layer quasi-geostrophic model is governed by a system of $m \in \mathbb{N}$ coupled barotropic vorticity equations, where the coupling between layers is given by a specific tridiagonal matrix. To the best of our knowledge, systems of nonlinear partial differential equations involving an arbitrary number of coupled equations have not yet been considered in the literature from the perspective of Lie group analysis. The arbitrariness of m in particular makes it impossible to directly apply specialized computer algebra packages for finding the algebra $\mathfrak{g}$.

This paper is organized as follows. In Section 4.2, we provide a comprehensive description of the multi-layer quasi-geostrophic problem, which is given by the system of equations (4.1). In particular, we present its vector representation (4.2) and study structural and spectral properties of the vertical coupling matrix F. We show that the matrix F has rank $m-1$, is diagonalizable and has pairwise distinct negative eigenvalues, except one eigenvalue, which is zero. These results are of great relevance to the physical interpretation of the model since even distinguishing between barotropic and baroclinic modes is based on the F-invariant subspace decomposition of the underlying space for values of the tuple of layer stream functions. We also present real geophysical data from [100] for parameters of the model of a stratified ocean with three layers, which are later used for physically relevant illustration of found invariant solutions.

Furthermore, in Section 4.3, we correctly construct for the first time families of conservation laws of the general case of the model (4.1) and its Hamiltonian structure.

Section 4.4 is devoted to computing the maximal Lie invariance algebra $\mathfrak{g}$ of the multi-layer quasi-geostrophic model (4.1). This computation is made possible due to two original tricks. The first trick is to formally change the independent spatial variables (x, y) to the complex conjugated ones $z = x+\mathrm{i}y$ and $\bar{z} = x-\mathrm{i}y$, which allows us to introduce a natural ordering for jet variables and reduce the size of involved

expressions. The second trick is to consider the class of systems of general form (4.1) as the lowest member in a chain of nested classes of systems of differential equations. In this way, the derivation of determining equations for Lie symmetries is simplified via the successive application of the infinitesimal invariance criterion to the classes in the chain starting from the largest one. Since a general system from a superclass has less constraints on its arbitrary elements than one from a proper subclass, it is easier to treat the corresponding invariance condition and split it only with respect to higher-order jet variables that are not involved in the arbitrary elements of the superclass as their arguments. At the same time, the determining equations for the superclass are a subsystem of the analogous system for the subclass. Higher-level classes in the hierarchy give more important determining equations; taking them into account significantly simplifies deriving the complete systems of determining equations in lower levels.

In Section 4.5, using the megaideal-based version of the algebraic method, we compute the pseudogroup G of point symmetries of the system (4.1) and thus exhaustively classify its discrete elements.

Then in Section 4.6, we consider the collection of systems of the form (4.1), where the vertical coupling matrix F and the Rossby parameter β run through the sets of admitted values, as a class $\mathcal{M}$ of systems of differential equations. Based on the results of Sections 4.4 and 4.5, we compute the equivalence groupoid, the equivalence group and the equivalence algebra of this class and carry out its exhaustive group classification. In particular, we show that the class $\mathcal{M}$ is normalized in the usual sense and describe its generalized equivalence group.

Towards the study of Lie reductions of the multi-layer quasi-geostrophic problem (4.1), in Section 4.7 we classify one- and two-dimensional subalgebras of the Lie algebra $\mathfrak{g}$ with respect to the action of the pseudogroup G. We also show that codimension-three Lie reductions of (4.1) lead to no interesting exact solutions of (4.1).

Section 4.8 is devoted to the systematic investigation of codimension-one Lie reductions of (4.1). We carry out the group classification of each class of the reduced systems, thus computing their maximal Lie invariance algebras, and determine their induced Lie symmetries. This allows us to exhaustively describe hidden Lie symmetries of the original nonlinear system (4.1). We thoroughly study the cases when the codimension-one reduced systems are linear and completely decoupled (at least under

additional differential constraints). In this setting, we derive solutions of the multi-layer quasi-geostrophic problem (4.1) expressed in terms of solutions of decoupled systems of well-known linear equations, including Helmholtz, modified Helmholtz, Laplace, Klein–Gordon, Whittaker, Bessel and linearized Benjamin–Bona–Mahony equations. Among them, the most physically prominent solutions are represented in terms of *Herglotz wave functions*; such functions associated with finite measures on the circle constitute the class of entirely bounded solutions of the Helmholtz equations. Particular cases of this representation include baroclinic Rossby waves, coherent baroclinic eddies, coherent hetons and their superpositions. Furthermore, by partitioning the domain and merging different solutions (even those originating from different reduced systems), we reconstruct representations for dipolar vortices, also known in the oceanological literature as modons. The illustrations of the above physically relevant solutions using real-world geophysical data from [100] for a three-layered ocean are presented as well. We also consider the generically coupled case of codimension-one reduced systems and further reduce them using appropriate one-dimensional subalgebras of their Lie invariance algebras, which, in some cases, allowed us to construct explicit exact solutions.

The study of codimension-two Lie reductions of the system (4.1) is carried out in Section 4.9. It turns out that each of them is just a two-step reduction, where the first step is a codimension-one Lie reduction of (4.1) and the second step is the further Lie reduction of the obtained reduced systems with respect to their induced Lie symmetry. Nevertheless, we present results on codimension-two reductions for the following two reasons. First and foremost, the codimension-two reduced systems are systems of linear ordinary differential equations (of order at most three) with constant coefficients, and thus they can be completely integrated explicitly or at least in quadratures. At the same time, it is optimal not to consider them as two-step reductions [148, Section 2]. Given that the codimension-one reduced systems in Section 4.8 are integrated in terms of elementary or special functions only for certain special cases or specific values of subalgebra parameters, the codimension-two Lie reduction may result in solutions that are not found in Section 4.8, and this is indeed the case for integrating three out of four families of corresponding reduced systems. The second reason is that including the results on codimension-two reductions in the paper maintains the systematic and comprehensive nature of the present symmetry

study and provides a complete solution to the problem of Lie reductions for the multi-layer quasi-geostrophic problem (4.1).

We summarize the results of the present paper and outline research perspectives on the multi-layer quasi-geostrophic problem (4.1) in Section 4.10.

For readers' convenience, the constructed exact solutions of the multi-layer quasi-geostrophic problem (4.1) are marked by the bullet symbol $\bullet$ and the constructed exact solutions of the reduced systems of (4.1) are marked by the circle symbol $\circ$.

4.2 Multi-layer quasi-geostrophic problem

The multi-layer quasi-geostrophic problem describes the dynamics of a stratified incompressible fluid with m layers of constant densities $\rho_1 < \cdots < \rho_m$, stacked according to increasing density (i.e., the density of the top layer is ρ_1), in the "rigid-lid" and flat-bottom approximation (i.e., the upper surface of the fluid is assumed to be fixed [145, Section 16.A.2] and the variation of bottom is negligible).

4.2.1 Mathematical model

The quasi-geostrophic approximation assumes that the vertical component of the velocity field is negligibly small and in the horizontal direction the motion satisfies a system of m ($m \geqslant 2$) coupled vorticity equations, that is, the third-order partial differential equations

$$\begin{aligned} &q^i_t + \{\psi^i, q^i\} = 0,\\ &q^i := \psi^i_{xx} + \psi^i_{yy} + f_{i,i-1}(\psi^{i-1} - \psi^i) - f_{i,i+1}(\psi^i - \psi^{i+1}) + \beta y,\ \ i = 1, \dots, m. \end{aligned} \tag{4.1}$$

Here $\psi^i = \psi^i(t,x,y)$ and $q^i = q^i(t,x,y)$ are the stream function and the quasi-geostrophic potential vorticity for the ith layer, respectively, and their Poisson bracket is given by their Jacobian with respect to (x,y), $\{\psi^i, q^i\} := \psi^i_x q^i_y - \psi^i_y q^i_x$. We define the stream function as in meteorology and oceanography such that the tuple $(-\psi^i_y, \psi^i_x)$ is the velocity field on the ith layer.

The ith layer of the stratified fluid is characterized by its mean thickness $H_i > 0$, the fluid density ρ_i in this layer and the corresponding reduced gravity $g'_i = g(\rho_{i+1} -$

$\rho_i)/\rho_0 > 0$, $i = 1, \dots, m-1$, where ρ_0 denotes the reference density[1] of the fluid. The coupling between ith and $(i-1)$th (resp. $(i+1)$th) layer is provided by the term $f_{i,i-1}(\psi^i - \psi^{i-1})$ with $f_{i,i-1} := f_0^2/(H_i g'_{i-1}) > 0$ (resp. $f_{i,i+1}(\psi^{i+1} - \psi^i)$ with $f_{i,i+1} := f_0^2/(H_i g'_i) > 0$), where $f_{1,0} := 0$ and $f_{m,m+1} := 0$. Therefore, for each i both the constants $f_{i,i-1}$ and $f_{i,i+1}$ are positive real numbers, except for $f_{1,0}$ and $f_{m,m+1}$, which are zero. Here, f_0 and β are the coefficients in the beta-plane approximation of the Coriolis parameter $f = 2\Omega \sin\vartheta$ about the latitude $\vartheta = \vartheta_0$, $f = f_0 + \beta y$. More specifically, the Coriolis parameter at ϑ_0 is $f_0 = 2\Omega \sin\vartheta_0$, the Rossby parameter at ϑ_0 is $\beta = 2\Omega \cos\vartheta_0/R > 0$, $y = R(\vartheta - \vartheta_0)$, where R is the Earth's radius and Ω is the angular rotation rate of the Earth. We note that constants $f_{i,i\pm1}$ and H_i satisfy a nontrivial relation, $H_i f_{i,i+1} = H_{i+1} f_{i+1,i}$. See [145, Section 16.A.2] and [116, Section 6.16] for the more detailed description of the model as well as Fig. 4.1 for graphic illustration in the particular case of three-layer models, $m = 3$.

Notation. Throughout the paper, the Latin indices i, j and k run through the sets $\{1, \dots, m\}$ unless the other range is specified. The summation over the repeated indices whenever they appear in the different multipliers is understood unless the other is specified, except for the index i. Capital sans serif letters denote, usually $m \times m$, matrices.

The expanded form of the system (4.1) is

$$\begin{aligned}
&\psi^i_{txx} + \psi^i_{tyy} + \psi^i_x(\psi^i_{xxy} + \psi^i_{yyy}) - \psi^i_y(\psi^i_{xxx} + \psi^i_{yyx}) \\
&\quad + f_{i,i-1}\big(\psi^{i-1}_t - \psi^i_t + \psi^i_x(\psi^{i-1}_y - \psi^i_y) - \psi^i_y(\psi^{i-1}_x - \psi^i_x)\big) \\
&\quad - f_{i,i+1}\big(\psi^i_t - \psi^{i+1}_t + \psi^i_x(\psi^i_y - \psi^{i+1}_y) - \psi^i_y(\psi^i_x - \psi^{i+1}_x)\big) + \beta\psi^i_x = 0.
\end{aligned}$$

[1] In practice, there are several ways to choose the reference density depending on physical context. An option is to take the weighted average of the layer densities with weights given by the layer thicknesses. In some ocean models, the density of the densest (bottom) layer is used as the reference one. Alternatively, depending on the medium under study, one may use a standard constant like the density of fresh water $1000\,\mathrm{kg/m^3}$, the density of salt water $1025\,\mathrm{kg/m^3}$ or the air density $1.225\,\mathrm{kg/m^3}$ at the sea level under the International Standard Atmosphere (ISA) model. Note that within the multi-layer quasi-geostrophic framework, which is based on the Boussinesq approximation, the relative variations of the layer densities in comparison with ρ_0 are usually small ($\approx 1\%$), hence the results are quite insensitive to the specific choice of ρ_0. The differences between the individual layer densities ρ_i are essential in the vertical direction in the interactions between neighboring layers because of gravity, which is captured via the definition of the reduced gravities or, equivalently, of the vertical coupling coefficients. In the horizontal direction, the density differences affect the inertia of the fluid negligibly, and thus the layer densities ρ_i are then replaced by the reference density ρ_0.

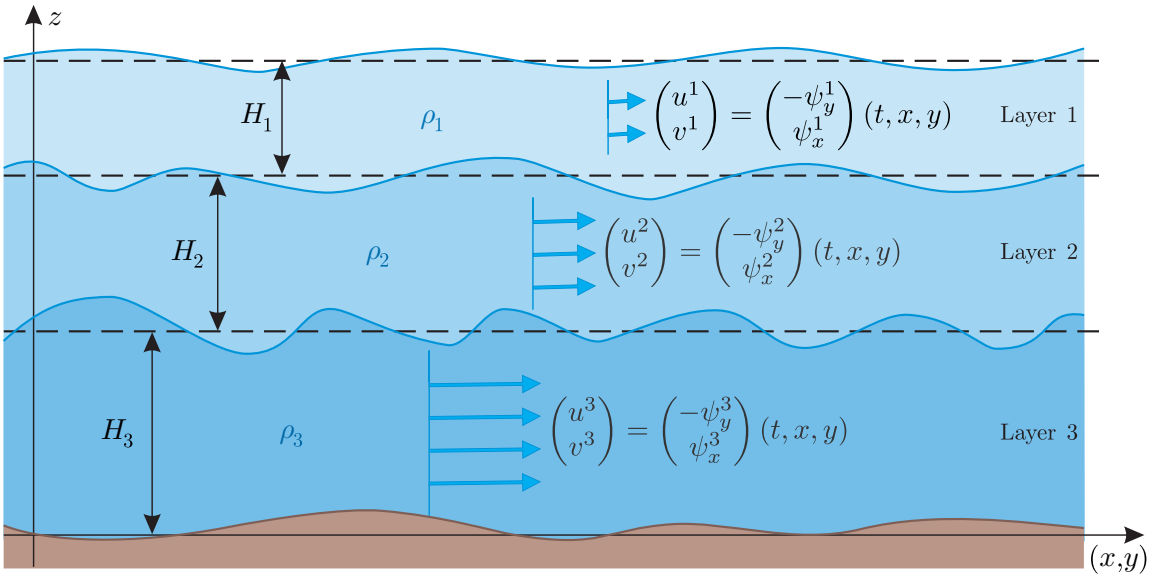


Figure 4.1: Three-layer model

More rigorously, this represents the class $\mathcal{M}$ of systems of m third-order partial differential equations for m unknown functions $\psi := (\psi^1, \dots, \psi^m)^{\mathsf{T}}$ in the three independent variables (t, x, y), where the constant parameters $f_{i,i-1}$, $f_{i,i+1}$, and β play the role of arbitrary elements of the class and are positive, except for the zero artificial parameters $f_{1,0}$ and $f_{m,m+1}$. In other words, the arbitrary-element tuple of the class $\mathcal{M}$ is

$$\theta := (f_{i+1,i}, f_{i,i+1}, i = 1, \dots, m-1, \beta),$$

and the system from $\mathcal{M}$ that corresponds to a value θ of the arbitrary-element tuple is denoted by $\mathcal{M}_\theta$. This is why all problems of finding symmetry-like objects associated with the system (4.1) should be considered classification problems for such objects within the class of systems of the form (4.1). Nevertheless, it turns out that all these systems have the same maximal Lie invariance algebra and the point symmetry pseudogroups, see Theorems 4.9 and 4.13, respectively, which leads to the degeneration of the corresponding classification problems to the regular computations of the above structures as for a single system. Note that the physical constraint of the parameters' positivity can be weakened, in the course of these computations in Sections 4.4 and 4.5 to the condition of being nonzero.

4.2.2 Vector representation of the model

One more convenient representation of the system (4.1), in particular for studying properties of invariant submodels, is that in the matrix form:

$$\begin{aligned} &q_t + \{\psi, q\} = 0,\\ &q := \psi_{xx} + \psi_{yy} + \mathsf{F}\psi + \beta y \bar{1}, \end{aligned} \tag{4.2}$$

where $\psi := (\psi^1, \dots, \psi^m)^{\mathsf{T}}$, $q := (q^1, \dots, q^m)^{\mathsf{T}}$, the Poisson bracket $\{\psi, q\}$ is defined componentwise, $\{\psi, q\} := (\{\psi^i, q^i\})^{\mathsf{T}} = \psi_x \odot q_y - \psi_y \odot q_x$, "$\odot$" is the Hadamard (componentwise) product of matrices, the vertical coupling matrix $\mathsf{F} := (f_{ij})$ is a constant m-by-m tridiagonal matrix with $f_{ii} := -f_{i,i-1} - f_{i,i+1}$, $f_{10} := 0$, $f_{m,m+1} := 0$ and $f_{ij} := 0$ if $|i-j| > 1$,

$$\mathsf{F} := \begin{pmatrix} -f_{12} & f_{12} & 0 & \cdots & 0 & 0 & 0\\ f_{21} & -f_{21} - f_{23} & f_{23} & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & f_{m-1,m-2} & -f_{m-1,m-2} - f_{m-1,m} & f_{m-1,m}\\ 0 & 0 & 0 & \cdots & 0 & f_{m,m-1} & -f_{m,m-1} \end{pmatrix},$$

and $\bar{1}$ denotes the all-ones m-column, $\bar{1} := (1, \dots, 1)^{\mathsf{T}}$. We call the components $f_{i+1,i}$ and $f_{i,i+1}$, $i = 1, \dots, m-1$, of the matrix F the essential ones. In what follows, E denotes the identity matrix of the same size as F.

4.2.3 Properties of the vertical coupling matrix

The matrix F is identified with the corresponding linear operator on the space $\mathbb{R}^m$. Under the minimal constraints of nonvanishing for the essential components of F, its rank is equal to $m-1$ and its kernel $\ker \mathsf{F}$ and image $\operatorname{im} \mathsf{F}$ are spanned by the vector $\bar{1}$ and by the columns of F, respectively. In other words, 0 is an eigenvalue of F whose eigenspace is $\langle \bar{1} \rangle$, and $m-1$ columns of F can be chosen as a basis of $\operatorname{im} \mathsf{F}$ since they are linearly independent.

In the course of carrying out Lie reductions and finding exact solutions of systems from the class $\mathcal{M}$, we also need less trivial properties of the matrix F. Under the physical constraints of positivity on the essential components of F, we can reduce the

matrix F by the similarity transformation $\mathsf{F} \to \mathsf{D}^{-1}\mathsf{F}\mathsf{D}$ with

$$\mathsf{D} := \operatorname{diag}(d_1, \dots, d_m), \quad d_i := \begin{cases} 1, & i = 1, \\ \sqrt{\dfrac{f_{21} \cdots f_{i,i-1}}{f_{12} \cdots f_{i-1,i}}} = \sqrt{\dfrac{H_1}{H_i}}, & i = 2, \dots, m, \end{cases}$$

where we use the relation $H_i f_{i,i+1} = H_{i+1} f_{i+1,i}$, to the symmetric tridiagonal matrix S with components

$$s_{ii} = -f_{i,i-1} - f_{i,i+1}, \quad s_{i,i+1} = s_{i+1,i} = \sqrt{f_{i,i+1} f_{i+1,i}}, \quad s_{ij} = 0, \quad |i-j| > 1.$$

(We can use the dimensional matrices $\hat{\mathsf{D}} := \operatorname{diag}(1/\sqrt{H_1}, \dots, 1/\sqrt{H_m})$ instead of the dimensionless matrix D.) This implies that F is diagonalizable and its eigenvalues are real. Moreover, since $f_{i,i+1}$ or $f_{i,i-1}$ are nonzero, for any $\lambda \in \mathbb{R}$, the matrix $\mathsf{F} - \lambda\mathsf{E}$ contains a submatrix of rank $m-1$ (e.g., the submatrix of the matrix $\mathsf{F} - \lambda\mathsf{E}$ obtained by deleting its first row and its last column), and thus by the rank-nullity theorem, the dimension of the kernel of $\mathsf{F} - \lambda\mathsf{E}$ is at most $m - (m-1) = 1$. If λ is an eigenvalue of F, then this dimension must be 1, i.e., the geometric multiplicity of each eigenvalue of F is equal to one. Since the matrix F is diagonalizable, for any eigenvalue of F, its algebraic multiplicity coincides with its geometric multiplicity and is thus equal to one as well. In other words, eigenvalues of F are pairwise distinct.

In fact, we can say more about the properties of eigenvalues of F. For this purpose, we need to study the characteristic polynomial $P(\lambda) := \det(\lambda\mathsf{E} - \mathsf{F})$ of the matrix F. The above discussion implies that P is divisible by λ and has no multiple roots.

Lemma 4.1. *If the physical constraints of positivity on the essential components $f_{i+1,i}$ and $f_{i,i+1}$, $i = 1, \dots, m-1$, of the matrix F is imposed, the coefficients of its characteristic polynomial $P(\lambda) := \det(\lambda\mathsf{E} - \mathsf{F})$ are positive, except the zero-degree one, which is equal to zero.*

Proof. Denote by $\hat{\mathsf{F}}_k$ the kth order leading principal submatrix of F,

$$\mathsf{F}_k := \hat{\mathsf{F}}_k|_{f_{k,k+1}=0}, \quad \hat{P}_k(\lambda) := \det\Big(\lambda\mathsf{E} - \hat{\mathsf{F}}_k\Big), \quad P_k(\lambda) := \det(\lambda\mathsf{E} - \mathsf{F}_k).$$

Thus, $\hat{\mathsf{F}}_m = \mathsf{F}_m = \mathsf{F}$ and $\hat{P}_m = P_m = P$. We prove by induction with respect to the submatrix size k that the coefficients of the polynomials $\hat{P}_k$, $k = 1, \dots, m-1$, and

the coefficients of the polynomials P_k, $k = 1, \ldots, m$, except the zero-degree one, are positive, and the zero-degree coefficient of P_k is equal to zero.

The induction base, where $k \leqslant 2$, is obvious since

$$P_1(\lambda) = \lambda, \quad \hat{P}_1(\lambda) = \lambda + f_{12},$$
$$P_2(\lambda) = \lambda^2 + (f_{12} + f_{21})\lambda, \quad \hat{P}_2(\lambda) = \lambda^2 + (f_{12} + f_{21} + f_{23})\lambda + f_{12}f_{23}.$$

For the induction step, suppose that the statement holds true for the submatrix sizes less than k. Then the recurrent formulas

$$P_k(\lambda) = (\lambda + f_{k,k-1})P_{k-1}(\lambda) + \lambda\hat{P}_{k-2}(\lambda),$$
$$\hat{P}_k(\lambda) = P_k(\lambda) + f_{k,k-1}\hat{P}_{k-1}(\lambda).$$

imply the statement for the submatrix size equal to k. □

Corollary 4.2. *All nonzero eigenvalues of* F *are negative real.*[2]

Proof. Since all the eigenvalues of F are real and the polynomial $\lambda^{-1}P(\lambda)$ has strictly positive coefficients, the nonzero eigenvalues should be negative. □

Remark 4.3. Corollary 4.2 also follows from the Gershgorin circle theorem. From this theorem, we can derive even a stronger statement that any eigenvalue of the matrix F belongs to the interval $[-R, 0]$, where $R := \min\big(2\max_i(f_{i,i-1} + f_{i,i+1}), \max_i(f_{i,i-1} + f_{i,i+1} + f_{i-1,i} + f_{i+1,i})\big)$, $f_{01} := 0$ and $f_{m+1,m} := 0$. In other words, the spectral radius $\rho(\mathsf{F}) = |\lambda_1|$ of F is not greater than R, and, moreover, $-R \leqslant \lambda_1 < \cdots < \lambda_{m-1} < \lambda_m = 0$. At the same time, $\lambda_1 < -\max_i(f_{i,i-1} + f_{i,i+1})$ according to the Rayleigh quotient argument.

To present the solutions of (4.1) derived via Lie reductions in a convenient form, we introduce the following useful notation. Denote by $e_1, \ldots, e_m$ the eigenvectors of the matrix F corresponding to its ordered eigenvalues $\lambda_1, \ldots \lambda_m$ of F, $\lambda_1 < \cdots < \lambda_m$, and let $\Lambda := \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$ be the matrix constituted by these eigenvalues.

[2] In some oceanological literature, see, e.g., [58, p. 56], the paper [59] is indicated as the source with the first proof of Corollary 4.2 for an arbitrary value of the number of layers m, but, unfortunately, this paper is not well accessible. This is why we decided to include the proofs of Lemma 4.1 and Corollary 4.2 in the present paper. The statement on the diagonalizability of tridiagonal matrices is commonly known.

Thus, $\lambda_m = 0$, and we can choose $e_m = \bar{1}$. Since the matrix F is tridiagonal and irreducible, the first and last components of any of its eigenvectors are guaranteed to be nonzero. We define $(\cdot,\cdot)_{\mathsf{W}}$ to be the weighted inner product on the space $\mathbb{R}^m$ with the weight matrix $\mathsf{W} := \mathsf{D}^{-2} = \operatorname{diag}(d_1^{-2}, \dots, d_m^{-2})$. The dimensional analogue of W is $\hat{\mathsf{W}} := \hat{\mathsf{D}}^{-2} = \operatorname{diag}(H_1, \dots, H_m)$. In this Euclidean space, the eigenvectors of F are orthogonal, $(e_i, e_j)_{\mathsf{W}} = 0$ for $i \neq j$. The eigenvectors e_1, …, e_m are columns of the transition matrix P from the matrix F to the matrix Λ, $\Lambda := \mathsf{P}^{-1}\mathsf{F}\mathsf{P}$. It is obvious that $\operatorname{im}\mathsf{F} = \langle e_1, \dots, e_{m-1}\rangle$, $\ker\mathsf{F} = \langle e_m\rangle$, $\mathbb{R}^m = \operatorname{im}\mathsf{F} \oplus \ker\mathsf{F}$ and, moreover, $\ker\mathsf{F} \perp_{\mathsf{W}} \operatorname{im}\mathsf{F}$. Due to this decomposition, the Moore–Penrose inverse[3] F^+ of the matrix F interpreted as an operator on $\mathbb{R}^m$ can be simply represented in the decomposed form,

$$\mathsf{F}^+ = (\mathsf{F}|_{\operatorname{im}\mathsf{F}})^{-1} \oplus \mathsf{0}|_{\ker\mathsf{F}}, \tag{4.3}$$

where $\mathsf{0}$ denotes the zero $m \times m$ matrix or, equivalently, the zero operator on $\mathbb{R}^m$.

Remark 4.4. For real-world multi-layer quasi-geostrophic problems, the vertical coupling coefficients $f_{i,i-1}$ and $f_{i,i+1}$ are all in general pairwise different and, moreover, the parameter f_{12} is usually much larger than $f_{m,m-1}$, cf. Section 4.2.5 below. For simplifying the interaction terms for every single inner layer while allowing the model to capture the asymmetric global structure (e.g., a highly stratified surface layer overlying a weakly stratified deep layer), the constraint $f_{i,i-1} = f_{i,i+1} := \alpha_i$, $i = 2, \dots, m-1$ is often set, see, e.g., [33]. Additionally denoting $\alpha_1 := f_{12}$ and $\alpha_m := f_{m,m-1}$, we can decompose the matrix F in the form $\mathsf{F} = \mathsf{A}\mathsf{L}$, where L is the $m \times m$ tridiagonal matrix with the main diagonal $(-1, -2, \dots, -2, -1)$ and the same sub- and superdiagonals $(1, \dots, 1)$, i.e., the matrix of the discrete Laplacian operator for an interval with Neumann boundary conditions at both ends, and the matrix $\mathsf{A} := \operatorname{diag}(\alpha_1, \dots, \alpha_m)$ can be interpreted as a weight matrix for this operator. Then $d_i = \sqrt{\alpha_i/\alpha_1}$, but there are still no explicit closed-form expressions for nonzero eigenvalues and corresponding eigenvectors of the matrix F for general $(\alpha_1, \dots, \alpha_m)$. In the course of discretizing a uniform fluid model in the vertical direction with equal steps for further numerical analysis,

[3] Given a linear operator A on a finite-dimensional space V with inner product, its Moore–Penrose inverse A^+ can be defined in terms of the inverse of the restriction $\hat{\mathsf{A}}$ of A to $(\ker\mathsf{F})^\perp$ with codomain restricted to $\operatorname{im}\mathsf{A}$, $\mathsf{A}^+ v = \hat{\mathsf{A}}^{-1}\operatorname{proj}_{\operatorname{im}\mathsf{A}} v$ for any $v \in V$, where $\operatorname{proj}_{\operatorname{im}\mathsf{A}}$ denotes the operator of orthogonal projection on $\operatorname{im}\mathsf{A}$, and hence $\mathsf{A}^+(\operatorname{im}\mathsf{A})^\perp = \{0\}$. If $\ker\mathsf{A} \perp \operatorname{im}\mathsf{A}$, then we have simpler expression for the Moore–Penrose inverse of A, $\mathsf{A}^+ = (\mathsf{A}|_{\operatorname{im}\mathsf{A}})^{-1} \oplus \mathsf{0}|_{\ker\mathsf{F}}$, where $\mathsf{0}$ is the zero operator on V. Algorithms for finding the Moore–Penrose inverse of an operator are implemented in different packages for symbolic or numerical computations, like Maple and Sage or the package Numpy for Python.

the constructed multi-layer quasi-geostrophic problem is even more simplified. More specifically, since then $H_i = H_j$ and $\rho_{i+1} - \rho_i = \rho_{j+1} - \rho_j$, $i, j = 1, \dots, m-1$, all the vertical coupling coefficients $f_{i,i-1}$ and $f_{i,i+1}$ are the same, except the artificial ones $f_{1,0}$ and $f_{m,m+1}$, and hence the matrix F, its eigenvalues and eigenvectors take the form

$$\mathsf{F} = f_{12}\mathsf{L}, \quad \lambda_i = -4f_{12}\sin^2\frac{m-i}{2m}\pi, \quad e_i = \left(\cos\frac{m-i}{m}\Big(j-\frac{1}{2}\Big)\pi,\ j = 1, \dots, m\right)^{\mathsf{T}}.$$

Remark 4.5. In the theory of continuous-time Markov processes, the matrix F arises as the rate matrix (or infinitesimal generator) of a birth–death process [61, 133]. The spectral properties of such matrices have been studied extensively in the literature. In particular, Sasaki [133] identified exact conditions on the entries of F under which its eigenvectors can be explicitly expressed in terms of orthogonal polynomials from the (q-)Askey scheme [66, Chapters 9, 14]. We review and adapt these results in Section 4.A.

4.2.4 Barotropic and baroclinic modes

It is often convenient to expand ψ and q with respect to the eigenbasis of the matrix F,

$$\psi = \tilde{\psi}^k e_k, \quad q = \tilde{q}^k e_k, \quad \text{where} \quad \tilde{\psi}^k = \frac{(\psi, e_k)_{\mathsf{W}}}{(e_k, e_k)_{\mathsf{W}}}, \quad \tilde{q}^k = \frac{(q, e_k)_{\mathsf{W}}}{(e_k, e_k)_{\mathsf{W}}}.$$

In other words, $\tilde{\psi} := (\tilde{\psi}^1, \dots, \tilde{\psi}^m) = \mathsf{P}^{-1}\psi$ and $\tilde{q} := (\tilde{q}^1, \dots, \tilde{q}^m) = \mathsf{P}^{-1}q$. In contrast to the pairs (ψ^i, q^i), the pairs $(\tilde{\psi}^i, \tilde{q}^i)$ are decoupled in the sense that for each fixed i, the expression for the ith recombined vorticity $\tilde{q}^i$ involves the ith recombined stream function $\tilde{\psi}^i$ only,

$$\tilde{q}^i = \tilde{\psi}^i_{xx} + \tilde{\psi}^i_{yy} + \lambda_i\tilde{\psi}^i, \quad i = 1, \dots, m-1, \quad \tilde{q}^m = \tilde{\psi}^m_{xx} + \tilde{\psi}^m_{yy} + \beta y.$$

The expansion with respect to the eigenbasis of F allows one to separate the *barotropic mode*, which corresponds to the zero eigenvalue λ_m and represents the depth-averaged flow of the stratified fluid, from the *baroclinic modes*, which correspond to the nonzero eigenvalues $\lambda_1, \dots \lambda_{m-1}$ and represent movements of the fluid layers with respect to

each other. In terms of $\tilde{\psi}$ and $\tilde{q}$, the system (4.2) takes the form

$$\begin{aligned} &\tilde{q}_t + \mathsf{P}^{-1}\{\mathsf{P}\tilde{\psi}, \mathsf{P}\tilde{q}\} = 0,\\ &\tilde{q} := \tilde{\psi}_{xx} + \tilde{\psi}_{yy} + \Lambda\tilde{\psi} + (0, \dots, 0, \beta y)^{\mathsf{T}}. \end{aligned} \tag{4.4}$$

The decoupling between the components in the expression for $\tilde{q}$ leads to the decoupling in derivatives with respect to t, but this is not the case for the nonlinearity given by the Poisson bracket, where the coupling becomes even more complicated.

At the same time, most of the Lie submodels of the multi-layer quasi-geostrophic problem (4.2) are linear, and the vertical couplings for a number of them are given by modifications of the matrix F with diagonal summands. Eigenvectors of these modified matrices are in general different from those of the matrix F, but the corresponding eigenbases are orthogonal in the same weighted inner product $(\cdot,\cdot)_{\mathsf{W}}$. For each submodel with such vertical couplings, the expansion of the tuple of unknown invariant functions with respect to the associated eigenbasis leads to the total decoupling of the submodel, see Sections 4.8 and 4.9 below.

4.2.5 Numerical example

As a specific illustrative example of the multi-layer quasi-geostrophic problem, throughout the paper we consider the case of three layers ($m = 3$) with realistic geophysical data from [100, Table 1] for model parameters,

$$\begin{aligned} &f_0 = 1.0 \cdot 10^{-4}\ \mathrm{s}^{-1}, && H_1 = 6.0 \cdot 10^{2}\ \mathrm{m},\ g_1' = 2.0 \cdot 10^{-2}\ \mathrm{m}\cdot\mathrm{s}^{-2},\\ &\beta = 1.6 \cdot 10^{-11}\ (\mathrm{m}\cdot\mathrm{s})^{-1}, && H_2 = 1.4 \cdot 10^{3}\ \mathrm{m},\ g_2' = 3.0 \cdot 10^{-2}\ \mathrm{m}\cdot\mathrm{s}^{-2},\\ &\rho_0 = 1.0 \cdot 10^{3}\ \mathrm{kg}\cdot\mathrm{m}^{-3}, && H_3 = 2.0 \cdot 10^{3}\ \mathrm{m}. \end{aligned} \tag{4.5}$$

In particular, the plots of found exact solutions are constructed for the values (4.5). If $m = 3$, there are only four essential vertical coupling parameters, f_{12}, f_{21}, f_{23} and f_{32}. For the specific model data (4.5), they take the values

$$\begin{aligned} &f_{12} = \frac{f_0^2}{H_1 g_1'} \approx 8.33 \cdot 10^{-10}\ \mathrm{m}^{-2}, && f_{21} = \frac{f_0^2}{H_2 g_1'} \approx 3.57 \cdot 10^{-10}\ \mathrm{m}^{-2},\\ &f_{23} = \frac{f_0^2}{H_2 g_2'} \approx 2.38 \cdot 10^{-10}\ \mathrm{m}^{-2}, && f_{32} = \frac{f_0^2}{H_3 g_2'} \approx 1.67 \cdot 10^{-10}\ \mathrm{m}^{-2}. \end{aligned}$$

Denote $b := (b_1, b_2, b_3)^{\mathsf{T}}$ and $c := (c_1, c_2, c_3)^{\mathsf{T}}$. The vertical coupling matrix F, its eigenvalues, the corresponding eigenvectors, the diagonal entries of the matrix $\mathsf{D} := \operatorname{diag}(d_1, d_2, d_3)$ and the weighted inner product $(\cdot,\cdot)_{\mathsf{W}}$ are respectively given by

$$\mathsf{F} \approx 10^{-10} \cdot \begin{pmatrix} -8.33 & 8.33 & 0 \\ 3.57 & -5.95 & 2.38 \\ 0 & 1.67 & -1.67 \end{pmatrix}, \quad \begin{aligned} &\lambda_1 \approx -12.9 \cdot 10^{-10}, && e_1 \approx (1, -0.54, 0.08)^{\mathsf{T}}, \\ &\lambda_2 \approx -3.1 \cdot 10^{-10}, && e_2 \approx (1, 0.63, -0.74)^{\mathsf{T}}, \\ &\lambda_3 = 0, && e_3 := (1, 1, 1)^{\mathsf{T}}, \end{aligned}$$

$$d_1 = 1, \quad d_2 \approx 0.65, \quad d_3 \approx 0.55, \quad (b, c)_{\mathsf{W}} \approx b_1 c_1 + 2.33 b_2 c_2 + 3.33 b_3 c_3.$$

4.3 Conservation laws and Hamiltonian structure

We extend the well-known results on local conservation laws and the Hamiltonian structure of the β-plane vorticity equation [135] to the multi-layer quasi-geostrophic problem (4.2). Below we list the names of conservation laws of (4.2) jointly with related objects, which include their canonical characteristics,[4] equivalent canonical forms of their conserved currents, which are usually not unique, and the corresponding conserved values.

In this section, we interpret functionals in the spirit of [103, Section 7.1] as equivalence classes of differential functions up to adding total divergences and use the notation $\iint F[\psi]\,\mathrm{d}x\mathrm{d}y$ for them, where $F[\psi]$ is a differential function from the corresponding equivalence class. For such a functional to be turned into the associated physical value, we need to fix a domain of integration and reflect this choice in the notation.

Conservation of generalized total weighted circulations. $\lambda = \kappa(t)\mathsf{W}\bar{1}$,

$$\begin{aligned} &\kappa\big(0,\ (\bar{1}, \psi_{xt})_{\mathsf{W}} - (\psi_y, q)_{\mathsf{W}},\ (\bar{1}, \psi_{yt})_{\mathsf{W}} + (\psi_x, q)_{\mathsf{W}}\big) \\ &\sim \big(\kappa(\bar{1}, \triangle\psi)_{\mathsf{W}},\ -\kappa_t(\bar{1}, \psi_x)_{\mathsf{W}} - \kappa(\psi_y, q)_{\mathsf{W}},\ -\kappa_t(\bar{1}, \psi_y)_{\mathsf{W}} + \kappa(\psi_x, q)_{\mathsf{W}}\big) \\ &\sim \big(\kappa(\bar{1}, q)_{\mathsf{W}},\ -\kappa(\psi_y, q)_{\mathsf{W}} - \kappa_t(\bar{1}, \psi_x)_{\mathsf{W}},\ \kappa(\psi_x, q)_{\mathsf{W}} - (\kappa_t\bar{1}, \psi_y + \tfrac{1}{2}\beta y^2\bar{1})_{\mathsf{W}}\big). \end{aligned}$$

The first conserved current shows that this conservation law for each fixed κ can be

[4] Conservation-law characteristics [103, Section 4.3] are also called multipliers for conservation laws, see, e.g., [4, 5] and [20, Section 1.3.3].

interpreted as a "short" one whose density is zero. This explains the presence and the negligibility of the multiplier κ that is an arbitrary sufficiently smooth function of t. The second conserved current multiplied by $H_1/(H_1+\cdots+H_m)$ can be interpreted as that representing the conservation of the generalized total weighted circulation, which is the weighted average of the generalized total circulations on single layers,

$$\iint \frac{\kappa(\bar{1},\triangle\psi)_{\hat{\mathsf{W}}}}{H_1+\cdots+H_m}\,\mathrm{d}x\mathrm{d}y=\sum_{j=1}^{m}\frac{H_j}{H_1+\cdots+H_m}\iint \kappa\triangle\psi^j\,\mathrm{d}x\mathrm{d}y.$$

Here and in what follows, we use the obvious equality $(\cdot,\cdot)_{\hat{\mathsf{W}}}=H_1(\cdot,\cdot)_{\mathsf{W}}$. The third conserved current leads to the representation of this conserved functional in terms of the potential vorticity, and thus this functional can be re-interpreted as a time-dependent distinguished functional [103, Section 7.2] for the Hamiltonian operator $\mathfrak{H}$ of the system (4.2), see below on a Hamiltonian structure for (4.2).

Conservation of generalized total zonal momentums. $\lambda=-\chi(t)y\mathsf{W}\bar{1}$,

$$\begin{aligned}
&\chi\big(0,\ y(\psi_y,q)_{\mathsf{W}}+\tfrac12|\psi_x|^2_{\mathsf{W}}+\tfrac12|\psi_y|^2_{\mathsf{W}}+\tfrac12(\psi,\mathsf{F}\psi)_{\mathsf{W}}+y(\bar{1},\beta\psi-\psi_{tx})_{\mathsf{W}},\\
&\quad -(\psi_x,yq-\psi_y)_{\mathsf{W}}-(\bar{1},y\psi_{ty}-\psi_t)_{\mathsf{W}}\big)\\
&\sim\big(-(\chi y\bar{1},\triangle\psi)_{\mathsf{W}},\\
&\quad \chi y(\psi_y,q)_{\mathsf{W}}+\tfrac12\chi(|\psi_x|^2_{\mathsf{W}}+|\psi_y|^2_{\mathsf{W}}+(\psi,\mathsf{F}\psi)_{\mathsf{W}})+y(\bar{1},\beta\chi\psi+\chi_t\psi_x)_{\mathsf{W}},\\
&\quad -\chi(\psi_x,yq-\psi_y)_{\mathsf{W}}+\chi_t(\bar{1},y\psi_y-\psi)_{\mathsf{W}}\big)\\
&\sim\big(\chi(\bar{1},\psi_y)_{\mathsf{W}},\ \chi\big(y(\psi_y,q)_{\mathsf{W}}-\tfrac12|\psi_x|^2_{\mathsf{W}}-\tfrac12|\psi_y|^2_{\mathsf{W}}-\tfrac12(\psi,\mathsf{F}\psi)_{\mathsf{W}}-y(\bar{1},\psi_{tx}+\beta\psi)_{\mathsf{W}}\big),\\
&\quad -\chi(\psi_x,yq-\psi_y)_{\mathsf{W}}-(\bar{1},\chi y\psi_{ty}+\chi_t\psi)_{\mathsf{W}}\big).
\end{aligned}$$

Analogously, the first conserved current shows that this conservation law for each fixed χ can be interpreted as a "short" one whose density is zero. This again explains the presence and the negligibility of the multiplier χ that is an arbitrary sufficiently smooth function of t. At the same time, in contrast to the previous family of conservation laws, the short form for this conservation law exists only in terms of the stream function, becoming nonlocal in terms of the fluid velocity. The density of the second conserved current is nicely represented in terms of the vorticity. For proper physical interpretation, we multiply the third conserved current by ρ_0H_1. The first component of the obtained tuple is the density of the generalized total zonal momentum, which

is the sum of the total zonal momenta on single layers,

$$\iint \chi\rho_0(\bar{1},\psi_y)_{\hat{\mathsf{W}}}\,\mathrm{d}x\mathrm{d}y = \sum_{i=1}^{m}\iint \chi\rho_0 H_i\psi^i_y\,\mathrm{d}x\mathrm{d}y.$$

Conservation of total energy. $\lambda = -\mathsf{W}\psi$,

$$\begin{aligned}&\big(\tfrac12(|\psi_x|^2_{\mathsf{W}}+|\psi_y|^2_{\mathsf{W}}-(\psi,\mathsf{F}\psi)_{\mathsf{W}}),\ -(\psi,\psi_{tx}+\tfrac12\psi\odot q_y)_{\mathsf{W}},\ -(\psi,\psi_{ty}-\tfrac12\psi\odot q_x)_{\mathsf{W}}\big)\\ &\sim \tfrac12\big(-(\psi,q-\beta y\bar{1})_{\mathsf{W}},\ -(\psi,\psi\odot q_y+\psi_{xt})_{\mathsf{W}}+(\psi_x,\psi_t)_{\mathsf{W}},\\ &\qquad (\psi,\psi\odot q_x+\psi_{yt})_{\mathsf{W}}-(\psi_y,\psi_t)_{\mathsf{W}}\big).\end{aligned}$$

After multiplying the first conserved current by $\rho_0 H_1$, its first component is the density of the total energy

$$\begin{aligned}\mathcal{E} &:= \iint \frac{\rho_0}{2}\big(|\psi_x|^2_{\hat{\mathsf{W}}}+|\psi_y|^2_{\hat{\mathsf{W}}}-(\psi,\mathsf{F}\psi)_{\hat{\mathsf{W}}}\big)\mathrm{d}x\mathrm{d}y\\ &= \sum_{i=1}^{m}\iint \frac{\rho_0 H_i}{2}\big((\nabla\psi^i)^2-\psi^i(\mathsf{F}\psi)^i\big)\mathrm{d}x\mathrm{d}y,\end{aligned}$$

which can of course be represented in several ways, e.g., as the sum of the total kinetic energy and the total energy of interaction between neighboring layers or as the sum of total energies of single layers.

The system (4.1) is Hamiltonian with a non-canonical Hamiltonian structure under treating the potential vorticity q as the tuple of unknown functions of this system, which is a common approach to models of fluid dynamics, cf. [103, Example 7.10]. Thus, a Hamiltonian representation for (4.1) is

$$q_t = \mathfrak{H}\frac{\delta\mathcal{H}}{\delta q},$$

where $\delta/\delta q := (\delta/\delta q^1,\dots,\delta/\delta q^m)^{\mathsf{T}}$, $\delta/\delta q^i$ denotes the variational derivative with respect to q^i,

$$\mathfrak{H} := \mathsf{W}^{-1}\operatorname{diag}(q_y\mathrm{D}_x - q_x\mathrm{D}_y) = \operatorname{diag}\big((\mathsf{D}^2 q_y)\mathrm{D}_x-(\mathsf{D}^2 q_x)\mathrm{D}_y\big)$$

is the associated Hamiltonian differential operator with $\mathsf{D} := \operatorname{diag}(d_1,\dots,d_m)$ and the total derivative operators D_x and D_y with respect to x and y, respectively, and the

Hamiltonian $\mathcal{H}$ of is given by the "mathematical total energy" (without the multiplier $\rho_0 H_1$),

$$\begin{aligned}\mathcal{H} &:= \iint \frac{1}{2}\big(|\psi_x|^2_{\mathsf{W}} + |\psi_y|^2_{\mathsf{W}} - (\psi, \mathsf{F}\psi)_{\mathsf{W}}\big)\mathrm{d}x\mathrm{d}y \\ &= \iint \sum_{i=1}^{m} \frac{1}{2d_i^2}\big((\nabla\psi^i)^2 - \psi^i(\mathsf{F}\psi)^i\big)\mathrm{d}x\mathrm{d}y = -\frac{1}{2}\iint (q - \beta y\bar{1}, \psi)_{\mathsf{W}}\,\mathrm{d}x\mathrm{d}y\end{aligned}$$

with the constant parameters d_i defined in Section 4.2.3, cf. [33, Section 2.1]. (The last expression of the Hamiltonian $\mathcal{H}$, which involves the potential vorticity q, justifies the presentation of the second conserved current.) Hence $\delta\mathcal{H}/\delta q = -\mathsf{W}\psi$. The corresponding Lie–Poisson vorticity bracket is given by

$$\{\mathcal{F}, \mathcal{G}\} := \sum_{i=1}^{m} d_i^2 \iint q^i \left\{\frac{\delta\mathcal{F}}{\delta q^i}, \frac{\delta\mathcal{G}}{\delta q^i}\right\}\mathrm{d}x\mathrm{d}y,$$

where $\mathcal{F}$ and $\mathcal{G}$ are interpreted as functionals of q. The constructed Hamiltonian structure extends the (one-layer) vorticity equation [135] to the general multi-layer model (4.1).

Conservation of generalized potential enstrophies on the ith layer, $i \in \{1, \dots, m\}$.

$$\lambda = \frac{\mathrm{d}\Phi^i}{\mathrm{d}q^i}\delta_i, \quad (\Phi^i, -\psi^i_y\Phi^i, \psi^i_x\Phi^i),$$

where $\delta_i = (\delta_{ij}, j = 1, \dots, m)^{\mathsf{T}}$ with the Kronecker delta δ_{ij}, Φ^i is an arbitrary smooth function of q^i. For each fixed i, particular elements of the corresponding family of conserved functionals

$$\mathcal{C}_{i,\Phi^i} := \iint \Phi^i(q^i)\mathrm{d}x\mathrm{d}y$$

are the total potential circulation on the ith layer, where $\Phi^i = q^i$, and the potential enstrophy on the ith layer, where $\Phi^i = \frac{1}{2}(q^i)^2$. For any (i, Φ^i), the conserved functional $\mathcal{C}_{i,\Phi^i}$ is a Casimir (or, in terminology of [103, Section 7.2], distinguished) functional for the Hamiltonian operator $\mathfrak{H}$, and the space of Casimir functionals of $\mathfrak{H}$ is exhausted by the sums $\mathcal{C}_{1,\Phi^1} + \dots + \mathcal{C}_{m,\Phi^m}$, where Φ^i runs through the set of arbitrary smooth functions of q^i for each $i \in \{1, \dots, m\}$. The space of time-dependent distinguished functionals of the Hamiltonian operator $\mathfrak{H}$ that are conserved for the

system (4.2) is exhausted by the sums of generalized potential enstrophies and generalized total weighted circulations, $\mathcal{C}_{1,\Phi^1}+\dots+\mathcal{C}_{m,\Phi^m}+\iint\kappa(\bar{1},q)_{\mathsf{W}}\,\mathrm{d}x\mathrm{d}y$, where in addition κ runs through the set of arbitrary smooth functions of t.

4.4 Lie invariance algebra and Hamiltonian symmetries

To find the maximal Lie invariance algebra $\mathfrak{g}$ of the multi-layer quasi-geostrophic problem (4.1), we employ the infinitesimal invariance criterion, see, e.g., [19, Theorem 4.1.1-1] or [103, Theorem 2.31]. For low values of the number of layers m, the Lie invariance algebra $\mathfrak{g}$ can be found by one of the numerous packages for symbolically computing Lie symmetries, like DESOLV [27] or GeM [29], but this is not the case for an arbitrary value of m. Below we construct the algebra $\mathfrak{g}$ by hand. Since the straightforward performance of this construction is highly complicated and inefficient, we use two tricks that significantly arrange and simplify the computations.

The first of these tricks is the formal replacement of the real independent variables (x,y) by the complex conjugate variables $z=x+\mathrm{i}y$ and $\bar z=x-\mathrm{i}y$, where i is the imaginary unit.[5] Throughout this section, bar denotes the complex conjugation. In the independent variables $(t,z,\bar z)$, the system (4.1) takes the form

$$\begin{aligned}&q^i_t-2\mathrm{i}(\psi^i_z q^i_{\bar z}-\psi^i_{\bar z}q^i_z)=0,\\ &q^i:=4\psi^i_{z\bar z}+f_{i,i-1}(\psi^i-\psi^{i-1})-f_{i,i+1}(\psi^{i+1}-\psi^i)+\frac{\beta}{2\mathrm{i}}(z-\bar z),\end{aligned}\tag{4.6}$$

where $\psi^i=\psi^i(t,z,\bar z)$ are real-valued functions of their arguments, $f_{1,0}=f_{m,m+1}=0$ and the rest of the constants $f_{i,i-1}$, $f_{i,i+1}$ are nonzero, $i=1,\dots,m$. Having found the maximal Lie invariance algebra for the system (4.6) and pulling it back with respect to the change of variables $(t,x,y,\psi)\mapsto(t,z,\bar z,\psi)$, we obtain the maximal Lie invariance algebra of the system (4.1). There are two advantages of the system (4.6) in comparison with the system (4.1). The expansion of the former is more concise due to

[5] In fact, the variables z and $\bar z$ are not independent to each other in the canonical sense since they are uniquely related by the complex conjugation. However, the equalities $\partial_{\bar z}z=\partial_z\bar z=0$ and $\partial_z z=\partial_{\bar z}\bar z=1$ with the Wirtinger derivatives $\partial_z=\frac12(\partial_x-\mathrm{i}\partial_y)$ and $\partial_{\bar z}=\frac12(\partial_x+\mathrm{i}\partial_y)$ allow one to formally treat z and $\bar z$ as independent variables and to simultaneously indicate them as arguments of relevant functions instead of (x,y).

replacing each of the Laplacians $\psi^i_{xx}+\psi^i_{yy}$ by the single mixed derivative $\psi^i_{z\bar z}$ (up to the multiplier 4). Moreover, in contrast to the system (4.1), its counterpart (4.6) allows for confining to its solution set while preserving the parity of the variables z and $\bar z$.

Notation. In addition in this section, the Greek indices μ, ν, κ and λ run through the set $\{t,z,\bar z\}$.

The second trick consists in introducing a chain of nested superclasses that contain the class (4.6). The largest considered class $\mathcal{V}$ is constituted by the systems of third-order partial differential equations of the form

$$\mathcal{V}_{F^i}\colon\quad \psi^i_{tz\bar z}+F^i(t,z,\bar z,\psi,\psi_t,\psi_z,\psi_{\bar z},\psi_{z\bar z},\psi_{zz\bar z},\psi_{z\bar z\bar z})=0 \tag{4.7}$$

for the real-valued unknown functions $\psi=(\psi^1,\dots,\psi^m)$ of independent variables $(t,z,\bar z)$. In view of their dependence at most on $(t,z,\bar z,\psi,\psi_t,\psi_z,\psi_{\bar z},\psi_{z\bar z},\psi_{zz\bar z},\psi_{z\bar z\bar z})$, the arbitrary elements F^1, …, F^m can be interpreted as real-valued third-order differential functions of ψ that run through the solution set of the auxiliary system of differential equations

$$F^i_{\psi^j_{\mu\mu}}=F^i_{\psi^j_{t\mu}}=F^i_{\psi^j_{t\nu\lambda}}=F^i_{\psi^j_{zzz}}=F^i_{\psi^j_{\bar z\bar z\bar z}}=0.$$

The second, narrower class $\hat{\mathcal{V}}$ consists of the systems of the form

$$\hat{\mathcal{V}}_{(a_i,\hat F^i)}\colon\quad \psi^i_{tz\bar z}+a_i\psi^i_{\bar z}\psi^i_{zz\bar z}+\bar a_i\psi^i_z\psi^i_{z\bar z\bar z}+\hat F^i(t,z,\bar z,\psi,\psi_t,\psi_z,\psi_{\bar z},\psi_{z\bar z})=0. \tag{4.8}$$

The class $\hat{\mathcal{V}}$ can be considered as the subclass of the class $\mathcal{V}$ that is singled out by the auxiliary system

$$\begin{aligned}&F^i_{\psi^j_{zz\bar z}}=F^i_{\psi^j_{z\bar z\bar z}}=0,\quad i\neq j,\\ &\left(\frac{F^i_{\psi^i_{zz\bar z}}}{\psi^i_{\bar z}}\right)_\upsilon=\left(\frac{F^i_{\psi^i_{z\bar z\bar z}}}{\psi^i_z}\right)_\upsilon=0,\quad \upsilon\in\{\mu,\psi_\mu,\psi_{\mu\nu},\psi_{\mu\nu\kappa}\}.\end{aligned}$$

We can also reparameterize the class $\hat{\mathcal{V}}$ and declare, instead of $F^i=a_i\psi^i_{\bar z}\psi^i_{zz\bar z}+\bar a_i\psi^i_z\psi^i_{z\bar z\bar z}+\hat F^i$, the nonzero complex constants a_i and the real-valued second-order differential functions $\hat F^i$ to be the arbitrary elements, which thus satisfy the auxiliary

system

$$a_{i,\upsilon}=0,\quad \upsilon\in\{\mu,\psi_\mu,\psi_{\mu\nu},\psi_{\mu\nu\kappa}\},\quad a_i\neq 0,\quad \hat F^i_{\psi_{\mu\nu\kappa}}=\hat F^i_{\psi_{t\mu}}=\hat F^i_{\psi_{zz}}=\hat F^i_{\psi_{\bar z\bar z}}=0.$$

The introduction of the classes $\hat{\mathcal{V}}$ and $\mathcal{V}$ allows us to simplify the computation of the maximal Lie invariance algebra of the system of the form (4.6) via splitting it into natural steps in the following way. Applying the infinitesimal invariance criterion to an arbitrary system from the most general class $\mathcal{V}$, we derive the principal constraints on the components of Lie symmetry vector fields, which hold for any system from this class and, consequently, for any system belonging to $\hat{\mathcal{V}}$. Using the obtained constraints, we proceed with successively specifying the form of Lie symmetry vector fields for systems from the class $\hat{\mathcal{V}}$ and for systems of the form (4.6). As a result, we need to expand the cumbersome expressions for F^i in the latter systems only on the very last stage of computing, which is one of the sources of the caused simplification. Moreover, sequentially deriving more and more restrictions on the components of the Lie symmetry vector fields allows us to avoid solving at once a vast overdetermined system of determining equations, which arise in the straightforward application of the infinitesimal invariance criterion to systems of the form (4.6).

The suggested approach results in the next chain of lemmas.

Lemma 4.6. *Any Lie symmetry vector field of an arbitrary system from the class $\mathcal{V}$ is of the form*

$$Q=\tau(t)\partial_t+\xi(t,z)\partial_z+\bar\xi(t,\bar z)\partial_{\bar z}+\big(\Psi^{jk}(t)\psi^k+\Phi^j(t,z,\bar z)\big)\partial_{\psi^j},$$

where τ, Ψ^{jk} and Φ^j are smooth real-valued functions of their arguments, and ξ is a smooth complex-valued function of (t,z).

Proof. In order to find the above form, we start from a vector field Q of the most general form on the space with the coordinates $(t,z,\bar z,\psi)$,

$$Q=\tau(t,z,\bar z,\psi)\partial_t+\xi(t,z,\bar z,\psi)\partial_z+\bar\xi(t,z,\bar z,\psi)\partial_{\bar z}+\eta^j(t,z,\bar z,\psi)\partial_{\psi^j},$$

where τ and $\eta^1,\dots,\eta^m$ (resp. ξ) are real-valued (resp. is complex-valued) smooth function(s) of $(t,z,\bar z,\psi)$. The necessary and sufficient condition for Q to be a Lie symmetry vector field of a fixed system $\mathcal{V}_F$: $\psi^i_{tz\bar z}+F^i=0$ from the class $\mathcal{V}$ with

a fixed tuple of parameter-functions $F=(F^1,\dots,F^m)$ is given by the infinitesimal invariance criterion, see, e.g., [103, Theorem 2.31]. In the specific case of Q and $\mathcal{V}_F$, the criterion implies the condition

$$Q_{(3)}(\psi^i_{tz\bar z}+F^i)|_{\mathcal{V}_F}=0,$$

where $Q_{(3)}$ is the third prolongation of Q,

$$Q_{(3)}=Q+\eta^{j,\mu}\partial_{\psi^j_\mu}+\eta^{j,\mu\nu}\partial_{\psi^j_{\mu\nu}}+\eta^{j,\mu\nu\kappa}\partial_{\psi^j_{\mu\nu\kappa}},$$

and the notation $|_{\mathcal{V}_F}$ means that the condition holds only on the solution set of the system $\mathcal{V}_F$. Here and in what follows, tuples of Greek indices are assumed unordered. The coefficients $\eta^{j,\mu}$, $\eta^{j,\mu\nu}$ and $\eta^{j,\mu\nu\kappa}$ are defined by the general prolongation formulas for vector fields. In particular,

$$\eta^{i,\mu\nu\kappa}=\mathrm{D}_\mu\mathrm{D}_\nu\mathrm{D}_\kappa(\eta^i-\tau\psi^i_t-\xi\psi^i_z-\bar\xi\psi^i_{\bar z})+\tau\psi^i_{t\mu\nu\kappa}+\xi\psi^i_{z\mu\nu\kappa}+\bar\xi\psi^i_{\bar z\mu\nu\kappa}.$$

Here D_μ denotes the total derivative operator with respect to $\mu\in\{t,z,\bar z\}$,

$$\mathrm{D}_\mu=\partial_\mu+\psi^j_\mu\partial_{\psi^j}+\psi^j_{\mu\nu}\partial_{\psi^j_\nu}+\psi^j_{\mu\nu\kappa}\partial_{\psi^j_{\nu\kappa}}+\psi^j_{\mu\nu\kappa\lambda}\partial_{\psi^j_{\nu\kappa\lambda}}+\cdots.$$

Expanding the invariance condition for $\mathcal{V}_F$, we obtain

$$\begin{aligned}&\eta^{i,tz\bar z}+\eta^{j,zz\bar z}F^i_{\psi^j_{zz\bar z}}+\eta^{j,z\bar z\bar z}F^i_{\psi^j_{z\bar z\bar z}}+\eta^{j,z\bar z}F^i_{\psi^j_{z\bar z}}\\&+\eta^{j,\mu}F^i_{\psi^j_\mu}+\eta^jF^i_{\psi^j}\ +\tau F^i_t+\xi F^i_z+\bar\xi F^i_{\bar z}=0\end{aligned}\tag{4.9}$$

whenever $\psi^i_{tz\bar z}+F^i=0$. After further expanding the system (4.9) and substituting $-F^i$ for $\psi^i_{tz\bar z}$, we split it with respect to the third- and second-order derivatives of ψ that are not among the arguments of F; the other jet variables are not appropriate to be used in the course of splitting. More specifically, separately collecting coefficients of summands involving $\psi^i_{tt\bar z}$ and ψ^i_{ttz}, $\psi^i_{t\bar z\bar z}$ and ψ^i_{tzz}, $\psi^i_{t\bar z}$ and ψ^i_{tz}, each of which originates only from the expansion of $\eta^{i,tz\bar z}$, results in the determining equations

$$\mathrm{D}_z\tau=\mathrm{D}_{\bar z}\tau=0,\quad \mathrm{D}_z\bar\xi=\mathrm{D}_{\bar z}\xi=0,\quad \eta^i_{\psi^i\psi^j}=\eta^i_{z\psi^i}=\eta^i_{\bar z\psi^i}=0,$$

respectively. The simultaneous integration of these equations gives the required form

of the components of Q. $\square$

In other words, for an arbitrary system from the class $\mathcal{V}$, any of its Lie symmetry vector fields is projectable to the space coordinatized by the independent variables $(t,z,\bar z)$ and is affine in ψ.

Lemma 4.7. *Any Lie symmetry vector field of an arbitrary system from the class $\hat{\mathcal{V}}$ is of the form*

$$\begin{aligned}Q&=\tau\partial_t+\big(\gamma z+\delta\big)\partial_z+\big(\bar\gamma\bar z+\bar\delta\big)\partial_{\bar z}\\&\quad+\big((\gamma+\bar\gamma-\tau_t)\psi^j+\tfrac12(a_j^{-1}\gamma_t+\bar a_j^{-1}\bar\gamma_t)z\bar z+a_j^{-1}\delta_t\bar z+\bar a_j^{-1}\bar\delta_t z+\phi^j\big)\partial_{\psi^j},\end{aligned}$$

where $\tau=\tau(t)$ and $\phi^j=\phi^j(t)$ (resp. $\gamma=\gamma(t)$ and $\delta=\delta(t)$) are real-valued (resp. complex-valued) smooth functions of t, and $\operatorname{Arg}\gamma_t=\pm\operatorname{Arg}a_j$ with Arg *denoting the principal value of the corresponding argument.*

Proof. The class $\hat{\mathcal{V}}$ is a subclass of $\mathcal{V}$. Hence we can substitute the expressions for the components of Lie symmetry vector fields from Lemma 4.6 for the arbitrary elements F^i within the subclass $\hat{\mathcal{V}}$, $F^i=a_i\psi^i_{\bar z}\psi^i_{zz\bar z}+\bar a_i\psi^i_z\psi^i_{z\bar z\bar z}+\hat F^i$, into (4.9) to derive the condition of invariance of a system in $\hat{\mathcal{V}}$ with respect a vector field Q,

$$\begin{aligned}&\eta^{i,tz\bar z}+a_i\eta^{i,zz\bar z}\psi^i_{\bar z}+\bar a_i\eta^{i,z\bar z\bar z}\psi^i_z+a_i\eta^{i,\bar z}\psi^i_{zz\bar z}+\bar a_i\eta^{i,z}\psi^i_{z\bar z\bar z}\\&\quad+\eta^{j,z\bar z}\hat F^i_{\psi^j_{z\bar z}}+\eta^{j,\alpha}\hat F^i_{\psi^j_\alpha}+\eta^j\hat F^i_{\psi^j}+\tau\hat F^i_t+\xi\hat F^i_z+\bar\xi\hat F^i_{\bar z}=0\end{aligned}\tag{4.10}$$

whenever $\psi^i_{tz\bar z}+a_i\psi^i_{\bar z}\psi^i_{zz\bar z}+\bar a_i\psi^i_z\psi^i_{z\bar z\bar z}+\hat F^i=0$.

We expand the invariance condition (4.10), substitute $-a_i\psi^i_{\bar z}\psi^i_{zz\bar z}-\bar a_i\psi^i_z\psi^i_{z\bar z\bar z}-\hat F^i$ for the derivative $\psi^i_{tz\bar z}$ whenever it appears and split the obtained equality with respect to the third-order derivatives $\psi^j_{zz\bar z}$ and $\psi^j_{z\bar z\bar z}$. (The rest of the jet variables are not appropriate for splitting since they are in the arguments of $\hat F^i$ or have been used in the course of splitting in Lemma 4.6.) The equations corresponding summands involving $\psi^j_{zz\bar z}$ or $\psi^j_{z\bar z\bar z}$ can be further split with respect to $(\psi^k_z,\psi^k_{\bar z})$, which leads to the following system on the parameters in η^i:

$$\Psi^{ij}=-\tau_t+\xi_z+\bar\xi_{\bar z},\quad i=j,\qquad \Psi^{ij}=0,\quad i\neq j,\tag{4.11a}$$

$$a_i\Phi^i_{\bar z}=\xi_t,\quad \bar a_i\Phi^i_z=\bar\xi_t.\tag{4.11b}$$

Since Ψ^{ii} depends only on t, differentiating (4.11a) with respect to z results in the equation $\xi_{zz}=0$, whose general solution is $\xi=\gamma z+\delta$, where $\gamma=\gamma(t)$ and $\delta=\delta(t)$ are smooth complex-valued functions. Therefore, $\Psi^{ii}=-\tau_t+\gamma+\bar\gamma$, and the equations (4.11b) are expanded as

$$a_i\Phi^i_{\bar z}=\gamma_t z+\delta_t,\quad \bar a_i\Phi^i_{z}=\bar\gamma_t\bar z+\bar\delta_t. \tag{4.11c}$$

The compatibility condition $\Phi^i_{\bar z z}=\Phi^i_{z\bar z}$ reduces to $\bar a_i\gamma_t=a_i\bar\gamma_t$, which means that $\bar a_i\gamma_t\in\mathbb{R}$, i.e., $\operatorname{Arg}\gamma_t=\pm\operatorname{Arg}a_i$. Integrating (4.11c) leads to the required form of the vector field Q. □

Any system of the form (4.6) belongs to the class $\hat{\mathcal{V}}$ and expands into

$$\begin{aligned}&\mathcal{M}^{\rm c}_\theta\colon\quad \psi^i_{tz\bar z}-2{\rm i}(\psi^i_z\psi^i_{z\bar z\bar z}-\psi^i_{\bar z}\psi^i_{zz\bar z})+\check F^i=0,\quad i=1,\dots,m,\\ &\qquad\check F^i:=\frac{\beta}{4}(\psi^i_z+\psi^i_{\bar z})+\frac14\sum_{k\in I_i}f_{ik}\big(\psi^i_t-\psi^k_t-2{\rm i}(\psi^k_z\psi^i_{\bar z}-\psi^i_z\psi^k_{\bar z})\big)\end{aligned} \tag{4.12}$$

with $I_i:=\{i-1,i+1\}$, $\beta,f_{i,i-1},f_{i,i+1}\neq0$, except $f_{1,0}:=0$ and $f_{m,m+1}=0$. Recall that $\theta:=(f_{i+1,i},f_{i,i+1},i=1,\dots,m-1,\beta)$.

Lemma 4.8. *The maximal Lie invariance algebra $\mathfrak{g}^{\rm c}$ of any system $\mathcal{M}^{\rm c}_\theta$ of the form (4.12) is spanned by the vector fields*

$$\begin{aligned}&\partial_t,\quad \partial_z-\partial_{\bar z},\quad \chi(t)(\partial_z+\partial_{\bar z})-\frac{1}{2{\rm i}}(z-\bar z)\chi_t\big(\partial_{\psi^1}+\dots+\partial_{\psi^m}\big),\\ &\partial_{\psi^1},\ \dots,\ \partial_{\psi^m},\quad \kappa(t)\big(\partial_{\psi^1}+\dots+\partial_{\psi^m}\big),\end{aligned}$$

where χ and κ are arbitrary real-valued smooth functions of t.

Proof. We use the fact the system $\mathcal{M}^{\rm c}_\theta$ belongs to the class $\hat{\mathcal{V}}$ and specify Lemma 4.7 for $\mathcal{M}^{\rm c}_\theta$, substituting $\check F^i$ and 2i for $\hat F^i$ and a_i, respectively. In particular, we proceed with further deriving the constraints on the parameter functions τ, γ, δ and ϕ^j from the specified invariance condition (4.10),

$$\eta^{i,tz\bar z}+2{\rm i}\eta^{i,zz\bar z}\psi^i_{\bar z}-2{\rm i}\eta^{i,z\bar z\bar z}\psi^i_z+2{\rm i}\eta^{i,\bar z}\psi^i_{zz\bar z}-2{\rm i}\eta^{i,z}\psi^i_{z\bar z\bar z}+\sum_{j=i-1}^{i+1}\eta^{j,\mu}\check F^i_{\psi^j_\mu}=0 \tag{4.13}$$

whenever $\psi^i_{tz\bar z}+2{\rm i}\psi^i_{zz\bar z}-2{\rm i}\psi^i_{z\bar z\bar z}+\check F^i=0$.

After expanding, substituting $-2\mathrm{i}\psi^i_{zz\bar z}+2\mathrm{i}\psi^i_{z\bar z\bar z}-\check F^i$ for $\psi^i_{tz\bar z}$ and simplifying, the system (4.13) does not contain the second- and third-order derivatives of ψ and can be split with respect to the first-order derivatives of ψ, which results in the equations

$$\begin{aligned}&f_{i,i-1}(\bar\gamma+\tau_t)=0,\quad f_{i,i+1}(\gamma+\tau_t)=0,\quad \beta(\bar\gamma+\tau_t)=0,\quad \beta(\gamma+\tau_t)=0,\\ &f_{i,i-1}(\gamma+\bar\gamma)=0,\quad f_{i,i+1}(\gamma+\bar\gamma)=0.\end{aligned}\tag{4.14}$$

$$f_{i,i-1}(\phi^{i-1}_t-\phi^i_t)+f_{i,i+1}(\phi^{i+1}_t-\phi^i_t)=\frac{\mathrm{i}}{2}\beta(\bar\gamma_t\bar z-\gamma_t z+\bar\delta_t-\delta_t).\tag{4.15}$$

In view of the fact that all constants $f_{i,i-1}$, $f_{i,i+1}$ (positive) and β are nonzero, the equations (4.14) imply that $\gamma=-\tau_t$, $\bar\gamma=-\tau_t$ and $\gamma=-\bar\gamma$, which in turn means $\gamma=\bar\gamma=\tau_t=0$.

We consider the system (4.15) as a system of linear (algebraic) equations with respect to the unknowns $(\phi^1_t,\phi^2_t,\dots,\phi^m_t)$. Its coefficient matrix A is given by $A:=(f_{ij})_{i,j=1}^m$, where $f_{ii}=-f_{i,i-1}-f_{i,i+1}$ and $f_{ij}=0$ when $|i-j|>1$. The rank of A is equal to $m-1$, and the linear span of the columns of A over $\mathbb{R}$ coincides with that of the columns v_j, $j\in\{1,\dots,m-1\}$ with the only nonzero entries $v_{j,j}=-f_{j,j+1}$ and $v_{j,j+1}=f_{j+1,j}$. In view of the Rouché–Capelli theorem and the consistency of the system (4.15), the rank of the augmented matrix of this system is equal to the rank of A, i.e., the column $\frac{\mathrm{i}}{2}\beta(\bar\delta_t-\delta_t)(1,\dots,1)^{\mathsf T}$, whose entries are real, belongs to the linear span $\langle v_1,\dots,v_{m-1}\rangle$ over $\mathbb{R}$. Suppose that $\bar\delta_t\neq\delta_t$. Then the above claim also holds true for the column $(1,\dots,1)^{\mathsf T}$, which is equivalent to the consistency of the system

$$\begin{aligned}&-\lambda_1f_{1,2}=1,\quad -\lambda_2f_{2,3}+\lambda_1f_{2,1}=1,\quad \dots,\\ &\qquad -\lambda_{m-1}f_{m-1,m}+\lambda_{m-2}f_{m-1,m-2}=1,\quad \lambda_{m-1}f_{m,m-1}=1\end{aligned}$$

with respect to the unknowns λ_1, …, λ_{m-1}. Successively solving the first $(m-1)$ equations, we obtain the solution in the recursive form

$$\lambda_1=-\frac{1}{f_{1,2}},\quad \lambda_j=-\frac{1-\lambda_{j-1}f_{j,j-1}}{f_{j,j+1}},\quad j\in\{2,\dots,m-1\},$$

which inductively implies $\lambda_1\leqslant 0$, …, $\lambda_{m-1}\leqslant 0$. However, this contradicts the last equation of the system. Hence $\delta_t=\bar\delta_t$, i.e., $\delta=\chi+\mathrm{i}c$. This means that in fact the right-hand sides of the equations of the system (4.15) vanish, and its general solution

is $\phi^i = \kappa + b^i$. Here $\chi = \chi(t)$ and $\kappa = \kappa(t)$ are arbitrary smooth real-valued functions of t, and b^i and c are arbitrary real constants. $\square$

Pulling back the vectors fields given in Lemma 4.8 with respect to the change of variables $\tilde{t} = t$, $z = x + \mathrm{i}y$, $\bar{z} = x - \mathrm{i}y$, we obtain a spanning set of vector fields for the maximal Lie invariance algebra of the system (4.1).

Theorem 4.9. *The maximal Lie invariance algebra $\mathfrak{g}$ of the system (4.1) is spanned by the vector fields*

$$\begin{aligned} &\mathcal{P}^t = \partial_t, \quad \mathcal{P}^y = \partial_y, \quad \mathcal{P}^x(\chi) = \chi(t)\partial_x - y\chi_t(\partial_{\psi^1} + \cdots + \partial_{\psi^m}),\\ &\mathcal{J}^1 = \partial_{\psi^1}, \quad \dots, \quad \mathcal{J}^m = \partial_{\psi^m}, \quad \mathcal{Z}(\kappa) = \kappa(t)(\partial_{\psi^1} + \cdots + \partial_{\psi^m}), \end{aligned} \tag{4.16}$$

where χ and κ are arbitrary smooth functions of t.

Remark 4.10. The algebra $\mathfrak{g}$ is infinite-dimensional. The vector fields (4.16) do not constitute a basis of $\mathfrak{g}$ since they are linearly dependent. There is no natural way to choose a basis in $\mathfrak{g}$. The main reason for this is that there is no natural basis in the space of smooth functions of t.

Remark 4.11. The coefficients in the expansion of any element of the algebra $\mathfrak{g}$ with respect to the collection of vector fields (4.16), $a_1\mathcal{P}^t + a_2\mathcal{P}^y + \mathcal{P}^x(\chi) + c_k\mathcal{J}^k + \mathcal{Z}(\kappa)$, is defined up to the gauge transformations $\tilde{c} = c + \epsilon\bar{1}$, $\tilde{\kappa} = \kappa - \epsilon$ with arbitrary $\epsilon \in \mathbb{R}$, where $c := (c_1, \dots, c_m)^{\mathsf{T}}$, and similarly for $\tilde{c}$. Due to the presence of these gauge transformations, we can set without loss of generality that $c \perp_{\mathsf{W}} \bar{1}$ *or, equivalently,* $c \in \operatorname{im}\mathsf{F}$ *in the expansion of any element of the algebra $\mathfrak{g}$ with respect to the collection of vector fields* (4.16). At the same time, in particular cases when constant summand α is singled out in the parameter function κ or, moreover, $\kappa = \alpha$, it may be convenient to change $c + \alpha\bar{1} \to c$ and $\kappa - \alpha \to \kappa$ and neglect the gauge $c \perp_{\mathsf{W}} \bar{1}$.

Up to antisymmetry of the Lie bracket, the nontrivial commutation relations between vector fields from the spanning set of $\mathfrak{g}$ from Theorem 4.9 are exhausted by

$$[\mathcal{P}^t, \mathcal{P}^x(\chi)] = \mathcal{P}^x(\chi_t), \quad [\mathcal{P}^t, \mathcal{Z}(\kappa)] = \mathcal{Z}(\kappa_t), \quad [\mathcal{P}^x(\chi), \mathcal{P}^y] = \mathcal{Z}(\chi_t).$$

It is thus obvious that the algebra $\mathfrak{g}$ is solvable and not nilpotent. Its nilradical is $\mathfrak{n} := \langle \mathcal{P}^y, \mathcal{P}^x(\chi), \mathcal{J}^1, \dots, \mathcal{J}^m, \mathcal{Z}(\kappa)\rangle$. To classify subalgebras of $\mathfrak{g}$ it is convenient to

decompose the algebra $\mathfrak{g}$ into the semidirect sum of the subalgebras as follows:

$$\mathfrak{g} = \langle \mathcal{P}^t, \mathcal{P}^y \rangle \in \langle \mathcal{P}^x(\chi), \mathcal{J}^1, \dots, \mathcal{J}^m, \mathcal{Z}(\kappa) \rangle.$$

Since the Hamiltonian structure of the system (4.2) is not canonical, see Section 4.3, it establishes a connection between conserved quantities and Hamiltonian symmetries in a specific way. The Hamiltonian operator $\mathfrak{H}$ of the system (4.2) maps characteristics of local conservation laws to the q-component tuple, $\hat{\mathcal{Q}}q = (\hat{\mathcal{Q}}q^1, \dots, \hat{\mathcal{Q}}q^m)^{\mathsf{T}}$, of prolonged evolution forms, $\hat{\mathcal{Q}}$, of local infinitesimal Hamiltonian symmetries, $\mathcal{Q}$. Given a generalized symmetry vector field $\mathcal{Q} = \tau\partial_t + \xi^x\partial_x + \xi^y\partial_y + \eta^i\partial_{\psi^i}$, each of whose components is a differential function of ψ, i.e., a smooth function of the independent variables t, x and y and a finite number of derivatives of ψ with respect to these variables, we have $\hat{\mathcal{Q}}q = (\mathrm{D}_x^2 + \mathrm{D}_y^2 + \mathsf{F})(\eta - \tau\psi_t - \xi^x\psi_x - \xi^y\psi_y)$. Hence

$$\begin{aligned}
&-\hat{\mathcal{P}}^t q = q_t = \mathfrak{H}(-\mathsf{W}\psi), \quad -\hat{\mathcal{P}}^x(\chi)q = \chi(t)q_x = \mathfrak{H}\big(-\chi(t)y\mathsf{W}\bar{1}\big),\\
&-\hat{\mathcal{Z}}(\kappa)q = 0 = \mathfrak{H}\big(\kappa(t)\mathsf{W}\bar{1}\big), \quad 0 = \mathfrak{H}\big(\Phi^i_{q^i}\delta_i\big),\\
&-\hat{\mathcal{P}}^y q = q_y, \quad -\hat{\mathcal{J}}^i q = -\mathsf{F}\delta_i,
\end{aligned}$$

where $\delta_i = (\delta_{ij}, j = 1, \dots, m)^{\mathsf{T}}$ with the Kronecker delta δ_{ij}. Thus, we have

- ◇ the standard relation of the invariance with respect to shifts in time ($\mathcal{Q} = \mathcal{P}^t$) with the conservation of energy ($\lambda = -\mathsf{W}\psi$),
- ◇ the standard relation of the invariance with respect to generalized (time-dependent) shifts in x ($\mathcal{Q} = \mathcal{P}^x(\chi)$) with the conservation of generalized total zonal momentums ($\lambda = -\chi(t)y\mathsf{W}\bar{1}$) and
- ◇ the degenerate (via zero) association of simultaneous equal time-dependent shifts of ψ^i ($\mathcal{Q} = \mathcal{Z}(\kappa)$) with generalized total weighted circulations ($\lambda = \kappa(t)\mathsf{W}\bar{1}$), which are conserved time-dependent Casimir functionals.

All the Casimir functionals ($\lambda = \Phi^i_{q^i}\delta_i$) are mapped by $\mathfrak{H}$ to zero and, except the total weighted circulation ($\lambda = \mathsf{W}\bar{1}$), has no relation to symmetries. The algebra of Hamiltonian Lie symmetries of the system (4.2) is $\langle \mathcal{P}^t, \mathcal{P}^x(\chi), \mathcal{Z}(\kappa) \rangle$, whereas $\mathcal{P}^y$ and $\mathcal{J}^1, \dots, \mathcal{J}^m$ are not such symmetries and thus have no counterparts among conserved quantities of (4.2).

4.5 Point symmetry pseudogroup

To find the point symmetry pseudogroup of the multi-layer quasi-geostrophic model (4.1), we use the megaideal-based version of the algebraic method [16]. The description and applications of this technique can be found, e.g., in [26,108] and references therein. It is an extension of the automorphism-based version of the algebraic method first introduced by Hydon [50, 51].

Both versions are based on the idea that the pushforward Φ_* of any point-symmetry transformation Φ of the system of differential equations is an automorphism of its maximal Lie invariance algebra $\mathfrak{g}$, $\Phi_*\mathfrak{g} = \mathfrak{g}$. But Hydon's approach requires the explicit computation of the automorphisms of the algebra $\mathfrak{g}$, which certainly cannot be done if its dimension is not finite; even for a finite-dimensional Lie algebra, the explicit construction of its automorphisms can be highly complicated. In contrast, the megaideal-based version of the algebraic method employs the notion of a megaideal $\mathfrak{m}$ of a Lie algebra $\mathfrak{g}$ (which is also called a fully characteristic ideal in [47, Exercise 14.1.1]). Recall that a vector subspace $\mathfrak{m}$ of a Lie algebra $\mathfrak{g}$ is called *megaideal* if it is invariant under all automorphisms of $\mathfrak{g}$ [125], that is, $\mathfrak{T}\mathfrak{m} \subseteq \mathfrak{m}$ for all $\mathfrak{T} \in \mathrm{Aut}(\mathfrak{g})$.

The first step in the megaideal-based version of the algebraic method is to construct a certain set of megaideals of the maximal Lie invariance algebra $\mathfrak{g}$. Then we consider a point transformation Φ acting on the space of independent and dependent variables and using property $\Phi_*\mathfrak{m} \subseteq \mathfrak{m}$ find the general constraints imposed on components of Φ for each megaideal $\mathfrak{m}$ from the found set. After deriving all possible constraints on the components of Φ in such a manner using all found megaideals, we check if the condition $\Phi_*\mathfrak{g} \subseteq \mathfrak{g}$ is satisfied, which guarantees that no more constraints within the algebraic method can be found. Then one finishes the computation by applying the direct method, which is usually not computationally expensive.

Elementary properties of megaideals allow us to find a large set of megaideals of the algebra $\mathfrak{g}$ without any knowledge of the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of $\mathfrak{g}$. For example, all elements of the derived series and lower and upper central series of $\mathfrak{g}$, including the center $\mathfrak{z}$ and the derivative $\mathfrak{g}'$ of $\mathfrak{g}$, as well as the radical $\mathfrak{r}$ and nilradical $\mathfrak{n}$ of $\mathfrak{g}$, are megaideals of $\mathfrak{g}$. Moreover, if $\mathfrak{i}_1$ and $\mathfrak{i}_2$ are megaideals of $\mathfrak{g}$, then so are $\mathfrak{i}_1 + \mathfrak{i}_2$, $\mathfrak{i}_1 \cap \mathfrak{i}_2$ and $[\mathfrak{i}_1, \mathfrak{i}_2]$, i.e., sums, intersections and commutators of megaideals are again megaideals. Note that for computing point symmetry transformations,

we should exclude megaideals that are sums of two proper submegaideals from the consideration, since they do not provide essentially new constraints on the components of the transformation in comparison to those obtained from the summands. If $\mathfrak{i}_2$ is a megaideal of $\mathfrak{i}_1$ and $\mathfrak{i}_1$ is a megaideal of $\mathfrak{g}$, then $\mathfrak{i}_2$ is a megaideal of $\mathfrak{g}$, i.e., megaideals of megaideals are also megaideals.

The following lemma from [37] is also used to construct megaideals from known ones.

Lemma 4.12. *If $\mathfrak{i}_0$, $\mathfrak{i}_1$ and $\mathfrak{i}_2$ are megaideals of $\mathfrak{g}$, then the set $\mathfrak{s}$ of elements from $\mathfrak{i}_0$ whose commutators with arbitrary elements from $\mathfrak{i}_1$ belong to $\mathfrak{i}_2$ is also a megaideal of $\mathfrak{g}$.*

Since the algebra $\mathfrak{g}$ is solvable the radical $\mathfrak{r}$ of $\mathfrak{g}$ coincides with $\mathfrak{g}$,

$$\mathfrak{m}_1 := \mathfrak{g} = \langle \mathcal{P}^t, \mathcal{P}^y, \mathcal{P}^x(\chi), \mathcal{J}^1, \dots, \mathcal{J}^m, \mathcal{Z}(\kappa)\rangle.$$

The next obvious proper megaideals are the derived algebra $\mathfrak{g}'$, the nilradical $\mathfrak{n}$ of $\mathfrak{g}$, its derived algebra $\mathfrak{n}'$ and the center $\mathfrak{z}$ of $\mathfrak{g}$, which respectively are

$$\mathfrak{g}' = \langle \mathcal{P}^x(\chi), \mathcal{Z}(\kappa)\rangle, \quad \mathfrak{m}_2 := \mathfrak{n} = \langle \mathcal{P}^y, \mathcal{P}^x(\chi), \mathcal{J}^1, \dots, \mathcal{J}^m, \mathcal{Z}(\kappa)\rangle,$$
$$\mathfrak{n}' = \langle \mathcal{Z}(\kappa)\rangle, \quad \mathfrak{m}_3 := \mathfrak{z} = \langle \mathcal{P}^x(1), \mathcal{J}^1, \dots, \mathcal{J}^m\rangle.$$

Pairwise intersecting the above megaideals, we get the next new ones

$$\mathfrak{m}_4 := \mathfrak{g}' \cap \mathfrak{m}_3 = \langle \mathcal{P}^x(1), \mathcal{Z}(1)\rangle, \quad \mathfrak{m}_5 := \mathfrak{n}' \cap \mathfrak{m}_3 = \langle \mathcal{Z}(1)\rangle.$$

In notations of Lemma 4.12, let the megaideals $\mathfrak{i}_0$ and $\mathfrak{i}_1$ be $\mathfrak{m}_2$ and $\mathfrak{m}_1$, respectively. Taking $\mathfrak{i}_2$ to be $\mathfrak{m}_5$ we obtain

$$\mathfrak{s} = \langle \mathcal{P}^x(1), \mathcal{P}^x(t), \mathcal{Z}(1), \mathcal{Z}(t)\rangle, \quad \mathfrak{m}_6 := \mathfrak{s} \cap \mathfrak{n}' = \langle \mathcal{Z}(1), \mathcal{Z}(t)\rangle.$$

The found megaideals that are denoted by $\mathfrak{m}_i$, $i \in \{1, \dots, 6\}$ are essential in computing the point symmetry pseudogroup G.

Theorem 4.13. *The point symmetry pseudogroup G of the system (4.1) consists of the point transformations of the form*

$$\tilde{t} = \varepsilon_1 t + T^0, \ \tilde{x} = \varepsilon_1 x + h(t), \ \tilde{y} = \varepsilon_2 y + Y^0, \ \tilde{\psi}^i = \varepsilon_2 \psi^i - \varepsilon_1\varepsilon_2 h_t(t) y + g(t) + \Psi^i,$$

where $\varepsilon_1,\varepsilon_2=\pm1$, T^0, Y^0 *and* Ψ^i, $i=1,\dots,m$, *are arbitrary real constants, and* h *and* g *are arbitrary smooth functions of* t.

Proof. The general form of point transformations acting on the space $\mathbb{R}^3_{t,x,y}\times\mathbb{R}^m_\psi$ is

$$\Phi\colon\quad (\tilde t,\tilde x,\tilde y,\tilde\psi^1,\dots,\tilde\psi^m)=(T,X,Y,\Psi^1,\dots,\Psi^m),$$

where T, X, Y, Ψ^1, $\dots$, Ψ^m are smooth functions of t, x, y, ψ^1, $\dots$, ψ^m, whose Jacobian is nonzero. If Φ is a point-symmetry transformation of the system (4.1), then $\Phi_*\mathfrak{m}_i\subseteq\mathfrak{m}_i$ for all $i\in\{1,\dots,6\}$, where Φ_* denotes the pushforward of vector fields by Φ. We use this necessary condition for deriving principal constraints on the components of such transformations.

We choose $\mathcal{Z}(1)$, $\mathcal{Z}(t)$, $\mathcal{P}^x(1)$, $\mathcal{J}^i$, $\mathcal{P}^y$ and $\mathcal{P}^t$ as linearly independent vector fields from $\mathfrak{g}$ for checking the above necessary condition. Since $\mathcal{Z}(1)\in\mathfrak{m}_5$, $\mathcal{Z}(t)\in\mathfrak{m}_6$, $\mathcal{P}^x(1)\in\mathfrak{m}_4$, $\mathcal{J}^i\in\mathfrak{m}_3$, $\mathcal{P}^y\in\mathfrak{m}_2$ and $\mathcal{P}^t\in\mathfrak{m}_1$, the above necessary condition implies that

$$\Phi_*\mathcal{Z}(1)=\tilde{\mathcal{Z}}(\tilde\kappa^1),\tag{4.17a}$$
$$\Phi_*\mathcal{Z}(t)=\tilde{\mathcal{Z}}(\tilde\kappa^2)+\tilde{\mathcal{Z}}(\hat\kappa^2\tilde t),\tag{4.17b}$$
$$\Phi_*\mathcal{P}^x(1)=\tilde{\mathcal{P}}^x(\tilde\chi^3)+\tilde{\mathcal{Z}}(\tilde\kappa^3),\tag{4.17c}$$
$$\Phi_*\mathcal{J}^i=\tilde{\mathcal{P}}^x(\tilde\chi^4)+\alpha^{4j}\tilde{\mathcal{J}}^j,\tag{4.17d}$$
$$\Phi_*\mathcal{P}^y=\beta^5\tilde{\mathcal{P}}^y+\tilde{\mathcal{P}}^x(\tilde\chi^5)+\alpha^{5j}\tilde{\mathcal{J}}^j+\tilde{\mathcal{Z}}(\tilde\kappa^5),\tag{4.17e}$$
$$\Phi_*\mathcal{P}^t=\gamma^6\tilde{\mathcal{P}}^t+\beta^6\tilde{\mathcal{P}}^y+\tilde{\mathcal{P}}^x(\tilde\chi^6)+\alpha^{6j}\tilde{\mathcal{J}}^j+\tilde{\mathcal{Z}}(\tilde\kappa^6),\tag{4.17f}$$

where $\tilde\chi^i$ and $\tilde\kappa^i$, $i\in\{5,6\}$, are smooth functions of $\tilde t$, the other parameters are real constants, and the tildes in the notation of the vector fields on the right-hand sides mean that these vector fields are defined in the coordinates with tildes. We expand the system (4.17) and split each of its equations componentwise In total, we obtain the following collection of constraints for the components of Φ:

$$\begin{aligned}
&T_{tt}=0,\quad T_x=T_y=T_{\psi^i}=0,\\
&X_x=X^1,\quad X_y=X^2,\quad X_{\psi^i}=X^{0i},\\
&Y_t=Y^2,\quad Y_y=Y^1,\quad Y_x=0,\quad Y_{\psi^i}=0\\
&\Psi^i_{\psi^j}=A^{ij},\quad \Psi^i_x=A^0,\quad \Psi^i_{yy}=0,\quad \Psi^i_{ty}=-T_t^{-1}X_{tt}Y_y,
\end{aligned}\tag{4.18}$$

where X^1, X^2, X^{0i}, Y^1, Y^2, A^{ij} and A^0 are arbitrary real constants with

$$\sum_{k=1}^{m} X^{0k} = 0, \quad \sum_{k=1}^{m} A^{ik} = \sum_{k=1}^{m} A^{jk}. \tag{4.19}$$

The solution set of this collection of constraints consists of the tuples with components of the form

$$\begin{aligned} &T = T^1 t + T^0,\ X = X^1 x + X^2 y + X^{0i}\psi^i + h(t),\ Y = Y^1 y + Y^2 t + Y^0,\\ &\Psi^i = A^{ij}\psi^j + A^0 x - \frac{Y^1}{T^1} f_t y + g(t) + b_i, \end{aligned} \tag{4.20}$$

where in addition to the above, T^0 and T^1 are arbitrary real constants, h and g are arbitrary smooth functions of t, and the nondegeneracy of the Jacobian matrix of Φ requires that $T^1 X^1 Y^1 \det A^{ij} \neq 0$.

One more further specification for the obtained form (4.18) for components of the arbitrary point-symmetry transformation Φ can be obtained using the megaideal $\mathfrak{g}'$,

$$X^1 Y^1 = T^1 \sum_{k=1}^{m} A^{ik}. \tag{4.21}$$

Under the derived restrictions on the components of Φ, we have $\Phi_*\mathfrak{g} \subseteq \mathfrak{g}$. This is why the further application of the algebraic method does not result in more constraints on the components of Φ.

We proceed with applying the direct method. In view of the obtained form (4.20) for Φ, the operators of partial derivatives are transformed according to the rule

$$\begin{aligned} &\partial_{\tilde t} = \frac{1}{\mathrm{D}_t T}\left(\mathrm{D}_t - \frac{\mathrm{D}_t X}{\mathrm{D}_x X}\mathrm{D}_x - \frac{\mathrm{D}_t Y}{\mathrm{D}_y Y}\left(\mathrm{D}_y - \frac{\mathrm{D}_y X}{\mathrm{D}_x X}\mathrm{D}_x\right)\right),\\ &\partial_{\tilde x} = \frac{1}{\mathrm{D}_x X}\mathrm{D}_x, \quad \partial_{\tilde y} = \frac{1}{\mathrm{D}_y Y}\left(\mathrm{D}_y - \frac{\mathrm{D}_y X}{\mathrm{D}_x X}\mathrm{D}_x\right). \end{aligned}$$

We label by ($\tilde{4.1}$) the system (4.1) written in the variables with tildes. The system obtained from ($\tilde{4.1}$) by substituting the expressions for jet variables with tildes in terms of jet variables without tildes and then substituting the expressions for ψ^i_{txx} in view of the system (4.1) is called the expanded system ($\tilde{4.1}$). The derivative ψ^i_{txy}

appears only in ith equation of this system in the term arising from $\tilde{\psi}^i_{\tilde{t}\tilde{y}\tilde{y}}$,

$$\tilde{\psi}^i_{\tilde{t}\tilde{y}\tilde{y}} = -\frac{2}{T^1(Y^1)^2}\frac{\mathrm{D}_y X}{\mathrm{D}_x X}A^{ij}\psi^j_{txy} + \cdots .$$

Hence $\mathrm{D}_y X = 0$ since $\det A^{ij} \neq 0$, and thus $X^2 = X^{0i} = 0$, which simplifies the expressions for the transformed operators of partial derivatives,

$$\partial_{\tilde{t}} = \frac{1}{T^1}\left(\mathrm{D}_t - \frac{f_t}{X^1}\mathrm{D}_x - \frac{Y^2}{Y^1}\mathrm{D}_y\right), \quad \partial_{\tilde{x}} = \frac{1}{X^1}\mathrm{D}_x, \quad \partial_{\tilde{y}} = \frac{1}{Y^1}\mathrm{D}_y.$$

Collecting the coefficients of ψ^j_{txx} in the ith equation of the expanded system (4.1) with $i \neq j$ gives the constraint $A^{ij} = 0$. Thus, the equation (4.21) implies $A^{ii} = X^1Y^1/T^1$. More constraints on Φ,

$$(f_{i,i-1} + f_{i,i+1})\frac{Y^2}{Y^1} = 0, \quad f_{i,i-1}\frac{A^0}{X^1Y^1} = 0,$$

are obtained after collecting, in the ith equation of the expanded system (4.1) for fixed i, the coefficients of ψ^i_y and ψ^{i-1}_y, which arise from the terms $(f_{i,i-1} + f_{i,i+1})\tilde{\psi}^i_{\tilde{t}}$ and $f_{i,i-1}\tilde{\psi}^i_{\tilde{x}}\tilde{\psi}^{i-1}_{\tilde{y}}$, respectively. Therefore, $Y^2 = A^0 = 0$. Under the derived constraints, the expanded system (4.1) takes the form

$$\begin{aligned}&(\psi^i_{tyy} + \psi^i_x\psi^i_{yyy} - \psi^i_y\psi^i_{yyx})\big((X^1)^2(Y^1)^{-2} - 1\big)\\ &\quad + f_{i,i-1}\big(\psi^i_t - \psi^{i-1}_t + \psi^{i-1}_x\psi^i_y - \psi^i_x\psi^{i-1}_y\big)\big((X^1)^2 - 1\big)\\ &\quad + f_{i,i+1}\big(\psi^i_t - \psi^{i+1}_t + \psi^{i+1}_x\psi^i_y - \psi^i_x\psi^{i+1}_y\big)\big((X^1)^2 - 1\big) + \beta\psi^i_x(T^1X^1 - 1) = 0.\end{aligned} \tag{4.22}$$

The obvious total splitting of (4.22) implies that $(X^1)^2 = (Y^1)^2 = 1$ and $T^1 = X^1$. □

Corollary 4.14. *The quotient of the pseudogroup G by its identity component G_{id} is isomorphic to the group $\mathbb{Z}_2^2$ and can be naturally identified with the subgroup of G generated by two involutions $\mathcal{I}^{tx}$ and $\mathcal{I}^{y\psi}$,*

$$\begin{aligned}&\mathcal{I}^{tx}\colon \quad (t,x,y,\psi^1,\dots,\psi^m) \mapsto (-t,-x,y,\psi^1,\dots,\psi^m),\\ &\mathcal{I}^{y\psi}\colon \quad (t,x,y,\psi^1,\dots,\psi^m) \mapsto (t,x,-y,-\psi^1,\dots,-\psi^m).\end{aligned}$$

The pseudosubgroups of the pseudogroup G each of which is parameterized by

a single functional or constant canonical parameter and associated with a family of vector fields listed in Theorem 4.9 are of the form

$$\begin{aligned}
\mathcal{P}^t(\epsilon) &:= (t,x,y,\psi^1,\dots,\psi^m)\mapsto(t+\epsilon,x,y,\psi^1,\dots,\psi^m),\\
\mathcal{P}^x(h) &:= (t,x,y,\psi^1,\dots,\psi^m)\mapsto(t,x+h,y,\psi^1-h_ty,\dots,\psi^m-h_ty),\\
\mathcal{P}^y(\epsilon) &:= (t,x,y,\psi^1,\dots,\psi^m)\mapsto(t,x,y+\epsilon,\psi^1,\dots,\psi^m),\\
\mathcal{J}^i(\epsilon) &:= (t,x,y,\psi^1,\dots,\psi^i,\dots,\psi^m)\mapsto(t,x,y,\psi^1,\dots,\psi^i+\epsilon,\dots,\psi^m),\\
\mathcal{Z}(g) &:= (t,x,y,\psi^1,\dots,\psi^m)\mapsto(t,x,y,\psi^1+g,\dots,\psi^m+g),
\end{aligned}\tag{4.23}$$

where the constant ϵ plays the role of a constant group parameter, and h and g are arbitrary smooth functions of t. The transformations $\mathcal{P}^t(\epsilon)$, $\mathcal{P}^y(\epsilon)$ and $\mathcal{P}^x(h)$ are respectively shifts with respect to t and y and generalized time-dependent shifts with respect to x. The transformations $\mathcal{J}^i(\epsilon)$ and $\mathcal{Z}(g)$ correspond to gaugings of the stream function tuple, where gauging summands for the stream functions on different layers differ at most by constants.

4.6 Equivalence groupoid, pseudogroup and algebra

Theorems 4.9 and 4.13 provide some additional insights about transformational properties of the system (4.1) when it is viewed as a class $\mathcal{M}$ (of systems) of differential equations, which is defined in Section 4.2. See [107, 109, 111, 146] for the required theoretical background on classes of differential equations and on point transformations within them. More specifically, Theorem 4.9 implies that the algebra $\mathfrak{g}$ actually coincides with the kernel of the maximal Lie invariance algebras of systems from the class $\mathcal{M}$. Furthermore, the trivial prolongation of the kernel algebra of a class of differential equations to its arbitrary elements is invariant under the action of the corresponding usual equivalence (pseudo)group, see [111, Section 2.2] for all the necessary notions and the proof of the mentioned property. This observation allows us to compute the equivalence pseudogroup $G^{\sim}$ of the class $\mathcal{M}$ using the megaideal-based version of the algebraic method, which was suggested in [13]. At the same time, the algebra $\mathfrak{g}$ is the maximal Lie invariance algebra for each system from the class $\mathcal{M}$, and

hence it is preserved when pushed forward by the transformational part of any admissible transformation within this class. This provides an algebraic tool for computing the equivalence groupoid $\mathcal{G}^{\sim}$ of $\mathcal{M}$. A nice feature of both computations, of $G^{\sim}$ and of $\mathcal{G}^{\sim}$, is that their major part repeats the proof of Theorem 4.13.

Theorem 4.15. *The equivalence groupoid $\mathcal{G}^{\sim}$ of the class $\mathcal{M}$ consists of the triples of the form $(\theta, \Phi, \tilde{\theta})$, where Φ is a point transformation in the space of independent and dependent variables, whose components are*

$$\tilde{t} = T^1 t + T^0, \quad \tilde{x} = X^1 x + h(t), \quad \tilde{y} = \varepsilon X^1 y + Y^0, \tag{4.24a}$$

$$\tilde{\psi}^i = \varepsilon \frac{(X^1)^2}{T^1}\psi^i - \varepsilon \frac{X^1}{T^1} h_t(t) y + g(t) + \Psi^i, \tag{4.24b}$$

and the target and source arbitrary-element tuples $\tilde{\theta} = (\tilde{f}_{i,i-1}, \tilde{f}_{i,i+1}, \tilde{\beta})$ and $\theta = (f_{i,i-1}, f_{i,i+1}, \beta)$ are related according to

$$\tilde{f}_{i,i-1} = \frac{f_{i,i-1}}{(X^1)^2}, \quad \tilde{f}_{i,i+1} = \frac{f_{i,i+1}}{(X^1)^2}, \quad \tilde{\beta} = \frac{\beta}{T^1 X^1}. \tag{4.24c}$$

Here $\varepsilon = \pm 1$, T^1, T^0, X^1 Y^0 and Ψ^i, $i = 1, \dots, m$, are arbitrary real constants satisfying $T^1 X^1 \neq 0$, and h and g are arbitrary smooth functions of t.

Proof. Given any two arbitrary-element tuples θ and $\tilde{\theta}$, let a point transformation

$$\Phi\colon \quad (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{\psi}^1, \dots, \tilde{\psi}^m) = (T, X, Y, \Psi^1, \dots, \Psi^m)(t, x, y, \psi^1, \dots, \psi^m)$$

connect the corresponding systems $\mathcal{M}_\theta$ and $\mathcal{M}_{\tilde{\theta}}$ from the class $\mathcal{M}$. In light of Theorem 4.9, the algebra $\mathfrak{g}$ is the maximal Lie invariance algebra of each system from the class $\mathcal{M}$. Therefore, $\Phi_*\mathfrak{g} = \mathfrak{g}$, which thus implies $\Phi_*\mathfrak{m} = \mathfrak{m}$ for every megaideal $\mathfrak{m} \subset \mathfrak{g}$. This returns us to the setting of the algebraic part of the proof of Theorem 4.13, which straightforwardly implies that the components of the transformation Φ are necessarily of the form (4.20), where the parameters satisfy the restrictions, indicated after this form, including (4.21).

The next step in the course of computing the explicit form of Φ is to apply the direct method. Within the framework of this method, we should express the transformed derivatives in terms of the initial ones and substitute the obtained expressions into the target system $\mathcal{M}_{\tilde{\theta}}$ and then substitute the expressions for ψ^i_{txx} in view of the

source system $\mathcal{M}_\theta$. The computations in this part are completely analogous to those carried out in the proof of Theorem 4.13. The only difference is that the expanded target system $\mathcal{M}_{\tilde\theta}$, which is the counterpart of (4.22), takes the form

$$\begin{aligned}&(\psi^i_{tyy}+\psi^i_x\psi^i_{yyy}-\psi^i_y\psi^i_{yyx})\big((X^1)^2(Y^1)^{-2}-1\big)\\&\quad+\big(\psi^i_t-\psi^{i-1}_t+\psi^{i-1}_x\psi^i_y-\psi^i_x\psi^{i-1}_y\big)\big(\tilde f_{i,i-1}(X^1)^2-f_{i,i-1}\big)\\&\quad+\big(\psi^i_t-\psi^{i+1}_t+\psi^{i+1}_x\psi^i_y-\psi^i_x\psi^{i+1}_y\big)\big(\tilde f_{i,i+1}(X^1)^2-f_{i,i+1}\big)\\&\quad+\psi^i_x(\tilde\beta T^1X^1-\beta)=0.\end{aligned}\tag{4.25}$$

This results in the remaining constraints, $(X^1)^2=(Y^1)^2$ and (4.24c). □

Theorem 4.16. *The usual equivalence pseudogroup $G^\sim$ of the class $\mathcal{M}$ consists of the point transformations in the foliated space $\mathbb{R}^{m+3}_{t,x,y,\psi}\times\mathbb{R}^{2m-1}_\theta$ with components of the form (4.24), where $\varepsilon=\pm1$, T^1, T^0, X^1 Y^0 and Ψ^i, $i=1,\dots,m$, are arbitrary real constants with $T^1X^1\neq0$, and f and g are arbitrary smooth functions of t.*

Proof. We start with the general form of fibre-preserving point transformation $\mathcal{T}$ acting on the foliated space $\mathbb{R}^{m+3}_{t,x,y,\psi}\times\mathbb{R}^{2m-1}_\theta$,

$$\begin{aligned}\mathcal{T}\colon\quad&(\tilde t,\tilde x,\tilde y,\tilde\psi^1,\dots,\tilde\psi^m)=(T,X,Y,\Psi^1,\dots,\Psi^m)(t,x,y,\psi^1,\dots,\psi^m),\\&(\tilde f_{i,i-1},\tilde f_{i,i+1},\tilde\beta)=(F^{i,i-1},F^{i,i+1},B)(t,x,y,\psi^1,\dots,\psi^m,f_{i,i-1},f_{i,i+1},\beta),\end{aligned}$$

where $i=1,\dots,m$ with the excluded index pairs $(1,0)$ and $(m,m+1)$. Denote by hat the trivial prolongation of objects defined in the space $\mathbb{R}^{m+3}_{t,x,y,\psi}$ to the space $\mathbb{R}^{m+3}_{t,x,y,\psi}\times\mathbb{R}^{2m-1}_\theta$. Theorem 4.9 implies that the algebra $\mathfrak{g}$ coincides with the kernel of maximal Lie invariance algebras of systems from the class $\mathcal{M}$. This is why the trivial prolongation $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ is preserved under the adjoint action of $G^\sim$, and thus $G^\sim$ also preserves the megaideals of $\hat{\mathfrak{g}}$. Hence the form of the (t,x,y,ψ)-components of $\mathcal{T}$ is given by (4.20), where the parameters satisfy the equations (4.21) and (4.19) the inequality $T^1X^1Y^1\det A^{ij}\neq0$. In notation of the proof of Theorem 4.13, the consideration of the necessary conditions $\mathcal{T}_*\hat{\mathcal{P}}^x(1)\in\hat{\mathfrak{m}}_4$, $\mathcal{T}_*\hat{\mathcal{J}}^i\in\hat{\mathfrak{m}}_3$, $\mathcal{T}_*\hat{\mathcal{P}}^y\in\hat{\mathfrak{m}}_2$ and $\mathcal{T}_*\hat{\mathcal{P}}^t\in\hat{\mathfrak{m}}_1$ also results in the following equations for the θ-components of $\mathcal{T}$:

$$\begin{aligned}&F^{i,i-1}_t=F^{i,i-1}_x=F^{i,i-1}_y=F^{i,i-1}_{\psi^j}=0,\quad F^{i,i+1}_t=F^{i,i+1}_x=F^{i,i+1}_y=F^{i,i+1}_{\psi^j}=0,\\&B_t=B_x=B_y=B_{\psi^j}=0,\end{aligned}$$

i.e., $F^{i,i-1}=F^{i,i-1}(f_{i,i-1},f_{i,i+1},\beta)$ and $B=B(f_{i,i-1},f_{i,i+1},\beta)$. The next step is to apply the direct method to find the remaining restrictions on the components of $\mathfrak{T}$. These computations are carried out analogously to those in the last part of the proof of Theorem 4.15, with the only difference that the parameters $\tilde f_{i,i-1}$, $\tilde f_{i,i+1}$ and $\tilde\beta$ in (4.25) are replaced with $F^{i,i-1}$, $F^{i,i+1}$ and B, respectively. □

Remark 4.17. Under the physical constraint $\beta>0$, we should set $T^1X^1>0$ in Theorems 4.15 and 4.16.

Corollary 4.18. *The class $\mathcal{M}$ is normalized in the usual sense.*

Proof. Theorems 4.15 and 4.16 jointly imply that the action groupoid of the equivalence group $G^{\sim}$ coincides with the equivalence groupoid $\mathcal{G}^{\sim}$. According to [146, Definition 2], this means that the class $\mathcal{M}$ is normalized in the usual sense. □

Remark 4.19. We could also derive Theorem 4.16 as a corollary of Theorem 4.15 since analyzing the equivalence groupoid $\mathcal{G}^{\sim}$ leads to the first claim in the proof of Corollary 4.18, which implies the form of equivalence transformations.

Remark 4.20. The generalized equivalence group of the class $\mathcal{M}$ consists of the point transformations in the foliated space $\mathbb{R}^{m+3}_{t,x,y,\psi}\times\mathbb{R}^{2m-1}_{\theta}$ with components of the form (4.24), where all parameters are allowed to depend on the arbitrary-element tuple θ. The effective generalized equivalence group of $\mathcal{M}$ coincides with its usual equivalence group $G^{\sim}$.

Corollary 4.21. *The equivalence algebra $\mathfrak{g}^{\sim}$ of the class $\mathcal{M}$ of systems of the form (4.1) is spanned by the vector fields*

$$\begin{aligned}&\hat{\mathcal{P}}^t,\quad \hat{\mathcal{P}}^y,\quad \hat{\mathcal{P}}^x(\chi),\quad \hat{\mathcal{J}}^1,\quad \dots,\quad \hat{\mathcal{J}}^m,\quad \hat{\mathcal{Z}}(\kappa),\\ &t\partial_t-(\psi^1\partial_{\psi^1}+\dots+\psi^m\partial_{\psi^m})-\beta\partial_\beta,\\ &x\partial_x+y\partial_y+2(\psi^1\partial_{\psi^1}+\dots+\psi^m\partial_{\psi^m})-2\sum_{i=1}^{m}(f_{i,i-1}\partial_{f_{i,i-1}}+f_{i,i+1}\partial_{f_{i,i+1}})-\beta\partial_\beta,\end{aligned}$$

where $\chi(t)$ and $\kappa(t)$ are arbitrary smooth functions of their argument.

We can summarize the results of Section 4.4 and this section as the solution of the group classification problem for the class $\mathcal{M}$.

Theorem 4.22. *The kernel of the maximal Lie invariance algebras of systems from the class $\mathcal{M}$ coincides with the algebra $\mathfrak{g}$. There are no Lie symmetry extensions in this class, i.e., each system from $\mathcal{M}$ is invariant exactly with respect to the algebra $\mathfrak{g}$. The usual equivalence pseudogroup $G^{\sim}$ of the class $\mathcal{M}$ is described in Theorem 4.16. As canonical representatives of $G^{\sim}$-equivalence classes of systems from the class $\mathcal{M}$, one can choose the systems with $\beta = 1$ and f_{12} equal to 1 or ± 1 depending on which condition, of the positivity or of nonvanishing, is assumed for the arbitrary elements.*

4.7 Classification of subalgebras

Since the multi-layer quasi-geostrophic model (4.1) is a system of partial differential equations with three independent variables, to classify its Lie reductions in an optimal way, it suffices to classify one- and two-dimensional Lie subalgebras of its Lie invariance algebra $\mathfrak{g}$ with respect to the adjoint action of its point-symmetry pseudogroup G. See [148, Section 2] for a description of the optimized procedure of Lie reductions in the case of three independent variables and Remark 4.25 below for the exhaustive study of codimension-three Lie reductions of the system (4.1). The pseudogroup G is generated by the continuous point transformations listed in (4.23) and the discrete point transformations listed in Corollary 4.14. They act on $\mathfrak{g}$ via pushing forward its elements as vector fields in the space $\mathbb{R}^3_{t,x,y} \times \mathbb{R}^m_{\psi}$, and the nonidentity actions on the elements spanning $\mathfrak{g}$ are exhausted by the following:

$$
\begin{aligned}
&\mathcal{P}^x(h)_*\mathcal{P}^t = \mathcal{P}^t + \mathcal{P}^x(h_t), \quad \mathcal{Z}(g)_*\mathcal{P}^t = \mathcal{P}^t + \mathcal{Z}(g_t),\\
&\mathcal{P}^x(h)_*\mathcal{P}^y = \mathcal{P}^y - \mathcal{Z}(h_t), \quad \mathcal{P}^y(\epsilon)_*\mathcal{P}^x(\chi) = \mathcal{P}^x(\chi) + \epsilon\mathcal{Z}(\chi_t),\\
&\mathcal{P}^t(\epsilon)_*\mathcal{Z}(\kappa) = \mathcal{Z}(\tilde{\kappa}^\epsilon), \quad \mathcal{P}^t(\epsilon)_*\mathcal{P}^x(\chi) = \mathcal{P}^x(\tilde{\chi}^\epsilon),\\
&\mathcal{I}^{tx}_*\mathcal{P}^t = -\mathcal{P}^t, \quad \mathcal{I}^{tx}_*\mathcal{P}^x(\chi) = -\mathcal{P}^x(\hat{\chi}), \quad \mathcal{I}^{tx}_*\mathcal{Z}(\kappa) = \mathcal{Z}(\hat{\kappa}),\\
&\mathcal{I}^{y\psi}_*\mathcal{P}^y = -\mathcal{P}^y, \quad \mathcal{I}^{y\psi}_*\mathcal{J}^k = -\mathcal{J}^k, \quad \mathcal{I}^{y\psi}_*\mathcal{Z}(\kappa) = -\mathcal{Z}(\kappa),
\end{aligned}
$$

where $\tilde{\kappa}^\epsilon(t) := \kappa(t-\epsilon)$, $\tilde{\chi}^\epsilon(t) := \chi(t-\epsilon)$, $\hat{\chi}(t) := \chi(-t)$ and $\hat{\kappa}(t) := \kappa(-t)$.

The algebra $\mathfrak{g}$ can be represented as the semidirect sum of its subalgebra $\mathfrak{g}_1 = \langle \mathcal{P}^t, \mathcal{P}^y \rangle$ acting on the ideal $\mathfrak{g}_2 = \langle \mathcal{P}^x(\chi), \mathcal{J}^1, \dots, \mathcal{J}^m, \mathcal{Z}(\kappa) \rangle$. Using this decomposition, we can apply the method for classifying subalgebras of semidirect sums of Lie algebras, which was suggested in [115], see also [28] and references therein for the

modern view on this method. Since each of the Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is abelian, its optimal list of subalgebras with respect to its inner automorphisms coincides with the list of its subspaces. Denote by π_1 the canonical projection $\pi_1\colon \mathfrak{g} \to \mathfrak{g}_1$ with respect to the vector space decomposition $\mathfrak{g} = \mathfrak{g}_1 \dotplus \mathfrak{g}_2$.

Theorem 4.23. *A complete list of G-inequivalent one-dimensional subalgebras of the algebra $\mathfrak{g}$ spanned by the vector fields (4.16) is exhausted by the following families. Below $c := (c_1, \dots, c_m)^{\mathsf{T}}$ and a are arbitrary constants with $c \in \operatorname{im}\mathsf{F}$ and χ and κ are arbitrary smooth functions of t that satisfy the indicated constraints.*

1. $\mathfrak{s}_{1.1}^{ac} := \langle \mathcal{P}^t + a\mathcal{P}^y + c_k\mathcal{J}^k\rangle$. *The subalgebras $\mathfrak{s}_{1.1}^{ac}$ and $\mathfrak{s}_{1.1}^{\tilde a\tilde c}$ are equivalent if and only if $\tilde a = \varepsilon a$ and $\tilde c = \varepsilon c$ for some $\varepsilon \in \{-1, 1\}$.*

2. $\mathfrak{s}_{1.2}^{\chi c} := \langle \mathcal{P}^y + \mathcal{P}^x(\chi) + c_k\mathcal{J}^k\rangle$. *The subalgebras $\mathfrak{s}_{1.2}^{\chi c}$ and $\mathfrak{s}_{1.2}^{\tilde\chi\tilde c}$ are equivalent if and only if $\tilde\chi(t) = \varepsilon'\chi(\varepsilon t - \epsilon)$ and $\tilde c = c$ for some $\varepsilon, \varepsilon' \in \{-1, 1\}$ and $\epsilon \in \mathbb{R}$.*

3. $\mathfrak{s}_{1.3}^{\chi c\kappa} := \langle \mathcal{P}^x(\chi) + c_k\mathcal{J}^k + \mathcal{Z}(\kappa)\rangle$ *with nonzero parameter tuple (χ, c, κ). The subalgebras $\mathfrak{s}_{1.3}^{\chi c\kappa}$ and $\mathfrak{s}_{1.3}^{\tilde\chi\tilde c\tilde\kappa}$ are equivalent if and only if $\tilde\chi(t) = \varepsilon\beta\chi(\varepsilon t - \epsilon)$, $\tilde\kappa(t) = \varepsilon'\beta\kappa(\varepsilon t - \epsilon) + \epsilon'\beta\dot\chi(\varepsilon t - \epsilon)$ and $\tilde c = \varepsilon'\beta c$ for some $\varepsilon, \varepsilon' \in \{-1, 1\}$, $\epsilon, \epsilon' \in \mathbb{R}$ and $\beta \in \mathbb{R}_{\neq 0}$ with $\dot\chi$ denoting the derivative of χ.*

Proof. Let $\mathfrak{s} \subset \mathfrak{g}$ be a one-dimensional subalgebra of $\mathfrak{g}$. Since $\dim \pi_1\mathfrak{s} \leqslant 1$, modulo the G-equivalence we can set $\pi_1\mathfrak{s} \in \big\{\langle \mathcal{P}^t + a\mathcal{P}^y\rangle, \langle \mathcal{P}^y\rangle, \{0\}\big\}$. We can set the gauge $a \geqslant 0$, acting by $\mathcal{I}_*^{y\psi}$ if necessary. According to Remark 4.11, we can also assume from the very beginning in each case below that $c \in \operatorname{im}\mathsf{F}$ in the fixed basis element Q of $\mathfrak{s}$.

$\boldsymbol{\pi_1\mathfrak{s} = \langle \mathcal{P}^t + a\mathcal{P}^y\rangle}$. Thus, we can choose $Q = \mathcal{P}^t + a\mathcal{P}^y + c_i\mathcal{J}^i + \mathcal{P}^x(\chi) + \mathcal{Z}(\kappa)$. Pushing forward Q by the transformation $\mathcal{Z}\big(-\int(\kappa + a\chi)\,\mathrm{d}t\big) \circ \mathcal{P}^x\big(-\int \chi\,\mathrm{d}t\big)$, we can set $\chi = \kappa = 0$, which results in the first subalgebra family in the theorem's statement. The equivalence within this family is generated by the pushforwards $\mathcal{Z}(\epsilon t)_*$ and $\mathcal{I}_*^{y\psi}$.

$\boldsymbol{\pi_1\mathfrak{s} = \langle \mathcal{P}^y\rangle}$. Taking $Q = \mathcal{P}^y + c_i\mathcal{J}^i + \mathcal{P}^x(\chi) + \mathcal{Z}(\kappa)$ as the basis vector of $\mathfrak{s}$, we simplify it by $\mathcal{P}^x\big(\int \kappa\,\mathrm{d}t\big)_*$, setting the gauge $\kappa = 0$ and thus obtaining the second listed subalgebra family. Furthermore, the pushforwards $\mathcal{P}^t(\epsilon)_*$, $\mathcal{I}_*^{tx}$ and $\mathcal{P}^x(-\epsilon' t)_*$ generate the equivalence within this family.

$\boldsymbol{\pi_1\mathfrak{s} = \{0\}}$. In other words, $\mathfrak{s} \subset \mathfrak{g}_2$. The one-dimensional subspaces of $\mathfrak{g}_2$ constitute

the third listed subalgebra family, the equivalence between which is generated by the action of G. □

Theorem 4.24. *A complete list of G-inequivalent two-dimensional subalgebras of the algebra $\mathfrak{g}$ spanned by the vector fields (4.16) is exhausted by the following subalgebra families. Below $b := (b_1, \dots, b_m)^{\mathsf{T}}$, $c := (c_1, \dots, c_m)^{\mathsf{T}}$, a and σ are arbitrary constants with $\sigma \neq 0$, $b \in \operatorname{im}\mathsf{F}$ and, except the first three algebras, $c \in \operatorname{im}\mathsf{F}$, and χ, χ^1, χ^2, κ, κ^1 and κ^2 are arbitrary smooth functions of t that satisfy the indicated constraints.*

1. $\mathfrak{s}_{2.1}^{abc} := \langle \mathcal{P}^t + b_k\mathcal{J}^k,\ \mathcal{P}^y + \mathcal{P}^x(a) + c_k\mathcal{J}^k\rangle$. *The subalgebras $\mathfrak{s}_{2.1}^{abc}$ and $\mathfrak{s}_{2.1}^{\tilde a\tilde b\tilde c}$ are equivalent if and only if $\tilde a = \varepsilon a$, $\tilde b = \varepsilon b$ and $\tilde c = c$ for some $\varepsilon \in \{-1, 1\}$.*

2. $\mathfrak{s}_{2.2}^{abc} := \langle \mathcal{P}^t + a\mathcal{P}^y + b_k\mathcal{J}^k,\ \mathcal{P}^x(1) + c_k\mathcal{J}^k\rangle$. *The subalgebras $\mathfrak{s}_{2.2}^{abc}$ and $\mathfrak{s}_{2.2}^{\tilde a\tilde b\tilde c}$ are equivalent if and only if $\tilde a = \varepsilon a$, $\tilde b = \varepsilon b$ and $\tilde c = \varepsilon c$ for some $\varepsilon \in \{-1, 1\}$.*

3. $\mathfrak{s}_{2.3}^{abc} := \langle \mathcal{P}^t + a\mathcal{P}^y + b_k\mathcal{J}^k,\ c_k\mathcal{J}^k\rangle$, *where the tuple c is nonzero. The subalgebras $\mathfrak{s}_{2.3}^{abc}$ and $\mathfrak{s}_{2.3}^{\tilde a\tilde b\tilde c}$ are equivalent if and only if $\tilde a = \varepsilon a$, $\tilde b = \varepsilon b + \epsilon c$ and $\tilde c = \gamma c$ for some $\varepsilon \in \{-1, 1\}$, $\epsilon \in \mathbb{R}$ and $\gamma \in \mathbb{R}_{\neq 0}$.*

4. $\mathfrak{s}_{2.4}^{ab\sigma} := \langle \mathcal{P}^t + a\mathcal{P}^y + b_k\mathcal{J}^k,\ \mathcal{P}^x(\mathrm{e}^{\sigma t}) + \mathcal{Z}(a\sigma t\mathrm{e}^{\sigma t})\rangle$. *The subalgebras $\mathfrak{s}_{2.4}^{ab\sigma}$ and $\mathfrak{s}_{2.4}^{\tilde a\tilde b\tilde\sigma}$ are equivalent if and only if $\tilde a = \varepsilon a$, $\tilde b = \varepsilon b$ and $\tilde\sigma = \varepsilon'\sigma$ for some $\varepsilon, \varepsilon' \in \{-1, 1\}$.*

5. $\mathfrak{s}_{2.5}^{ab\sigma} := \langle \mathcal{P}^t + a\mathcal{P}^y + b_k\mathcal{J}^k,\ \mathcal{Z}(\mathrm{e}^{\sigma t})\rangle$, *The subalgebras $\mathfrak{s}_{2.5}^{ab\sigma}$ and $\mathfrak{s}_{2.5}^{\tilde a\tilde b\tilde\sigma}$ are equivalent if and only if $\tilde a = \varepsilon a$, $\tilde b = \varepsilon b$ and $\tilde\sigma = \varepsilon'\sigma$ for some $\varepsilon, \varepsilon' \in \{-1, 1\}$.*

6. $\mathfrak{s}_{2.6}^{\chi bc\kappa} := \langle \mathcal{P}^y + \mathcal{P}^x(\chi) + b_k\mathcal{J}^k,\ \mathcal{P}^x(1) + c_k\mathcal{J}^k + \mathcal{Z}(\kappa)\rangle$. *Such subalgebras with parameter values (χ, b, c, κ) and $(\tilde\chi, \tilde b, \tilde c, \tilde\kappa)$ are equivalent if and only if $\tilde\chi(t) = \varepsilon\varepsilon'\chi(\varepsilon t - \epsilon) + \epsilon'$, $\tilde b = b + \varepsilon'\epsilon' c$, $\tilde c = \varepsilon' c$, $\tilde\kappa(t) = \varepsilon'\kappa(\varepsilon t - \epsilon)$ for some $\varepsilon, \varepsilon' \in \{-1, 1\}$ and $\epsilon, \epsilon' \in \mathbb{R}$.*

7. $\mathfrak{s}_{2.7}^{\chi bc\kappa} := \langle \mathcal{P}^y + \mathcal{P}^x(\chi) + b_k\mathcal{J}^k,\ c_k\mathcal{J}^k + \mathcal{Z}(\kappa)\rangle$, *where the tuple (c, κ) is nonzero. Such subalgebras with parameter values (χ, b, c, κ) and $(\tilde\chi, \tilde b, \tilde c, \tilde\kappa)$ are equivalent if and only if $\tilde\chi(t) = \varepsilon'\chi(\varepsilon t - \epsilon)$, $\tilde b = b + \epsilon' c$, $\tilde c = \alpha c$, $\tilde\kappa(t) = \alpha\kappa(\varepsilon t - \epsilon)$ for some $\varepsilon, \varepsilon' \in \{-1, 1\}$, $\epsilon, \epsilon' \in \mathbb{R}$ and $\alpha \in \mathbb{R}_{\neq 0}$.*

8. $\mathfrak{s}_{2.8}^{\upsilon} := \big\langle \mathcal{P}^x(\chi^1) + b_k\mathcal{J}^k + \mathcal{Z}(\kappa^1),\ \mathcal{P}^x(\chi^2) + c_k\mathcal{J}^k + \mathcal{Z}(\kappa^2)\big\rangle$, $\upsilon := (\chi^1, b, \kappa^1, \chi^2, c, \kappa^2)$, *and the subtuples (χ^1, b, κ^1) and (χ^2, c, κ^2) are linearly independent. Such subalgebras with parameter values υ and $\tilde\upsilon$ are equivalent if and only if*

$$\tilde\chi^k(t) = \varepsilon\beta_{kl}\chi^l(\varepsilon t - \epsilon), \quad \tilde\kappa^k(t) = \beta_{kl}\big(\varepsilon'\kappa^l(\varepsilon t - \epsilon) + \epsilon'\chi^l_t(\varepsilon t - \epsilon)\big),$$

$$\tilde b = \varepsilon'\beta_{11}b + \varepsilon'\beta_{12}c, \quad \tilde c = \varepsilon'\beta_{21}b + \varepsilon'\beta_{22}c,$$

where $\varepsilon, \varepsilon' \in \{-1, 1\}$, $\epsilon, \epsilon' \in \mathbb{R}$, (β_{kl}) *is a constant nondegenerate matrix, and the indices* k *and* l *run from 1 to 2.*

Proof. Let $\mathfrak{s}$ be a two-dimensional subalgebra of $\mathfrak{g}$. Modulo the G-equivalence we can set $\pi_1\mathfrak{s} \in \{\langle \mathcal{P}^t, \mathcal{P}^y\rangle, \langle \mathcal{P}^t + a\mathcal{P}^y\rangle, \langle \mathcal{P}^y\rangle, \{0\}\}$. Assume that $\mathfrak{s}$ is spanned by vector fields Q_1 and Q_2, whose forms will be specified depending on the case under the consideration.

$\boldsymbol{\pi_1\mathfrak{s} = \langle \mathcal{P}^t, \mathcal{P}^y\rangle}$. The subalgebra $\mathfrak{s}$ is spanned by the vector fields

$$Q_1 = \mathcal{P}^t + b_k\mathcal{J}^k + \mathcal{P}^x(\chi^1) + \mathcal{Z}(\kappa^1), \quad Q_2 = \mathcal{P}^y + c_k\mathcal{J}^k + \mathcal{P}^x(\chi^2) + \mathcal{Z}(\kappa^2).$$

Pushing forward the basis vector fields by the transformation $\mathcal{Z}\big(-\int \kappa^1 \mathrm{d}t\big) \circ \mathcal{P}^x\big(-\int \chi^1 \mathrm{d}t\big)$, we can set $\kappa^1 = \chi^1 = 0$. Since $\pi_1\mathfrak{s}$ is abelian, the algebra $\mathfrak{s}$ is abelian as well. The commutation relation $[Q_1, Q_2] = 0$ implies $\chi^2_t = \kappa^2_t = 0$. Denoting the constants $c_i + \kappa^2$ by c_i and χ^2 by a, we obtain the first subalgebra family listed in the theorem's statement. The equivalence within this family is generated by the pushforwards $\mathcal{Z}(\epsilon t)_*$ and $\mathcal{I}^{y\psi}_*$.

$\boldsymbol{\pi_1\mathfrak{s} = \langle \mathcal{P}^t + a\mathcal{P}^y\rangle}$. We can assume the vector fields Q_1 and Q_2 take the following form:

$$Q_1 = \mathcal{P}^t + a\mathcal{P}^y + \mathcal{P}^x(\chi^1) + b_k\mathcal{J}^k + \mathcal{Z}(\kappa^1), \quad Q_2 = \mathcal{P}^x(\chi^2) + c_k\mathcal{J}^k + \mathcal{Z}(\kappa^2).$$

Acting on $\mathfrak{s}$ by $\mathcal{Z}\big(-\int(\kappa^1 + a\chi^1)\,\mathrm{d}t\big)_* \circ \mathcal{P}^x\big(-\int \chi^1\,\mathrm{d}t\big)_*$, we set $\chi^1 = \kappa^1 = 0$.

If the subalgebra $\mathfrak{s}$ is abelian, the commutation relation $[Q_1, Q_2] = 0$ implies $\chi^2_t = 0$ and $\kappa^2_t = 0$. Denoting the constants $c_i + \kappa^2$ by c_i, we obtain the second and the third listed subalgebra families depending on whether the constant χ^2 is nonzero or zero, respectively. If $\chi^2 \neq 0$, then we can set $\chi^2 = 1$ by rescaling the vector field Q_2, and the equivalence within the corresponding family of subalgebras is generated by the pushforward $\mathcal{I}^{y\psi}_*$. Within the family with $\chi^2 = 0$, the equivalence additionally involves linear recombination of the basis elements.

If the subalgebra $\mathfrak{s}$ is nonabelian, then the commutation relation $[Q_1, Q_2] = \sigma Q_2$

with nonzero σ gives the system

$$c_1 = \cdots = c_m = 0, \quad \chi^2_t = \sigma\chi^2, \quad \kappa^2_t - a\chi^2_t = \sigma\kappa^2.$$

Hence $\chi^2 = \beta \mathrm{e}^{\sigma t}$, $\kappa^2 = (\beta a\sigma t + \delta)\mathrm{e}^{\sigma t}$ for some real constants β and δ with $(\beta, \delta) \neq (0, 0)$. Depending on whether β is zero or not, we obtain the fifth and fourth listed families. In the former case, we rescale Q_2 to set $\delta = 1$. In the latter case, we can set $\delta = 0$ and $\beta = 1$ by the pushforward $\mathcal{P}^y(-\beta^{-1}\delta)_*$ and rescaling Q_2. The equivalence within each of these families is generated by the pushforwards $\mathcal{I}^{tx}_*$ and $\mathcal{I}^{y\psi}_*$.

$\boldsymbol{\pi_1\mathfrak{s} = \langle \mathcal{P}^y \rangle}$. In view of Theorem 4.23 and modulo linearly combining Q_1 and Q_2, we can set $Q_1 = \mathcal{P}^y + \mathcal{P}^x(\chi^1) + b_k\mathcal{J}^k$ and $Q_2 = \mathcal{P}^x(\chi^2) + c_k\mathcal{J}^k + \mathcal{Z}(\kappa^2)$. The commutation relation $[Q_1, Q_2] = \mathcal{Z}(\chi^2_t)$ implies $\chi^2_t = 0$. The cases with $\chi^2 \neq 0$ and $\chi^2 = 0$ correspond to the sixth and the seventh subalgebra families, respectively. We also relabel (χ^1, κ^2) as (χ, κ) and, for the sixth subalgebra family, set $\chi^2 = 1$ by scaling the vector field Q_2. Note that in addition to the G-equivalence within each of these subalgebra families, we can also replace Q_1 with the linear combination $Q_1 + \gamma Q_2$, where $\gamma \in \mathbb{R}$, and use the pushforward $\mathcal{P}^x\big(\int \gamma\kappa \,\mathrm{d}t\big)_*$ to make the $\mathcal{Z}$-summand in this vector field zero.

$\boldsymbol{\pi_1\mathfrak{s} = \{0\}}$. This means that $\mathfrak{s}$ is a subalgebra of the abelian algebra $\mathfrak{g}_2$, which gives us the eighth subalgebra family. The equivalence within this subalgebra family is generated by the action of G and the linear recombination of the basis vector fields. This equivalence is described in the theorem's statement. □

Remark 4.25. A three-dimensional subalgebra of $\mathfrak{g}$ satisfies the transversality condition and thus is appropriate for Lie reduction if and only if it is G-equivalent to a subalgebra of the form

$$\langle \mathcal{P}^t + a_k\mathcal{J}^k,\ \mathcal{P}^y + b_k\mathcal{J}^k,\ \mathcal{P}^x(1) + c_k\mathcal{J}^k \rangle,$$

where $a := (a_1, \dots, a_m)^{\mathsf{T}}$, $b := (b_1, \dots, b_m)^{\mathsf{T}}$ and $c := (c_1, \dots, c_m)^{\mathsf{T}}$ are arbitrary constant tuples with $a, b \in \operatorname{im}\mathsf{F}$. An ansatz constructed with respect to such a subalgebra is $\psi = \phi + xc + yb + ta$, where ϕ is the m-tuple of unknown constants, which can be set to zero by the point symmetry transformation $\psi - \phi \to \psi$ of the system (4.1). The corresponding reduced system is thus trivial; this is just the compatibility condition of the ansatz with the system (4.1), which constrains the subalgebra parameters,

$\mathsf{F}a+c\odot\mathsf{F}b-b\odot\mathsf{F}c+\beta c=0$. This implies that $c\perp_{\mathsf{W}}\langle\bar{1}\rangle=\ker\mathsf{F}$, i.e., $c\in\operatorname{im}\mathsf{F}$ as well, and $a=-\mathsf{F}^{+}(c\odot\mathsf{F}b-b\odot\mathsf{F}c+\beta c)$, where the Moore–Penrose inverse F^{+} of F is given by (4.3). All the solutions of (4.1) constructed in this way are very simple. They are $\mathfrak{s}_{2.6}^{0bc0}$-invariant and belong to the solution family (4.94), see Section 4.9.4 below. This completes the study of codimension-three Lie reductions of the system (4.1).

4.8 Codimension-one Lie reductions

To construct a Lie invariant ansatz for a system $\mathcal{L}$ of differential equations with n independent variables $x=(x_1,\dots,x_n)$ and m dependent variables $u=(u^1,\dots,u^m)$, one selects a Lie subalgebra $\mathfrak{s}$ of the maximal Lie invariance algebra $\mathfrak{g}$ of $\mathcal{L}$ satisfying the local transversality condition. This condition means that the dimension of $\mathfrak{s}$ is equal to the dimension of the pushforward of $\mathfrak{s}$ by the canonical projection from the space with the coordinates (x,u) to the space with the coordinates x. If $(Q_1,\dots,Q_d)$ is a basis of the subalgebra $\mathfrak{s}$, then an $\mathfrak{s}$-invariant ansatz is a representation of the general solution of the system $\mathcal{Q}$ of quasilinear first-order partial differential equations $Q_1[u]=0,\ \dots,\ Q_d[u]=0$, where $Q[u]$ denotes the characteristic of a vector field Q. More specifically, given $n+m-d$ functionally independent particular solutions of $\mathcal{Q}$, which constitute a functional basis of $\mathfrak{s}$-invariants, m of them are assumed to be the new (invariant) dependent variables $v=(v^1,\dots,v^m)$ of the other $n-d$ solutions, which are considered as the new (invariant) independent variables $z=(z_1,\dots,z_{n-d})$. Substituting the ansatz into the system $\mathcal{L}$, one obtains a codimension-d reduced system in terms of the invariant variables (x,v). The substitution of any solution of this reduced system into the ansatz gives an $\mathfrak{s}$-invariant solution of $\mathcal{L}$. See [20, Section 5.2.1], [103, Chapter 3] or [112, Chapter V] for a related theoretical background as well as [148, Section 2] and [74, Section B] for the descriptions of the optimized procedures of Lie reduction in the case of three independent variables and of step-by-step reductions with involving hidden symmetries, respectively.

Using the subalgebras listed in Theorem 4.23, we construct a complete list of G-inequivalent group-invariant codimension-one submodels of the system (4.1). The systematic study of these submodels, even within the framework of classical symmetry analysis, is too involved. In particular, for each reduced system, the complexity of computing its Lie invariance algebra is comparable to that presented in Section 4.4 for

the system (4.1). Nevertheless, we were able to compute the Lie invariance algebras for all of the above submodels by hand, and in addition, we checked these results using several packages for symbolic computation of Lie symmetries for low values of the number of layers m.

In this section, $v^1, \dots, v^m$ denote the new dependent variables of the two new independent variables (z_1, z_2), $v := (v^1, \dots, v^m)^{\mathsf{T}}$ and the subscripts 1 and 2 of functions denote derivatives with respect to z_1 and z_2, respectively

4.8.1 Subalgebra family 1.1

Reduced system

An ansatz constructed with the one-dimensional subalgebra $\mathfrak{s}^{ac}_{1.1} := \langle \mathcal{P}^t + a\mathcal{P}^y + c_k\mathcal{J}^k \rangle$ of $\mathfrak{g}$, where a and $c := (c_1, \dots, c_m)^{\mathsf{T}}$ are arbitrary constants with $c \in \operatorname{im}\mathsf{F}$, is $\psi^i = v^i + c_i t + ax$ or, in the vector notation,

$$\psi = v + tc + ax\bar{1} \quad \text{with} \quad z_1 = x, \quad z_2 = y - at,$$

where $\bar{1}$ is the all-ones m-column defined in Section 4.2.2. The corresponding reduced system takes the form

$$\begin{aligned} &\{v^i, w^i\} + f_{i,i-1}\big(c_{i-1} - c_i\big) - f_{i,i+1}\big(c_i - c_{i+1}\big) + a\beta = 0,\\ &w^i := v^i_{11} + v^i_{22} + f_{i,i-1}(v^{i-1} - v^i) - f_{i,i+1}(v^i - v^{i+1}) + \beta z_2, \end{aligned} \tag{4.26}$$

where $\{v^i, w^i\} := v^i_1 w^i_2 - v^i_2 w^i_1$ is the Poisson bracket of v^i and w^i, and $c_0, c_{m+1} := 0$. The matrix form of the system (4.26) is

$$\begin{aligned} &\{v, w\} + \mathsf{F}c + a\beta\bar{1} = 0,\\ &w := v_{11} + v_{22} + \mathsf{F}v + \beta z_2\bar{1} \end{aligned} \tag{4.27}$$

with the componentwise Poisson bracket $\{v, w\}$, $\{v, w\} := (\{v^i, w^i\})^{\mathsf{T}}$ and the matrix $\mathsf{F} = (f_{ij})$ defined in Section 4.2.2. Using the package DESOLV [27] for the computer algebra system Maple we computed the maximal Lie invariance algebra of the system (4.26) for low values of the number of layers m, $m \in \{2, 3, 4\}$, but such a computation is not possible for an arbitrary value of m. Nevertheless, we can make it by hand.

Theorem 4.26. *The maximal Lie invariance algebra of the system* (4.26) *is the linear span* $\langle \partial_{z_1}, \partial_{z_2}, \partial_{v^1}, \dots, \partial_{v^m} \rangle$*, and it is entirely induced by the maximal Lie invariance algebra* $\mathfrak{g}$ *of the system* (4.1).

Proof. We follow the consideration of Section 4.4. More specifically, we formally replace the real independent variables (z_1, z_2) by the complex conjugate variables $z = z_1 + \mathrm{i}z_2$ and $\bar{z} = z_1 - \mathrm{i}z_2$, where i is the imaginary unit, which reduces the system (4.26) to the form

$$\begin{aligned} &v^i_z v^i_{z\bar{z}\bar{z}} - v^i_{\bar{z}} v^i_{zz\bar{z}} + G^i(z, \bar{z}, v, v_z, v_{\bar{z}}) = 0,\\ &G^i := \frac{1}{4}\big(v^i_z(\mathsf{F}v_{\bar{z}})^i - v^i_{\bar{z}}(\mathsf{F}v_z)^i\big) + \frac{\beta\mathrm{i}}{8}(v^i_z + v^i_{\bar{z}} + a) + \frac{\mathrm{i}}{8}(\mathsf{F}c)^i. \end{aligned} \tag{4.28}$$

The condition of infinitesimal invariance of the system (4.28) with respect to a vector field of the form $Q = \xi(z, \bar{z}, v)\partial_z + \bar{\xi}(z, \bar{z}, v)\partial_{\bar{z}} + \eta^j(z, \bar{z}, v)\partial_{v^j}$,

$$\eta^{i,z} v^i_{z\bar{z}\bar{z}} + v^i_z \eta^{i,z\bar{z}\bar{z}} - \eta^{i,\bar{z}} v^i_{zz\bar{z}} - v^i_{\bar{z}} \eta^{i,zz\bar{z}} + Q_{(1)}G^i = 0 \quad \text{on solutions of (4.28)},$$

is successively split with respect to parametric derivatives when assuming $v^i_{z\bar{z}\bar{z}}$ as the leading derivatives and taking into account the constraints derived for its components on the previous steps. Here and in what follows, $\bar{\xi}$ is the conjugate of ξ, $Q_{(1)}$ is the first prolongation of Q, $\eta^{i,\mu\nu\kappa}$ are the components of the third prolongation of Q for third-order derivatives, D_μ denotes the total derivative operator with respect to $\mu \in \{z, \bar{z}\}$,

$$\begin{aligned} &Q_{(1)} = Q + \eta^{j,\mu}\partial_{v^j_\mu} \quad \text{with} \quad \eta^{i,\mu} = \mathrm{D}_\mu(\eta^i - \xi v^i_z - \bar{\xi} v^i_{\bar{z}}) + \xi v^i_{z\mu} + \bar{\xi} v^i_{\bar{z}\mu},\\ &\eta^{i,\mu\nu\kappa} = \mathrm{D}_\mu\mathrm{D}_\nu\mathrm{D}_\kappa(\eta^i - \xi v^i_z - \bar{\xi} v^i_{\bar{z}}) + \xi v^i_{z\mu\nu\kappa} + \bar{\xi} v^i_{\bar{z}\mu\nu\kappa},\\ &\mathrm{D}_\mu = \partial_\mu + v^j_\mu \partial_{v^j} + v^j_{\mu\nu}\partial_{v^j_\nu} + v^j_{\mu\nu\kappa}\partial_{v^j_{\nu\kappa}} + v^j_{\mu\nu\kappa\lambda}\partial_{v^j_{\nu\kappa\lambda}} + \cdots, \end{aligned}$$

and the indices κ, λ and ν run through the set $\{z, \bar{z}\}$. Separately collecting the coefficients of the third-order jet variables v^i_{zzz}, $v^i_{\bar{z}\bar{z}\bar{z}}$, $v^j_{zz\bar{z}}$ with $j \neq i$ and $v^i_{zz\bar{z}}$ in the above ith condition, we derive the equations $\mathrm{D}_z\bar{\xi} = 0$, $\mathrm{D}_{\bar{z}}\xi = 0$, $\eta^i_{v^j} = 0$, $j \neq i$, and $v^i_{\bar{z}}\eta^i_z = v^i_z\eta^i_{\bar{z}}$, which further splits to $\bar{\xi}_z = \xi_{\bar{z}} = \xi_{v^k} = 0$ and $\eta^i_z = \eta^i_{\bar{z}} = 0$. Similarly continuing with the second-order jet variables v^i_{zz}, $v^i_{\bar{z}\bar{z}}$ and $v^i_{z\bar{z}}$, we obtain $\mathrm{D}_{\bar{z}}\eta^i_{v^i} = 0$, $\mathrm{D}_z\eta^i_{v^i} = 0$, $v^i_{\bar{z}}\xi_{zz} = v^i_z\bar{\xi}_{\bar{z}\bar{z}}$, which implies $\eta^i_{v^iv^i} = 0$ and $\xi_{zz} = \bar{\xi}_{\bar{z}\bar{z}} = 0$. The next step is to collect the coefficients of the first-degree terms with respect to the jet variables v^i_z

and $v^i_{\bar z}$, giving $\xi_z = \bar\xi_{\bar z}$ and $\eta^i_{v^i} = 3\xi_z$. Finally, analyzing the quadratic terms with respect to $(v^k_z, v^k_{\bar z})$, results in the equation $\xi_z = 0$. Therefore, all the components of Q are constants, which is an obvious solution of the determining equations for Q.

The normalizer $\mathrm{N}_{\mathfrak g}(\mathfrak s^{ac}_{1.1})$ of the subalgebra $\mathfrak s^{ac}_{1.1}$ in the algebra $\mathfrak g$ is spanned by the vector fields $\mathcal P^t$, $\mathcal P^y$, $\mathcal P^x(1)$, $\mathcal J^1, \dots, \mathcal J^m$. Therefore, the algebra of induced symmetries of the reduced system (4.26) is the linear span $\langle \partial_{z_1}, \partial_{z_2}, \partial_{v^1}, \dots, \partial_{v^m}\rangle$, which coincides with the maximal Lie invariance algebra of this system. □

Case of affine dependence

An important particular case of the submodels of the form (4.26) corresponds to the zero values of the parameter combinations $f_{i,i-1}(c_{i-1} - c_i) - f_{i,i+1}(c_i - c_{i+1}) + a\beta$, $i = 1, \dots, m$. The matrix form of this vanishing condition is $\mathsf F c + a\beta\bar 1 = 0$. Under the physical constraints of positivity on the essential components of the matrix $\mathsf F$ and the parameter β, we have $\ker \mathsf F \perp_{\mathsf W} \operatorname{im}\mathsf F$ and can split the vanishing condition into $a = 0$ and $\mathsf F c = 0$, i.e., $c \in \langle \bar 1\rangle$, and thus $c = 0$ since $c \perp_{\mathsf W} \bar 1$ here. This gives stationary solutions of the original system (4.1). For each i, the equation $\{v^i, w^i\} = 0$ implies that v^i and w^i are functionally dependent.

Each solution for which this dependence is affine satisfies a linear system of partial differential equations of the form $w^i = b_{1i}v^i + b_{0i}$ with some real constants b_{1i} and b_{0i}. Collecting the parameters b_{1i} and b_{0i} into the matrix $\mathsf B := \operatorname{diag}(b_{11}, \dots, b_{1m})$ and the column $b := (b_{01}, \dots, b_{0m})^{\mathsf T}$, respectively, we rewrite the above system in the matrix form

$$v_{11} + v_{22} + (\mathsf F - \mathsf B)v = b - \beta z_2 \bar 1. \tag{4.29}$$

Since the matrix $\mathsf F - \mathsf B$ is tridiagonal with the same subdiagonal and the superdiagonal entries as those of $\mathsf F$, which are positive, it is also similar to a symmetric tridiagonal matrix via the same matrix $\mathsf D$ as the matrix $\mathsf F$ is, see Section 4.2.3. Hence, it is diagonalizable and its eigenvalues ν_i are real and pairwise distinct. Thus, we can assume that $\nu_1 < \dots < \nu_m$. Consider the weighted inner product $(\cdot,\cdot)_{\mathsf W}$ with the weight matrix $\mathsf W := \mathsf D^{-2}$ and denote by $\tilde e_{\nu_i}$ an eigenvector of $\mathsf F - \mathsf B$ with the eigenvalue ν_i and $\|\tilde e_{\nu_i}\|_{\mathsf W} = 1$. Since $\mathsf F - \mathsf B$ is irreducible, the first and the last components of each of the eigenvectors are nonzero. According to the Perron–Frobenius theorem, one can

choose $\tilde{e}_{\nu_m}$ with all components to be positive, and all the other eigenvectors $\tilde{e}_{\nu_i}$ (with $i < m$) necessarily have components with opposite signs. These eigenvectors constitute an orthonormal basis in the W-weighted inner product $(\cdot,\cdot)_{\mathsf{W}}$. The transition matrix R to the obtained eigenbasis admits the representation $\mathsf{R} = \mathsf{D}\tilde{\mathsf{O}}$, where the matrix $\tilde{\mathsf{O}}$ is orthogonal with respect to the standard inner product and the diagonal matrix D is defined in Section 4.2.3, and $\mathsf{N} := \operatorname{diag}(\nu_1,\dots,\nu_m) = \mathsf{R}^{-1}(\mathsf{F}-\mathsf{B})\mathsf{R}$.

Under the change of dependent variables $\tilde{v} = \mathsf{R}^{-1}(v - v^0)$, where v^0 is a particular solution of the system (4.29), this system takes the form

$$\tilde{v}_{11} + \tilde{v}_{22} + \mathsf{N}\tilde{v} = 0, \tag{4.30}$$

which is a decoupled system of homogeneous linear partial differential equations, where the kind of the ith equation depends on the sign of ν_i, the (two-dimensional) modified Helmholtz, the Laplace or the Helmholtz equation if ν_i is negative, zero or positive, respectively. The form of a particular solution v^0 of the system (4.30) depends on whether the matrix $\mathsf{F}-\mathsf{B}$ is degenerate or not.

If $\det(\mathsf{F}-\mathsf{B}) = 0$, the image of the operator $\mathsf{F}-\mathsf{B}$ is orthogonal to its kernel with respect to the above weighted inner product, $\ker(\mathsf{F}-\mathsf{B}) = \langle\tilde{e}_0\rangle$ and the restriction of the operator $\mathsf{F}-\mathsf{B}$ on its image $\langle\tilde{e}_{\nu_i}, \nu_i \neq 0\rangle$ is invertible. Then we can take

$$v^0 = (3b - \beta z_2\bar{1}, \tilde{e}_0)_{\mathsf{W}}\frac{z_2^2}{6}\tilde{e}_0 + (\mathsf{F}-\mathsf{B})^+(b - \beta z_2\bar{1}),$$

where $(\mathsf{F}-\mathsf{B})^+$ is the Moore–Penrose inverse of $\mathsf{F}-\mathsf{B}$, see Footnote 3. As a result, we construct a wide family of solutions of the original system (4.1) that are expressed in terms of the general solutions of the modified Helmholtz, the Laplace or the Helmholtz equations,

- $$\psi = \mathsf{R}\tilde{v}(x,y) - \beta(\bar{1},\tilde{e}_0)_{\mathsf{W}}\frac{y^3}{6}\tilde{e}_0 + \gamma y^2\tilde{e}_0 - \beta y(\mathsf{F}-\mathsf{B})^+\bar{1}, \tag{4.31}$$

where $\mathsf{B} := \operatorname{diag}(b_{11},\dots,b_{1m})$, b_{11}, …, b_{1m} and γ are arbitrary constants, $\tilde{v}_{xx} + \tilde{v}_{yy} + \mathsf{N}\tilde{v} = 0$, $\mathsf{N} := \operatorname{diag}(\nu_1,\dots,\nu_m)$, ν_i are the eigenvalues of $\mathsf{F}-\mathsf{B}$, the columns of R are the corresponding eigenvectors $\tilde{e}_{\nu_i}$ of the matrix $\mathsf{F}-\mathsf{B}$ with $\|\tilde{e}_{\nu_i}\|_{\mathsf{W}} = 1$, $(\cdot,\cdot)_{\mathsf{W}}$ is the weighted inner product with the weight matrix $\mathsf{W} := \mathsf{D}^{-2}$, and $(\mathsf{F}-\mathsf{B})^+$ is the Moore–Penrose inverse of $\mathsf{F}-\mathsf{B}$. We omitted the constant vector summand since it is

inessential up to the G-equivalence or, more specifically, the stream function tuple ψ is defined up to adding a constant tuple.

For the system (4.29) with nondegenerate matrix $\mathsf{F}-\mathsf{B}$, we can take a particular solution of much simpler form, $v^0=(\mathsf{F}-\mathsf{B})^{-1}\big(b-\beta z_2\bar{1}\big)$. This leads to a wide family of solutions of the original system (4.1) that are expressed in terms of the general solutions of the modified Helmholtz or the Helmholtz equations,

- $\psi=\mathsf{R}\tilde{v}(x,y)-\beta y(\mathsf{F}-\mathsf{B})^{-1}\bar{1}$, (4.32)

where again $\mathsf{B}:=\operatorname{diag}(b_{11},\dots,b_{1m})$, b_{11}, $\dots$, b_{1m} are arbitrary constants, $\tilde{v}_{xx}+\tilde{v}_{yy}+\mathsf{N}\tilde{v}=0$, $\mathsf{N}:=\operatorname{diag}(\nu_1,\dots,\nu_m)$, ν_i are the eigenvalues of $\mathsf{F}-\mathsf{B}$, the columns of R are the corresponding eigenvectors of the matrix $\mathsf{F}-\mathsf{B}$, and we omitted the constant vector summand since it is inessential up to the G-equivalence or, more specifically, the stream function tuple ψ is defined up to adding a constant tuple.

While the found solutions (4.31) and (4.32) of the multi-layer quasi-geostrophic system (4.1) are stationary, they can be extended to time-evolving solutions using (generalized) Galilean boosts in the x-direction, which are point symmetries of (4.1). More specifically, given a stationary solution ψ of (4.1), its pushforward $\mathcal{P}(\chi)_*\psi$ by the generalized Galilean transformation $\mathcal{P}(\chi)$ with an arbitrary sufficiently smooth function χ of t, which has been defined in Section (4.5), is also a solution of (4.1). In order for $\mathcal{P}(\chi)_*\psi$ to represent a physically meaningful solution, the parameter function χ should satisfy additional constraints, in particular, on the sign of its derivative.

Many closed-form solutions of the modified Helmholtz, the Laplace and the Helmholtz equations were presented in the literature. In particular, wide families of such solutions were constructed in [24] and [93, Sections 1.1–1.3] using separation of variables based on symmetries. See also [118, Chapter 7].

One can select various physically relevant solutions from the above families or use them as building blocks for merging to more complicated solutions. The signs of the eigenvalues of the matrix $\mathsf{F}-\mathsf{B}$ play an important role in this selection. A sufficient condition for this matrix to have at least one positive eigenvalue is $f_{i,i-1}+f_{i,i+1}+b^{i1}<0$ for some $i\in\{1,\dots,m\}$, which follows from application of the Poincaré separation theorem to the symmetric matrix $\mathsf{D}^{-1}(\mathsf{F}-\mathsf{B})\mathsf{D}$. Sylvester's criterion implies that all the eigenvalues of the matrix $\mathsf{F}-\mathsf{B}$ are positive if all the leading principal minors

of its symmetric counterpart $\mathsf{D}^{-1}(\mathsf{F}-\mathsf{B})\mathsf{D}$ positive. In view of the Gershgorin circle theorem, a sufficient condition for the positivity of all the eigenvalues of the matrix $\mathsf{F}-\mathsf{B}$ is that $b_i < -\min\big(2|f_{i,i-1}+f_{i,i+1}|, |f_{i,i-1}+f_{i,i+1}+f_{i-1,i}+f_{i+1,i}|\big)$ for any i. Analogously, all the eigenvalues of the matrix $\mathsf{F}-\mathsf{B}$ are negative if all the odd and even leading principal minors are negative and positive, respectively, and a sufficient condition for this is that all b_i are positive.

Solutions associated with entirely bounded solutions of Helmholtz equations

Among the solutions constructed in Section 4.8.1, we single out those corresponding to velocity fields that are defined on the entire (x,y)-plane and bounded. Denote $\alpha_i := -\beta\big((\mathsf{F}-\mathsf{B})^+\bar{1}, \tilde{e}_{\nu_i}\big)_{\mathsf{W}}$. If $\nu_i<0$, then the corresponding $\tilde{e}_{\nu_i}$-component is $\tilde{v}^i+\alpha_i y$ for some solution $\tilde{v}^i$ of the modified Helmholtz equation $\tilde{v}^i_{xx}+\tilde{v}^i_{yy}-|\nu_i|\tilde{v}^i=0$. Since the nonzero solutions of this equation that are defined on the entire (x,y)-plane have exponential growth at infinity, the only choice for $\tilde{v}^i$ is $\tilde{v}^i\equiv 0$. If $\nu_i=0$, then $\alpha_i=0$ and the corresponding $\tilde{e}_{\nu_i}$-component is $\tilde{v}^i-\frac{1}{6}\beta(\bar{1},\tilde{e}_0)_{\mathsf{W}}y^3+\gamma y^2$ for some solution $\tilde{v}^i$ of the Laplace equation $\tilde{v}^i_{xx}+\tilde{v}^i_{yy}=0$ and some constant γ. The first derivatives of this component are bounded if and only if $\gamma=0$, $(\bar{1},\tilde{e}_0)_{\mathsf{W}}=0$ and both $\tilde{v}^i_x$ and $\tilde{v}^i_y$ are constant, that is, the function $\tilde{v}^i$ is affine with respect to (x,y).[6] For $\nu_i>0$, the corresponding $\tilde{e}_{\nu_i}$-component is $\tilde{v}^i+\alpha_i y$ for some solution $\tilde{v}^i$ of the Helmholtz equation $\tilde{v}^i_{xx}+\tilde{v}^i_{yy}+\nu_i\tilde{v}^i=0$. The first derivatives of this component are bounded if and only if the first derivatives of $\tilde{v}^i$ are bounded, which is equivalent to the fact that the function $\tilde{v}^i$ itself is bounded. The latter means that we have the representation of $\tilde{v}^i$ as a *generalized Herglotz wave function*,

$$\tilde{v}^i = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}\sqrt{\nu_i}(x\cos\theta+y\sin\theta)}\mathrm{d}\mu_i(\theta)$$

[6]The derivative $\tilde{v}^i_x$ itself is bounded and satisfies the Laplace equation; hence it is constant in view of the Liouville theorem, $\tilde{v}^i_x =: \sigma_1 = \text{const}$. The boundedness of the first y-derivative of the component implies that the solution $\tilde{v}^i_y$ of the Laplace equation grows not faster than a quadratic polynomial. It follows from the extended Liouville theorem [143, Theorem 5.9] that actually $\tilde{v}^i_y$ is a quadratic polynomial. Again recalling about the boundedness of the first y-derivative of the component, we have that $\tilde{v}^i_y=\frac{1}{2}\beta(\bar{1},\tilde{e}_0)_{\mathsf{W}}y^2-2\gamma y+\sigma_2$ for some constant σ_2. Then the function $\tilde{v}^i$ satisfies the Laplace equation if and only if $\gamma=0$ and $(\bar{1},\tilde{e}_0)_{\mathsf{W}}=0$, i.e., the derivative $\tilde{v}^i_y$ is constant as well.

for some finite complex Radon measure μ_i on the circle.[7] More details on Herglotz wave functions can be found in [32, 46] and references therein. For the function $\tilde{v}^i$ to be real-valued, the measure μ_i should satisfy, roughly speaking, the condition $\mu_i(\theta+\pi) = \bar{\mu}_i(\theta)$, where $\bar{\mu}_i$ denotes the complex conjugate of the measure μ_i. The alternative way to construct real-valued solutions is to take the real parts of generalized Herglotz wave functions, where the corresponding measures do, in general, not satisfy the above condition.

Summing up, the stationary solution of the original system (4.1) with an affine componentwise dependence between the stream function ψ and the potential vorticity q that are defined in the entire (x,y)-plane and correspond to bounded velocity fields are exhausted by the tuples

$$\bullet\quad \psi = \sum_{j\colon \nu_j>0} \operatorname{Re}\left(\int_0^{2\pi} \mathrm{e}^{\mathrm{i}\sqrt{\nu_j}(x\cos\theta+y\sin\theta)}\mathrm{d}\mu_j(\theta)\right)\tilde{e}_{\nu_j} + (\sigma_1 x+\sigma_2 y)\tilde{e}_0 - \beta y(\mathsf{F}-\mathsf{B})^+\bar{1}. \tag{4.33}$$

Here ν_j runs through the subset of $\{\nu_1,\dots,\nu_m\}$ constituted by the positive eigenvalues of $\mathsf{F}-\mathsf{B}$, $\tilde{e}_{\nu_i}$ is an eigenvector of $\mathsf{F}-\mathsf{B}$ with the eigenvalue ν_i and $\|\tilde{e}_{\nu_i}\|_{\mathsf{W}}=1$, $\tilde{e}_0=0$ if the matrix $\mathsf{F}-\mathsf{B}$ is invertible, σ_1 and σ_2 are arbitrary constants, and μ_i are arbitrary finite complex Radon measures on the circle. The operator $(\mathsf{F}-\mathsf{B})^+$ is the Moore–Penrose inverse of $\mathsf{F}-\mathsf{B}$, see Footnote 3.

Pushing the stationary solutions (4.33) forward using, e.g., Galilean boosts $\mathcal{P}(\gamma t)$ with $\gamma<0$, we obtain the travelling-wave solutions of the original system (4.1),

$$\bullet\quad \psi = \sum_{j\colon \nu_j>0} \operatorname{Re}\left(\int_0^{2\pi} \mathrm{e}^{\mathrm{i}\sqrt{\nu_j}((x-\gamma t)\cos\theta+y\sin\theta)}\mathrm{d}\mu_j(\theta)\right)\tilde{e}_{\nu_j} + (\sigma_1(x-\gamma t)+\sigma_2 y)\tilde{e}_0 - \beta y(\mathsf{F}-\mathsf{B})^+\bar{1} - \gamma y\bar{1},$$

[7] For any tempered solution u of the (homogeneous) Helmholtz equation $u_{xx}+u_{yy}+\kappa^2 u=0$, its Fourier transform $\hat{u}(\xi)$, $\xi:=(\xi_1,\xi_2)$, satisfies the equation $(|\xi|^2-\kappa^2)\hat{u}(\xi)=0$, and thus $\hat{u}(\xi)$ is a distribution supported on the circle $\{\xi \mid |\xi|=\kappa\}$. If u is a bounded solution, then $\hat{u}(\xi)$ is a finite measure supported on this circle. The solution u of the Helmholtz equation is the Herglotz wave function with kernel $g\in L^2(S^1)$, $u(x,y)=\int_0^{2\pi} g(\theta)\mathrm{e}^{\mathrm{i}\kappa(x\cos\theta+y\sin\theta)}\mathrm{d}\theta$, if and only if it has L^2-averaged $r^{-1/2}$-decay, i.e., there exists a (finite) limit of $\kappa(4\pi r)^{-1}\int_{x^2+y^2\leqslant r^2}|u(x,y)|^2\mathrm{d}x\mathrm{d}y$ as $r\to\infty$ [46], and it coincides with the square of the L^2-norm of g. Recall that every distributional solution of the homogeneous Helmholtz equation is a real analytic function.

which propagates in the negative direction of x. In the geophysical context, this corresponds to westward wave propagation relative to the Earth's surface.

Suitably specifying the form of the (finite) measures μ_i in (4.33), we construct various types of wave solutions. To demonstrate the physical capability of the solution family (4.33), we construct several distinct classes of solutions bounded on the entire plane: baroclinic simple plane waves, coherent baroclinic eddies, coherent baroclinic hetons and superpositions of solutions of the above kinds. We also plot some simple solutions among the obtained ones in the case of three layers with the numerical data from the Section 4.2.5 and specific values of solution parameters, neglecting the tuples $u_{\mathrm{bg}} = \beta(\mathsf{F}-\mathsf{B})^{+}\bar{1}$ of constant background flows in the layers from these solutions for a better illustration. The multiplier $\psi_0 := 15000\ \mathrm{m}^2/\mathrm{s}$ is chosen for solutions to be of the realistic scale.

Baroclinic plane waves. If the density of μ_i is $\sum_{n=0}^{\infty} A_{in}\delta(\theta-\theta_n)$, where A_{in} are arbitrary complex constants with $\sum_{n=0}^{\infty}|A_{in}|<\infty$, $0\leqslant\theta_n<2\pi$ and δ denotes the Dirac delta-function, then the function $\tilde{v}^i$ and its Galilean boost are the simple and the travelling two-dimensional plane harmonics,

$$\tilde{v}^i = \mathrm{Re}\sum_{n=0}^{\infty} A_{in}\mathrm{e}^{\mathrm{i}\sqrt{\nu_i}(x\cos\theta_n+y\sin\theta_n)}, \quad \mathcal{P}^x(\gamma t)_*\tilde{v}^i = \mathrm{Re}\sum_{n=0}^{\infty} A_{in}\mathrm{e}^{\mathrm{i}\sqrt{\nu_i}((x-\gamma t)\cos\theta_n+y\sin\theta_n)},$$

respectively, and these solutions describe stationary and travelling geophysical Rossby waves. Note that each fixed harmonic can be derived via the multiplicative separation of variables in the Cartesian coordinate system.

To plot a representative of this solution family, we select $\mathsf{B} = \mathrm{diag}(-7,4,-1)\cdot 10^{-10}$. The eigenvalues and corresponding eigenvectors of the matrix $\mathsf{F}-\mathsf{B}$ are given by

$$\nu_1 \approx -12.9\cdot 10^{-10}, \quad \nu_2 \approx -0.76\cdot 10^{-10}, \quad \nu_3 \approx 1.67\cdot 10^{-10},$$
$$\tilde{e}_{\nu_1} \approx (0.74,-1.02,0.14)^{\mathsf{T}},\ \tilde{e}_{\nu_2} \approx (-0.82,-0.06,1.02)^{\mathsf{T}},\ \tilde{e}_{\nu_3} \approx (0.96,0.35,0.25)^{\mathsf{T}}.$$

Note that in this case the matrix $\mathsf{F}-\mathsf{B}$ is invertible, so $\tilde{e}_0 = 0$, and the tuple of constant background flows in the layers is $u_{\mathrm{bg}} := \beta(\mathsf{F}-\mathsf{B})^{-1}\bar{1} \approx (0.28,0.06,-0.08)^{\mathsf{T}}$. Evidently, only the third mode is relevant for constructing globally bounded solutions. We choose the measure μ_3 to be proportional to the sum of two Dirac delta functions centered at a propagation angle $\pi/4$, $\mathrm{d}\mu_3(\theta) = \frac{1}{2}\psi_0\big(\delta(\theta-\pi/4)+\delta(\theta-5\pi/4)\big)\mathrm{d}\theta$. As

a result, we recover the plane wave solution

$$\psi = \psi_0 \cos\big(\sqrt{\nu_3/2}(x+y)\big)\tilde{e}_{\nu_3} - y u_{\mathrm{bg}} \quad \text{with } \psi_0 := 15000\ \mathrm{m}^2/\mathrm{s}. \tag{4.34}$$

The plot of (4.34) is given, up to the summand that gives constant flows in layers, in Figure 4.2. Physically, this solution corresponds to a stationary Rossby wave pattern where the westward phase propagation of the wave is exactly balanced by the eastward advection of the background current.

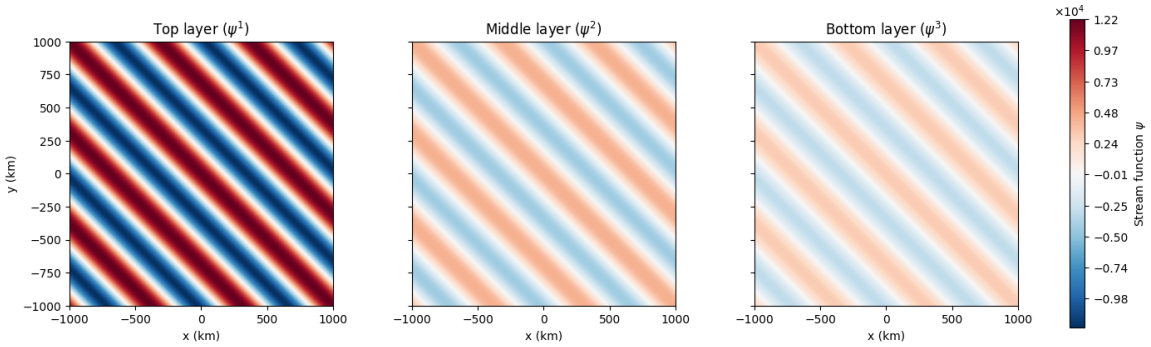


Figure 4.2: Baroclinic simple plane waves (4.34).

Coherent baroclinic eddies and hetons. The density $\sum_{n=0}^{\infty} A_{in} \mathrm{e}^{\mathrm{i}n\theta}$ of μ_i, where $(A_{in}, n \in \mathbb{N}_0)$ is an arbitrary tempered sequence of complex constants, corresponds to a linear superposition of two-dimensional radial harmonics,

$$\tilde{v}^i = 2\pi\,\mathrm{Re} \sum_{n=0}^{\infty} \mathrm{i}^n A_{in} J_n(\sqrt{\nu_i} r) \mathrm{e}^{\mathrm{i}n\vartheta}, \quad r := \sqrt{x^2+y^2}, \quad \vartheta := \arccos\frac{x}{r},$$

where J_n denotes the nth order Bessel function of the first kind. Note that each fixed harmonic can be derived via the multiplicative separation of variables in the polar coordinate system.

For instance, using the same matrix $\mathsf{B} = \mathrm{diag}(-7, 4, -1) \cdot 10^{-10}$ as above and selecting measure μ_3 to be given by a constant density on the spectral circle, $\mathrm{d}\mu_3(\theta) = \psi_0 \mathrm{d}\theta/2\pi$, we recover a rotationally symmetric isolated vortex represented in terms of the zeroth-order Bessel function of the first kind,

$$\psi = \psi_0 J_0(\sqrt{\nu_3} r)\tilde{e}_{\nu_3} - y u_{\mathrm{bg}} \quad \text{with } \psi_0 := 15000\ \mathrm{m}^2/\mathrm{s}. \tag{4.35}$$

The plot of this solution (up to the tuple $u_{\mathrm{bg}} \approx (0.28, 0.06, -0.08)^{\mathsf{T}}$ of constant background flows in layers) is given in Figure 4.3. Physically, it represents a standing baroclinic eddy embedded in a uniform zonal current.

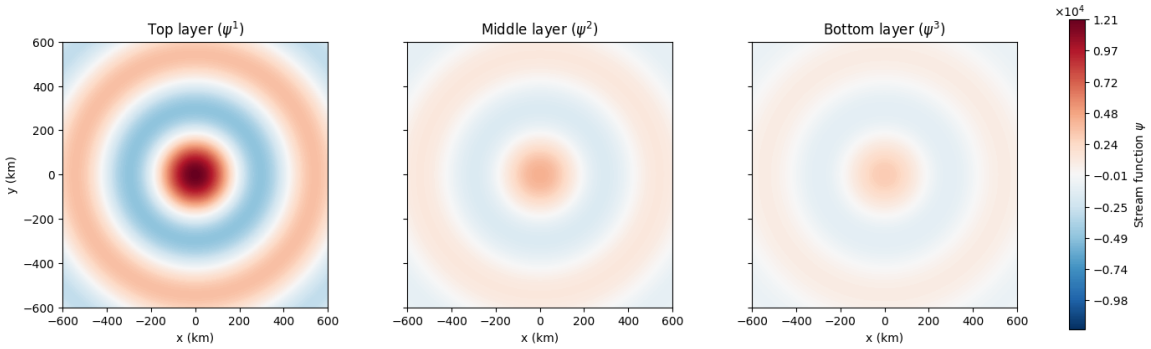


Figure 4.3: Coherent baroclinic eddy (4.35).

To obtain a heton solution, we take the matrix $\mathsf{B} = \mathrm{diag}(-10, -8, -5) \cdot 10^{-10}$ The appropriate eigenvalue and eigenvector of $\mathsf{F} - \mathsf{B}$ for constructing a heton are then $\nu_2 \approx 3.13 \cdot 10^{-10}$ and $\tilde{e}_{\nu_2} \approx (0.79, 0.14, -1.12)^{\mathsf{T}}$ since the first and the second components of $\tilde{e}_{\nu_2}$ have the same order and opposite signs. The measure μ_2 is chosen by a constant density on the spectral circle as above, $\mathrm{d}\mu_2(\theta) = \psi_0 \mathrm{d}\theta/2\pi$, and $\mu_3 = 0$ ($\nu_3 > 0$ as well). While the solution arising from this setting is formally still of the form (4.35) up to the permutation of ν_2 and ν_3,

$$\psi = \psi_0 J_0(\sqrt{\nu_2} r)\tilde{e}_{\nu_2} - y u_{\mathrm{bg}} \quad \text{with } \psi_0 := 15000 \text{ m}^2/\text{s}, \tag{4.36}$$

it has a different physical interpretation, see, e.g., [128]. We give its plot in Figure 4.4, neglecting the tuple $u_{\mathrm{bg}} \approx (0.86, 1.75, 3.92)^{\mathsf{T}} \cdot 10^{-2}$ of constant background flows in layers.

Superpositions of Rossby waves with different modes. A more complicated dynamics can be obtained via linear superposition of solutions. Consider now the matrix $\mathsf{B} := \mathrm{diag}(-8, 3, -2) \cdot 10^{-10}$. The relevant eigenvalues and eigenvectors of the matrix $\mathsf{F} - \mathsf{B}$ are $\nu_2 \approx 2.40 \cdot 10^{-10}$, $\nu_3 \approx 2.67 \cdot 10^{-10}$ and $\tilde{e}_{\nu_2} \approx (-0.83, -0.06, 1.02)^{\mathsf{T}}$, $\tilde{e}_{\nu_3} \approx (0.96, 0.35, 0.25)^{\mathsf{T}}$. The plane wave solution of the form (4.33) with

$$\mathrm{d}\mu_2(\theta) = \frac{\psi_0}{2}\big(\delta(\theta - \pi/3) + \delta(\theta - 2\pi/3)\big)\mathrm{d}\theta \text{ and } \mathrm{d}\mu_3(\theta) = \frac{\psi_0}{2}\big(\delta(\theta - \pi/4) + \delta(\theta - 5\pi/4)\big)\mathrm{d}\theta,$$

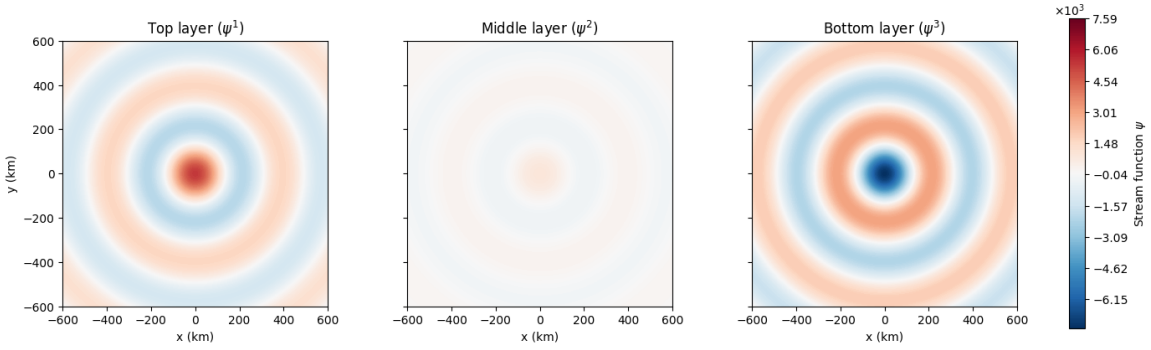


Figure 4.4: Coherent baroclinic heton (4.36).

where $\psi_0 := 15000\ \mathrm{m}^2/\mathrm{s}$, is

$$\psi(x,y) = \psi_0 \cos\Big(\frac{\sqrt{\nu_2}}{2}(\sqrt{3}x+y)\Big)\tilde{e}_{\nu_2} + \psi_0 \cos\big(\sqrt{\nu_3/2}(x+y)\big)\tilde{e}_{\nu_3} - y u_{\mathrm{bg}}, \qquad (4.37)$$

The plot of this solution (up to neglecting the tuple $u_{\mathrm{bg}} \approx (-0.21, 0.01, 0.43)^{\mathsf{T}} \cdot 10^{-2}$ of constant background flows in layers) is given in Figure 4.5. This picture represents the formation of cyclone and anticyclone in the bottom layer, while the top and the middle layers have a Rossby wave pattern.

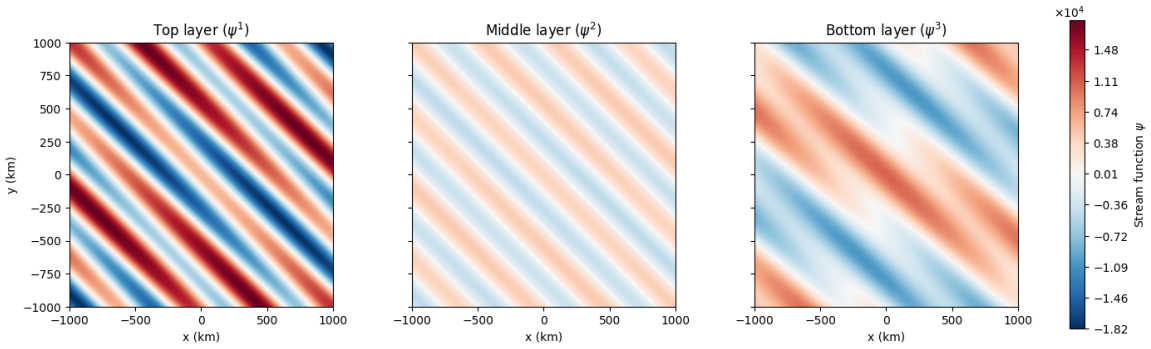


Figure 4.5: Superposition of two baroclinic Rossby waves (4.37).

General linear superpositions. One can construct exact solutions of the system (4.1) with quite irregular behaviour using merely the linear superposition of a particular solution of the system (4.29) with the above regular solutions patterns of the homogeneous counterpart of this system with the same parameter matrix B. The involved solution patterns, which are two-dimensional plane and radial harmonics, can in addition be shifted and scaled before superposing. An example of such an irregular

solution is presented in Figure 4.6. It is just the linear superposition of three plane and seven radial harmonics of low order.

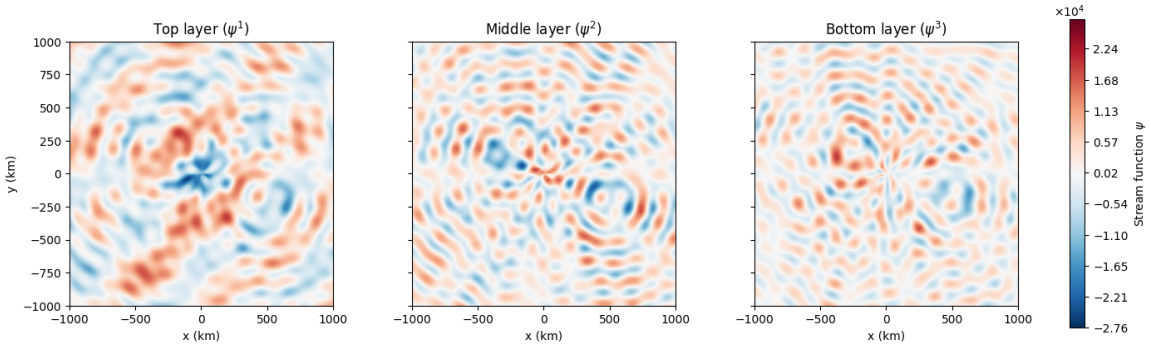


Figure 4.6: An example of linear superposition of various solution patterns.

Merging solutions and modons

Another way to construct a wide class of physically relevant (usually weak) solutions of the multi-layer quasi-geostrophic problem (4.1) using the Lie reduction with respect to the subalgebra $\mathfrak{s}_{1.1}^{00}$ is to partition the solution domain into several subdomains and then consider different solutions of the form (4.31) or (4.32) for the same or even different systems of the form (4.29) on the subdomains, merging these solutions by imposing interface boundary conditions between subdomains.

In this section, plotting merged solutions with the numerical data from the Section 4.2.5 and specific values of solution parameters, we do not neglect the tuples $u_{\rm bg} = \beta(\mathsf{F}-\mathsf{B})^{+}\bar{1}$ of constant background flows in the layer subdomains since otherwise the plots are discontinuous.

Dipolar vortices (modons). Consider arbitrary diagonal matrices $\tilde{\mathsf{B}}$ and $\hat{\mathsf{B}}$ such that all eigenvalues ν_i of $\mathsf{F}-\tilde{\mathsf{B}}$ are positive, $0<\nu_1<\dots<\nu_m$, and all eigenvalues ς_i of $\mathsf{F}-\hat{\mathsf{B}}$ are negative, $\varsigma_1<\dots<\varsigma_m<0$. Denote by $\tilde{e}_{\nu_j}$ and $\hat{e}_{\varsigma_j}$ W-normalized eigenvectors of the matrices $\mathsf{F}-\tilde{\mathsf{B}}$ and $\mathsf{F}-\hat{\mathsf{B}}$ associated with the eigenvalues ν_i and ς_i, respectively. Both collections of eigenvectors $(\tilde{e}_{\nu_1},\dots,\tilde{e}_{\nu_m})$ and $(\hat{e}_{\varsigma_1},\dots,\hat{e}_{\varsigma_m})$ are orthonormal bases in $\mathbb{R}^m$ with respect to the W-weighted inner product $(\cdot,\cdot)_{\mathsf{W}}$. We partition the (x,y)-plane into a disk of radius r_0 with center at zero and its exterior. As the values of the stream function ψ inside and outside the disk, we take the solutions $\tilde{\psi}$ and $\hat{\psi}$ defined using the first radial harmonics for the Helmholtz and modified Helmholtz equations

that are bounded on the corresponding domains, respectively,

$$\tilde{\psi} = \sin\theta\,\big(\tilde{\alpha}_j J_1(\sqrt{\nu_j}r)\tilde{e}_{\nu_j} - r\tilde{u}_{\mathrm{bg}}\big) \quad \text{with} \quad \tilde{u}_{\mathrm{bg}} := \beta(\mathsf{F}-\tilde{\mathsf{B}})^{-1}\bar{1}, \quad r \leqslant r_0,$$
$$\hat{\psi} = \sin\theta\,\big(\hat{\alpha}_j K_1(\sqrt{-\varsigma_j}r)\hat{e}_{\varsigma_j} - r\hat{u}_{\mathrm{bg}}\big) \quad \text{with} \quad \hat{u}_{\mathrm{bg}} := \beta(\mathsf{F}-\hat{\mathsf{B}})^{-1}\bar{1}, \quad r > r_0.$$

Here J_1 and K_1 denote the first-order Bessel functions of the first and the second kinds, respectively. Note that the Bessel function J_1 is slowly decaying as $r \to \infty$ and the Bessel function K_1 has a singularity at $r = 0$. But the partition of the domain has accounted for these problems. Moreover, $\tilde{\psi}$ and $\hat{\psi}$ are solutions of two different Helmholtz equations with different eigenvalues.

The parameters $\tilde{\mathsf{B}}$, $\hat{\mathsf{B}}$, r_0, $\tilde{\alpha}_j$ and $\hat{\alpha}_j$ are determined by enforcing the matching conditions $\tilde{\psi} = \hat{\psi} = 0$ and $\tilde{\psi}_r = \hat{\psi}_r$ at the interface $r = r_0$, which leads to a system for these parameters,

$$\begin{aligned}
&\tilde{\alpha}_j J_1(\sqrt{\nu_j}r_0)\tilde{e}_{\nu_j} = r_0\tilde{u}_{\mathrm{bg}}, \quad \hat{\alpha}_j K_1(\sqrt{-\varsigma_j}r_0)\hat{e}_{\varsigma_j} = r_0\hat{u}_{\mathrm{bg}}.\\
&\tilde{\alpha}_j\sqrt{\nu_j}J_1'(\sqrt{\nu_j}r_0)\tilde{e}_{\nu_j} - \hat{\alpha}_j\sqrt{-\varsigma_j}K_1'(\sqrt{-\varsigma_j}r_0)\hat{e}_{\varsigma_j} = \tilde{u}_{\mathrm{bg}} - \hat{u}_{\mathrm{bg}}.
\end{aligned} \tag{4.38}$$

We need the condition of vanishing both the functions $\tilde{\psi}$ and $\hat{\psi}$ on the entire circle $r = r_0$ to construct a solution of the system (4.1) at least in the weak sense since

$$\begin{aligned}
\tilde{\psi}_{rr} &= -r^{-1}\tilde{\psi}_r - r^{-2}\tilde{\psi}_{\theta\theta} - (\mathsf{F}-\tilde{\mathsf{B}})\tilde{\psi} - \beta r\sin\theta\,\bar{1},\\
\hat{\psi}_{rr} &= -r^{-1}\hat{\psi}_r - r^{-2}\hat{\psi}_{\theta\theta} - (\mathsf{F}-\hat{\mathsf{B}})\hat{\psi} - \beta r\sin\theta\,\bar{1},
\end{aligned}$$

$\hat{\mathsf{B}} \neq \tilde{\mathsf{B}}$, and thus otherwise $\tilde{\psi}_{rr} \neq \hat{\psi}_{rr}$ on the entire circle $r = r_0$ except the two points with $\theta \in \{0, \pi\}$, which leads to the discontinuity of ψ_{rr} and the vorticity q therein. In general, the third derivative ψ_{rrr} then has a finite discontinuity on the circle $r = r_0$, which fits well in the weak setting, cf. [81]. Therefore, the obtained modons are solutions of the system (4.1) in the weak sense. The system (4.38) implies that

$$\begin{aligned}
\tilde{\alpha}_j &= \frac{r_0\beta}{J_1(\sqrt{\nu_j}r_0)}\big((\mathsf{F}-\tilde{\mathsf{B}})^{-1}\bar{1}, \tilde{e}_{\nu_j}\big)_{\mathsf{W}} = \frac{r_0\beta}{J_1(\sqrt{\nu_j}r_0)}\big(\bar{1}, (\mathsf{F}-\tilde{\mathsf{B}})^{-1}\tilde{e}_{\nu_j}\big)_{\mathsf{W}}\\
&= \frac{r_0\beta(\bar{1}, \tilde{e}_{\nu_j})_{\mathsf{W}}}{\nu_j J_1(\sqrt{\nu_j}r_0)},\\
\hat{\alpha}_j &= \frac{r_0\beta}{K_1(\sqrt{-\varsigma_j}r_0)}\big((\mathsf{F}-\hat{\mathsf{B}})^{-1}\bar{1}, \hat{e}_{\varsigma_j}\big)_{\mathsf{W}} = \frac{r_0\beta}{K_1(\sqrt{-\varsigma_j}r_0)}\big(\bar{1}, (\mathsf{F}-\hat{\mathsf{B}})^{-1}\hat{e}_{\varsigma_j}\big)_{\mathsf{W}}
\end{aligned}$$

$$= \frac{r_0\beta(\bar{1},\hat{e}_{\varsigma_j})_{\mathsf{W}}}{\varsigma_j K_1(\sqrt{-\varsigma_j}r_0)},$$

$$\frac{r_0\beta}{\sqrt{\nu_j}}\frac{J_1'(\sqrt{\nu_j}r_0)}{J_1(\sqrt{\nu_j}r_0)}(\bar{1},\tilde{e}_{\nu_j})_{\mathsf{W}}\tilde{e}_{\nu_j} + \frac{r_0\beta}{\sqrt{-\varsigma_j}}\frac{K_1'(\sqrt{-\varsigma_j}r_0)}{K_1(\sqrt{-\varsigma_j}r_0)}(\bar{1},\hat{e}_{\varsigma_j})_{\mathsf{W}}\hat{e}_{\varsigma_j} = \tilde{u}_{\mathrm{bg}} - \hat{u}_{\mathrm{bg}}. \quad (4.39)$$

The equality (4.39) can be interpreted as a system of m equations with respect to the parameters $\tilde{b}_1, \dots, \tilde{b}_m, \hat{b}_1, \dots, \hat{b}_m$ and r_0. Thus, in general, $m+1$ of these parameters can be arbitrarily chosen. For practical computations, one should consider the equality (4.39) jointly with the determining equations $(\mathsf{F}-\tilde{\mathsf{B}})\tilde{e}_{\nu_j} = \nu_j\tilde{e}_{\nu_j}$, $(\mathsf{F}-\hat{\mathsf{B}})\hat{e}_{\varsigma_j} = \varsigma_j\hat{e}_{\varsigma_j}$ for the corresponding eigenvalues and eigenvectors, which are then also considered as additional parameters to be chosen or found. The solutions constructed in the above way are extensions, to the case of an arbitrary number of layers, of the Larichev–Reznik modons [81] for the single-layer model ($m = 1$), see also [34, Section 4.1] and [39,64] as well as for a precursory construction in [141]. Despite various differences and possible involvement of both baroclinic and barotropic modes, the shape of these solutions has common features. The structure is vertically locked and contains matched pairs of counter-rotating vortices (dipoles). It may exhibit baroclinic shear, modeling coherent structures like "mushroom" vortices often observed in satellite imagery.

In some special cases, the computation of particular solutions of the system (4.39) can be simplified. If $\hat{\mathsf{B}} = \tilde{\mathsf{B}} + \varrho\mathsf{E}$, then $\varsigma_i = \nu_i - \varrho$ and $\hat{e}_{\varsigma_i} = \tilde{e}_{\nu_i}$, i.e., the operators $\mathsf{F}-\tilde{\mathsf{B}}$ and $\mathsf{F}-\hat{\mathsf{B}}$ share the eigenbasis,

$$\tilde{\alpha}_j = \frac{r_0\beta(\bar{1},\tilde{e}_{\nu_j})_{\mathsf{W}}}{\nu_j J_1(\sqrt{\nu_j}r_0)}, \quad \hat{\alpha}_j = \frac{-r_0\beta(\bar{1},\tilde{e}_{\nu_j})_{\mathsf{W}}}{(\varrho-\nu_j)K_1(\sqrt{\varrho-\nu_j}r_0)},$$

and the condition (4.39) reduces to the single equation

$$\frac{r_0}{\sqrt{\nu}}\frac{J_0(\sqrt{\nu}r_0)}{J_1(\sqrt{\nu}r_0)} - \frac{r_0}{\sqrt{\varrho-\nu}}\frac{K_0(\sqrt{\varrho-\nu}r_0)}{K_1(\sqrt{\varrho-\nu}r_0)} = \frac{2}{\nu} + \frac{2}{\varrho-\nu},$$

and $\nu_1, \dots, \nu_m$ are m distinct positive roots of this equation for chosen r_0 and ϱ. Given these roots, we find the matrix $\tilde{\mathsf{B}}$ from the equality of polynomials

$$\det\Bigl(\mathsf{F}-\tilde{\mathsf{B}}-\nu\mathsf{E}\Bigr) = \prod_{i=1}^{m}(\nu-\nu_i),$$

which gives a system of m equations for the m diagonal entries of $\tilde{\mathsf{B}}$. Then we find

the eigenvectors $\tilde{e}_{\nu_1}, \dots, \tilde{e}_{\nu_m}$ of the matrix $\mathsf{F}-\tilde{\mathsf{B}}$. In Figure 4.7, we plot these specific modons for the numerical data from the Section 4.2.5 and the following values of parameters:

(a) chosen: $r_0 = 170$ km, $\varrho = 4.5\cdot 10^{-9}$,

found: $\mathsf{B} \approx \operatorname{diag}(-1.71, -4.26, -2.23)\cdot 10^{-9}$, $\nu = (1.74, 3.59, 9.4)\cdot 10^{-10}$,

$\tilde{e}_{\nu_1} \approx (-0.98, 0.12, -0.016)^{\mathsf{T}}$, $\tilde{e}_{\nu_2} \approx -(0.18, 0.64, 0.062)^{\mathsf{T}}$,

$\tilde{e}_{\nu_3} \approx (-0.05, -0.069, 0.54)^{\mathsf{T}}$,

$\tilde{\alpha} \approx (0.94, 1.87, 1.2)\cdot 10^{4}$, $\hat{\alpha} \approx (4.39, 0.104, -1.8)\cdot 10^{7}$;

(b) chosen: $r_0 = 150$ km, $\varrho = 8\cdot 10^{-9}$,

found: $\mathsf{B} \approx \operatorname{diag}(-3.34, -1.82, -5.25)\cdot 10^{-9}$, $\nu = (1.01, 2.7, 5.09)\cdot 10^{-9}$,

$\tilde{e}_{\nu_1} \approx (-0.34, 0.61, -0.025)^{\mathsf{T}}$, $\tilde{e}_{\nu_2} \approx (-0.94, -0.22, 0.016)^{\mathsf{T}}$,

$\tilde{e}_{\nu_3} \approx (0.011, 0.035, 0.55)^{\mathsf{T}}$,

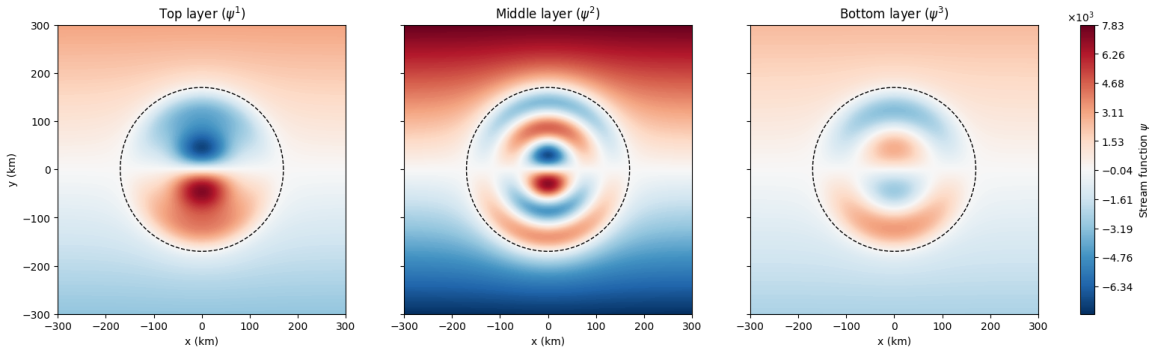
$\tilde{\alpha} \approx (-8.12, -6.22, -7.36)\cdot 10^{3}$, $\hat{\alpha} \approx (-26.46, 8.97, -1.12)\cdot 10^{7}$.

The simplest dipolar vortex solution of the system (4.1) is the *Larichev–Reznik barotropic modon*, which most directly extends the Larichev–Reznik modon [81] for the single-layer model to the case of arbitrary number of layers. We choose $\tilde{\mathsf{B}} = -\tilde{\varrho}\mathsf{E}$ and $\hat{\mathsf{B}} = \hat{\varrho}\mathsf{E}$ with $\tilde{\varrho} > \rho(\mathsf{F}) = |\lambda_1|$ and $\hat{\varrho} > 0$, where $\rho(\mathsf{F})$ denotes the spectral radius of F. Then $\nu_i = \lambda_i + \tilde{\varrho} > 0$, $\varsigma_i = \lambda_i - \hat{\varrho} < 0$ and $\tilde{e}_{\nu_i} = \hat{e}_{\varsigma_i} = e_i/\|e_i\|_{\mathsf{W}}$; in particular, $\nu_m = \tilde{\varrho}$, $\varsigma_m = -\hat{\varrho}$ and $\tilde{e}_{\nu_m} = \hat{e}_{\varsigma_m} = \bar{1}/\|\bar{1}\|_{\mathsf{W}}$. Hence $\tilde{u}_{\mathrm{bg}} = \beta\tilde{\varrho}^{-1}\bar{1}$, $\hat{u}_{\mathrm{bg}} := -\beta\hat{\varrho}^{-1}\bar{1}$, $\tilde{\alpha}_i = \hat{\alpha}_i = 0$, $i = 1, \dots, m-1$,

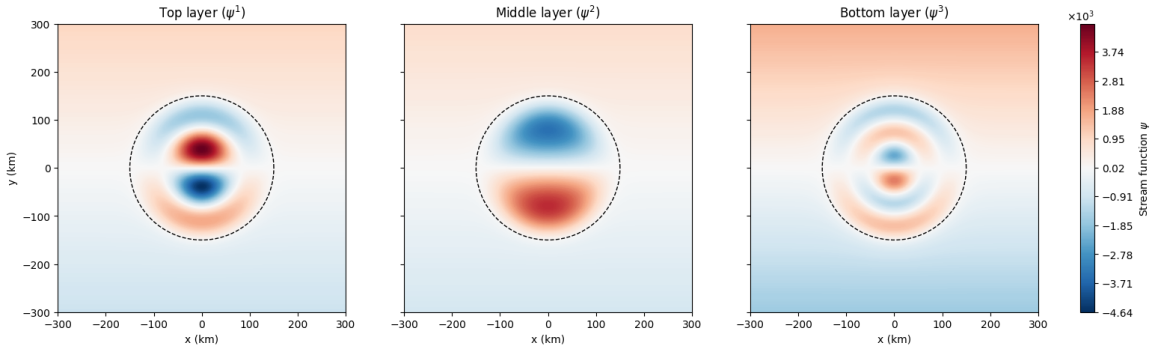
$$\begin{gathered}\tilde{\alpha}_m = \frac{r_0\beta\|\bar{1}\|_{\mathsf{W}}}{\tilde{\varrho}J_1(\sqrt{\tilde{\varrho}}r_0)}, \quad \hat{\alpha}_m = -\frac{r_0\beta\|\bar{1}\|_{\mathsf{W}}}{\hat{\varrho}K_1(\sqrt{\hat{\varrho}}r_0)},\\ \frac{r_0}{\sqrt{\tilde{\varrho}}}\frac{J_0(\sqrt{\tilde{\varrho}}r_0)}{J_1(\sqrt{\tilde{\varrho}}r_0)} - \frac{r_0}{\sqrt{\hat{\varrho}}}\frac{K_0(\sqrt{\hat{\varrho}}r_0)}{K_1(\sqrt{\hat{\varrho}}r_0)} = \frac{2}{\tilde{\varrho}} + \frac{2}{\hat{\varrho}}.\end{gathered} \tag{4.40}$$

We consider the last equality as an equation with respect to one of the parameters r_0, $\tilde{\varrho}$ and $\hat{\varrho}$ when the other two parameters are chosen. The corresponding Larichev–Reznik barotropic modon takes the form

$$\psi = \begin{cases} \dfrac{\beta}{\tilde{\varrho}}\left(r_0\dfrac{J_1(\sqrt{\tilde{\varrho}}r)}{J_1(\sqrt{\tilde{\varrho}}r_0)}\sin\theta - y\right)\bar{1}, & r \leqslant r_0,\\ -\dfrac{\beta}{\hat{\varrho}}\left(r_0\dfrac{K_1(\sqrt{\hat{\varrho}}r)}{K_1(\sqrt{\hat{\varrho}}r_0)}\sin\theta - y\right)\bar{1}, & r > r_0 \end{cases} \qquad \text{with (4.40).} \tag{4.41}$$



(a)



(b)

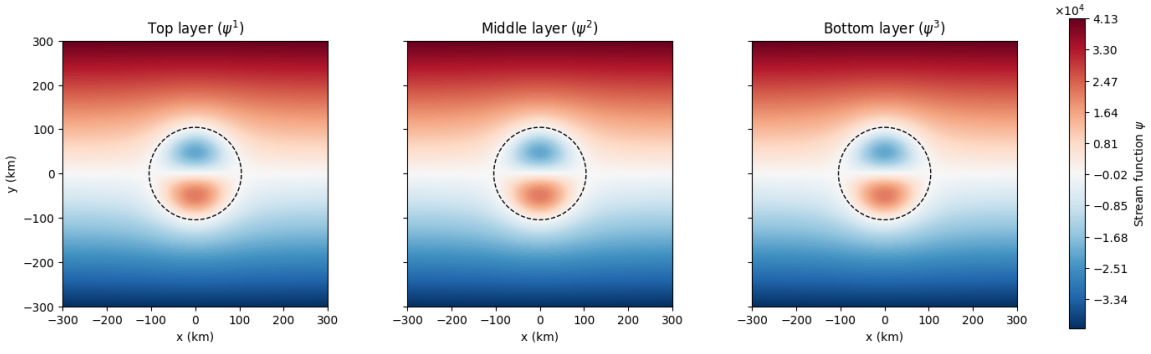
Figure 4.7: Larichev–Reznik baroclinic modon (4.41) in the case of the same eigenbasis in the inner and outer domains for two values of the parameter tuples (r_0, ϱ).

The structure is vertically locked and purely barotropic. On Figure 4.8, we plot the solution (4.41) for the numerical data from the Section 4.2.5 and the following values of parameters:

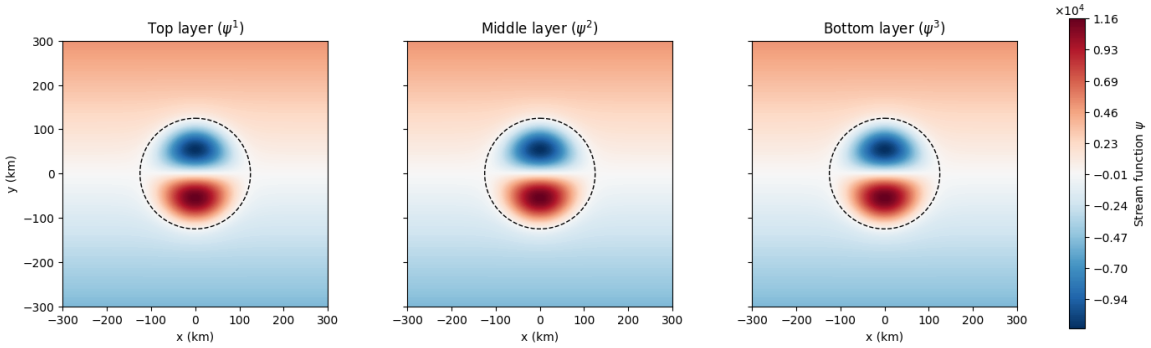
(a) chosen: $\tilde{\varrho} = 1.5 \cdot 10^{-9}$, $\hat{\varrho} = 3.0 \cdot 10^{-10}$,
found: $r_0 \approx 104.43$ km, $\tilde{\alpha}_3 \approx -3.47 \cdot 10^4$, $\hat{\alpha}_3 \approx -7.97 \cdot 10^5$;

(b) chosen: $\tilde{\varrho} = 1.3 \cdot 10^{-9}$, $r_0 = 125$ km,
found: $\hat{\varrho} \approx 2.39 \cdot 10^{-9}$, $\tilde{\alpha}_3 \approx -1.71 \cdot 10^4$, $\hat{\alpha}_3 \approx -1.81 \cdot 10^6$.

Remark 4.27. The above construction of modons is closer to the pioneering paper [81] on the single-layer model rather than to their recent studies, e.g., in [34]. In particular, in the quasi-geostrophic model considered in [34], it is allowed that the parameter β may essentially vary from layer to layer, which increases the possible freedom in the course of merging solutions in the inner and outer domains. Moreover,



(a)



(b)

Figure 4.8: Larichev–Reznik barotropic modon (4.41) for two values of the parameter tuples $(\tilde{\varrho}, \hat{\varrho}, r_0)$.

as far as we understand, the continuity of the stream function is chosen in [34] as the only matching condition at the interface. Then the stream function is in general nonzero for almost all points of the interface, which implies a finite discontinuity of the potential vorticity at these points. Therefore, the modons presented in [34] are not weak solutions of the original multi-layer quasi-geostrophic model [34, Eq. (2.1)–(2.4)] and, if the derivative ψ_r is not continuous, even of the reduced problem [34, Eq. (2.6)].

Remark 4.28. In view of the enforced matching conditions $\tilde{\psi} = \hat{\psi} = 0$ and $\tilde{\psi}_r = \hat{\psi}_r$ at the interface $r = r_0$, the common values of $\tilde{\psi}_r$ and $\hat{\psi}_r$ at the interface $r = r_0$ for any nonzero merged solution are necessarily nonzero, except those at the points $(\pm r_0, 0)$. This explains the opposite signs of the stream function in the inner and outer parts of neighborhoods of most of the interface points, cf. Figures 4.7 and 4.8.

4.8.2 Subalgebra family 1.2

Reduced system

For the one-dimensional subalgebra $\mathfrak{s}_{1.2}^{\chi c} := \langle \mathcal{P}^y + \mathcal{P}^x(\chi) + c_k\mathcal{J}^k\rangle$ of $\mathfrak{g}$, where $c := (c_1,\dots,c_m)^{\mathsf{T}}$ is an arbitrary element of $\operatorname{im}\mathsf{F}$ and χ is an arbitrary smooth function of t, an associated Lie ansatz is $\psi^i = v^i - \frac{1}{2}\chi_t(t)y^2 + c_i y$ or, in the vector notation,

$$\psi = v - \frac{1}{2}\chi_t(t)y^2\bar{1} + yc \quad \text{with} \quad z_1 = t, \quad z_2 = x - \chi(t)y.$$

It reduces the system (4.1) to the system

$$\begin{aligned}\big((1+\chi^2)v^i_{22}\big)_1 - c_i(1+\chi^2)v^i_{222} - f_{i,i-1}(c_iv_2^{i-1} - c_{i-1}v_2^i - v_1^{i-1} + v_1^i)\\ + f_{i,i+1}(c_{i+1}v_2^i - c_iv_2^{i+1} - v_1^i + v_1^{i+1}) + \beta v_2^i - \chi_{11} = 0.\end{aligned} \tag{4.42}$$

The condition $c \in \operatorname{im}\mathsf{F}$ modulo the G-equivalence is, in fact, inessential in the course of the above Lie reduction and for its further study and is neglected. Recall that $\bar{1}$ is the all-ones m-column and $\mathsf{F} = (f_{ij})$ is the tridiagonal matrix that are defined in Section 4.2.2, and E is the $m \times m$ identity matrix. Denote $\mathsf{C} := \operatorname{diag} c$ and $\mathsf{B} := \operatorname{diag}\mathsf{F}c - \mathsf{CF} + \beta\mathsf{E}$, which is a tridiagonal matrix as well. Using this notation, the reduced system (4.42) can be represented in a form that is more convenient for analysis,

$$((1+\chi^2)v_{22})_1 - (1+\chi^2)\mathsf{C}v_{222} + \mathsf{B}v_2 + \mathsf{F}v_1 = \chi_{11}\bar{1}. \tag{4.43}$$

Depending on whether $\chi_t = 0$ or $\chi_t \neq 0$, the normalizer of the subalgebra $\mathfrak{s}_{1.2}^{\chi c}$ in the algebra $\mathfrak{g}$ is

$$\langle \mathcal{P}^t, \mathcal{P}^y, \mathcal{P}^x(1), \mathcal{J}^1, \dots, \mathcal{J}^m, \mathcal{Z}(\kappa)\rangle \quad \text{or} \quad \langle \mathcal{P}^y + \mathcal{P}^x(\chi), \mathcal{P}^x(1), \mathcal{J}^1, \dots, \mathcal{J}^m, \mathcal{Z}(\kappa)\rangle,$$

respectively, where the parameter function κ runs through the set of smooth functions of t. Therefore, the algebra of induced symmetries of the system (4.44) coincides with

$$\begin{aligned}&\big\langle \partial_{z_2},\, \partial_{v^1},\, \dots,\, \partial_{v^m},\, \kappa(z_1)(\partial_{v_1} + \dots + \partial_{v_m})\big\rangle \quad \text{if} \quad \chi_t \neq 0,\\ &\big\langle \partial_{z_1},\, \partial_{z_2},\, \partial_{v^1},\, \dots,\, \partial_{v^m},\, \kappa(z_1)(\partial_{v_1} + \dots + \partial_{v_m})\big\rangle \quad \text{if} \quad \chi_t = 0,\end{aligned}$$

where the parameter function κ runs through the set of smooth functions of z_1.

The reduced system (4.43) is an inhomogeneous linear system of partial differential equations. Therefore, it admits Lie symmetries arising from the linear superposition of solutions of its homogeneous counterpart

$$((1+\chi^2)v_{22})_1-(1+\chi^2)\mathsf{C}v_{222}+\mathsf{B}v_2+\mathsf{F}v_1=0. \tag{4.44}$$

Most of these symmetries are genuine hidden symmetries of the original nonlinear system (4.1), which in particular shows that genuine hidden symmetries associated with the Lie reduction with respect to any subalgebra from the family $\{\mathfrak{s}_{1.2}^{\chi c}\}$ definitely exist. To exhaustively study hidden Lie symmetries of (4.1) associated with this reduction, it is necessary to compute the maximal Lie invariance algebra of (4.44). At the same time, the system (4.44) is parameterized by an arbitrary function $\chi=\chi(z_1)$ and the components of $\theta=(\mathsf{F},\beta)$ and c, so it is actually a class of systems of differential equations. To properly describe Lie symmetries of systems from the class (4.44), it is in fact necessary to solve the group classification problem for this class.

Since $\ker\mathsf{F}=\langle\bar{1}\rangle$ and $(\operatorname{diag}\mathsf{F}c)\bar{1}=\mathsf{F}c$, we can readily verify that

$$v=\frac{\chi_{11}}{\beta}z_2\bar{1}-\frac{\chi_1}{\beta}c$$

is a particular solution of (4.43). This is why the system (4.43) is reduced by the transformation

$$v-\beta^{-1}\chi_{11}z_2\bar{1}+\beta^{-1}\chi_1c\mapsto v$$

to the homogeneous system (4.44), and the symmetry analysis of the former thus reduces to the symmetry analysis of the latter. Another interpretation for this is to consider the modified ansatz

$$\psi=v(z_1,z_2)-\frac{\chi_t}{2}y^2\bar{1}+yc+\frac{\chi_{tt}}{\beta}(x-\chi y)\bar{1}-\frac{\chi_t}{\beta}c,\quad z_1=t,\quad z_2=x-\chi y,$$

which straightforwardly reduces the system (4.1) to the system (4.44). For the purpose of symmetry classification, we denote the class of systems of the form (4.44) by $\mathcal{R}$, its arbitrary-element tuple by $\varrho:=(\mathsf{F},\mathsf{C},\chi,\beta)$, and the system from the class $\mathcal{R}$ with a fixed value of the arbitrary-element tuple ϱ by $\mathcal{R}_\varrho$.

Group classification

We begin with the most general form of Lie-symmetry vector fields of the system $\mathcal{R}_\varrho$, which constitute the maximal Lie invariance algebra $\mathfrak{g}^{\max}_\varrho$ of this system,

$$\tau\partial_{z_1}+\xi\partial_{z_2}+\eta^j\partial_{v^j},$$

where the components τ, ξ and η^j are smooth functions depending on (z_1,z_2,v). Similarly to the proof of Lemma 4.6, we apply the Lie invariance criterion, see, e.g., [103, Theorem 2.31], to this vector field and the system $\mathcal{R}_\varrho$. In the obtained equation that associated with the ith equation of the system $\mathcal{R}_\varrho$, we separately collect coefficients of the jet monomial v^k_{112}, $v^j_2v^k_{112}$, $v^i_{12}v^j_{22}$, $v^j_{12}v^k_2$, v^j_{12}, which results in the following determining equations:

$$\tau_2=\tau_{v^j}=0,\quad \xi_{v^j}=0,\quad \eta^i_{v^jv^k}=0,\quad \eta^i_{z_2v^i}=\xi_{22},\quad \eta^i_{z_2v^j}=0,\ j\neq i.$$

These equations do not involve parameters χ, c and θ. Integrating them, we find that

$$\tau=\tau(z_1),\quad \xi=\xi(z_1,z_2),\quad \eta^i=\eta^{ij}(z_1,z_2)v^j+\eta^{i0}(z_1,z_2) \tag{4.45}$$

with $\eta^{ii}_2=\frac12\xi_{22}$ and $\eta^{ij}_2=0$, $j\neq i$. Collecting summands without the unknown functions v^j and their derivatives results in the system $\mathcal{R}_\varrho$ on the tuple $(\eta^{10},\dots,\eta^{m0})^{\mathsf T}$. Moreover, the components of this tuple do not appear in the other determining equations.

It is convenient to carry out the further computations in matrix form, using the following representation for Lie symmetry vector fields of the system $\mathcal{R}_\varrho$ in view of (4.45):

$$\tau\partial_{z_1}+\xi\partial_{z_2}+(\mathsf{H}v)^j\partial_{v^j}$$

with the matrix $\mathsf{H}=(\eta^{jk}(z_1,z_2))_{j,k=1}^m$, where $\tau=\tau(z_1)$, $\xi=\xi(z_1,z_2)$. Successively collecting the coefficients of v_{12} (we repeat this part of collecting for convenience), v_{222}, v_{22}, v_2, v_1 and v in the matrix equation obtained from the Lie invariance criterion for such a vector field and the system $\mathcal{R}_\varrho$, we derive the system

$$2\mathsf{H}_2=\xi_{22}\mathsf{E}, \tag{4.46a}$$

$$\xi_1=0,\quad [\mathsf{H},\mathsf{C}]=0,\quad (\tau_1-\xi_2)\mathsf{C}=\mathsf{0}, \tag{4.46b}$$

$$\mathsf{H}_1 + \big(\tau(\chi^2)_1(1+\chi^2)^{-1}\big)_1\mathsf{E} = \mathsf{0}, \tag{4.46c}$$

$$[\mathsf{H},\mathsf{B}] = \big(\tau_1 + \xi_2 - \tau(\chi^2)_1(1+\chi^2)^{-1}\big)\mathsf{B}, \tag{4.46d}$$

$$(1+\chi^2)\mathsf{H}_{22} - [\mathsf{H},\mathsf{F}] + (2\xi_2 - \tau(\chi^2)_1(1+\chi^2)^{-1})\mathsf{F} = \mathsf{0}, \tag{4.46e}$$

$$(\chi^2)_1\mathsf{H}_{22} + \mathsf{F}\mathsf{H}_1 + \beta\mathsf{H}_2 = \mathsf{0}, \tag{4.46f}$$

We have in addition arranged the system in view of the equation $\xi_{22}\mathsf{C} = \mathsf{0}$, which is a differential consequence of the equation $(\tau_1 - \xi_2)\mathsf{C} = \mathsf{0}$.

Substituting the expressions for H_1 and H_2 arising from (4.46a) and (4.46c) into (4.46f), we straightforwardly derive $\big(\tau(\chi^2)_1(1+\chi^2)^{-1}\big)_1 = 0$ and thus $\mathsf{H}_1 = \mathsf{0}$ since the matrix F has nonzero non-diagonal entries while H_2 and H_{22} are diagonal. Differentiating (4.46e) with respect to z_1 results in the equation $(\chi^2)_1\mathsf{H}_{22} = \mathsf{0}$. In view of this constraint, the equation (4.46f) implies $\mathsf{H}_2 = \mathsf{0}$. Therefore, H is a constant matrix and, in view of (4.46a), $\xi_{22} = 0$. The equation (4.46e) takes the form

$$[\mathsf{H},\mathsf{F}] = \big(2\xi_2 - \tau(\chi^2)_1(1+\chi^2)^{-1}\big)\mathsf{F},$$

where the coefficient $2\xi_2 - \tau(\chi^2)_1(1+\chi^2)^{-1}$ is a real constant. If this constant differs from zero, the equation is compatible if and only if F is nilpotent, see [57, Lemma 4, p. 44] or [132, Theorem II]. The discussion in Section 4.2 implies that the matrix F is not nilpotent under the assumed physical constraint of positivity of the essential components of F. Hence,

$$2\xi_2 - \tau(\chi^2)_1(1+\chi^2)^{-1} = 0 \quad \text{and} \quad [\mathsf{H},\mathsf{F}] = \mathsf{0}.$$

Combining the former equation with (4.46d) leads to the equation

$$[\mathsf{H},\mathsf{B}] = (\tau_1 - \xi_2)\mathsf{B}. \tag{4.47}$$

If $\mathsf{C} = \mathsf{0}$, then $\mathsf{B} = \beta\mathsf{E}$ by the construction, and thus $[\mathsf{H},\mathsf{B}] = 0$ and $\tau_1 = \xi_2$. Otherwise, the third equation from (4.46b) directly gives $\tau_1 = \xi_2$, and hence again $[\mathsf{H},\mathsf{B}] = 0$.

The above discussion implies that the system (4.46) is equivalent to the system

$$\tau_{11} = \tau_2 = 0, \quad \xi_1 = \xi_{22} = 0, \quad \tau_1 = \xi_2, \quad \tau(\chi^2)_1(1+\chi^2)^{-1} = 2\xi_2,$$
$$\mathsf{H}_1 = \mathsf{H}_2 = 0, \quad [\mathsf{H},\mathsf{F}] = [\mathsf{H},\mathsf{C}] = [\mathsf{H},\mathsf{B}] = 0,$$

and thus $[\mathsf{H}, \operatorname{diag}\mathsf{F}c] = 0$. Analyzing this system along with (4.45), we solve the group classification problem for the class $\mathcal{R}$.

Denote by $\mathfrak{H}$ the space of matrices H that commute with the matrices F, C and $\operatorname{diag}\mathsf{F}c$. It is obvious that $\mathfrak{H} \supseteq \langle \mathsf{E} \rangle$. Since the matrix F has m distinct eigenvalues, the condition $[\mathsf{H}, \mathsf{F}] = 0$ implies that $\mathfrak{H}$ is contained in the space $\mathbb{R}_{m-1}[\mathsf{F}]$ of polynomials of F whose degrees are not greater than $m-1$. Hence $\dim \mathfrak{H} \leqslant \dim \mathbb{R}_{m-1}[\mathsf{F}] = m$.

Lemma 4.29. *$\mathfrak{H} = \mathbb{R}_{m-1}[\mathsf{F}]$ if and only if $c \in \langle \bar{1} \rangle$, and thus $c = 0$ if in addition $c \perp_{\mathsf{W}} \bar{1}$.*

Proof. If $c \in \langle \bar{1} \rangle$, then $\mathsf{C} \in \langle \mathsf{E} \rangle$, $\mathsf{F}c = 0$. Hence the commutation conditions $[\mathsf{H}, \mathsf{C}] = [\mathsf{H}, \mathsf{B}] = 0$ imply no constraints for H. Conversely, if $\mathfrak{H} = \mathbb{R}_{m-1}[\mathsf{F}]$, then in particular $[\mathsf{H}, \mathsf{C}] = 0$ for $\mathsf{H} = \mathsf{F}$, which implies the equations $f_{i,i+1}(c_{i+1} - c_i) = 0$, $i = 1, \dots, m-1$. In view of the inequalities $f_{i,i+1} > 0$, $i = 1, \dots, m-1$, this means that $c_1 = \dots = c_m$, i.e., $c \in \langle \bar{1} \rangle$. □

Lemma 4.30. *$\mathfrak{H} = \langle \mathsf{E} \rangle$ if all c_i are pairwise distinct.*

Proof. The matrix $\tilde{\mathsf{F}} := -\Omega\mathsf{F}\Omega$ with $\Omega := \operatorname{diag}((-1)^k, k = 1, \dots, m)$ is tridiagonal with strictly positive subdiagonal, diagonal and superdiagonal entries. Its kth power $\tilde{\mathsf{F}}^k$ is a $(2k+1)$-diagonal matrix with strictly positive entries on these diagonals. Then, $\mathsf{F}^k = (-1)^k \Omega \tilde{\mathsf{F}}^k \Omega$, i.e., $(\mathsf{F}^k)_{ij} = (-1)^{j-i+k} (\tilde{\mathsf{F}}^k)_{ij}$. Therefore, F^k is a $(2k+1)$-diagonal matrix with nonzero entries on these diagonals and a chessboard sign pattern on them.

Take an arbitrary matrix $H \in \mathfrak{H}$. Suppose that the degree l of H as a polynomial of F is greater than zero and a_l is the leading coefficient of this polynomial. Then the condition $[\mathsf{H}, \mathsf{C}] = 0$ implies that $h_{i,i+l}(c_{i+l} - c_i) = 0$. At the same time, $c_{i+l} \neq c_i$ and $h_{i,i+l} = a_l (\mathsf{F}^l)_{i,i+l} \neq 0$ since $a_l \neq 0$ and $(\mathsf{F}^l)_{i,i+l} \neq 0$, which gives a contradiction. Therefore $l = 0$, i.e., $\mathsf{H} \in \langle \mathsf{E} \rangle$. □

It follows from Lemmas 4.29 and 4.30 that there are two outermost situations, all c_i are the same, where $\mathfrak{H} = \mathbb{R}_1[\mathsf{F}]$, and all c_i are pairwise distinct, where $\mathfrak{H} = \langle \mathsf{E} \rangle$. For $m = 2$, only these situations are possible. If $m > 2$ and c_i are not all the same and are not all pairwise distinct, then $\langle \mathsf{E} \rangle \subseteq \mathfrak{H} \subsetneq \mathbb{R}_{m-1}[\mathsf{F}]$, where in the first inclusion we can have equality, but, according to Lemma 4.29, not in the second one.

We exhaustively describe the space $\mathfrak{H}$ depending on F and C for low values of m,

$\mathfrak{H}$ for $m=2$: $\circ\ \mathfrak{H}=\mathbb{R}_1[\mathsf{F}]$ if $c\in\langle\bar1\rangle$ and $\circ\ \mathfrak{H}=\langle\mathsf{E}\rangle$ otherwise.

$\mathfrak{H}$ for $m=3$:

- $\circ\ \mathbb{R}_2[\mathsf{F}]$ if $c\in\langle\bar1\rangle$,
- $\circ\ \big\langle\mathsf{E},\mathsf{F}^2+(f_{12}+f_{21}+f_{23})\mathsf{F}\big\rangle$ if $f_{12}=f_{32},\quad c_1=c_3\neq c_2$,
- $\circ\ \langle\mathsf{E}\rangle$ otherwise.

$\mathfrak{H}$ for $m=4$:

- $\circ\ \mathbb{R}_2[\mathsf{F}]$ if $c\in\langle\bar1\rangle$,
- $\circ\ \langle\mathsf{E},\mathsf{F}^3+(f_{12}+2f_{21}+f_{23}+f_{32})\mathsf{F}^2+(f_{21}+f_{23}+f_{32})(f_{12}+f_{21})\mathsf{F}\rangle$ if $f_{43}=f_{12},\ f_{34}=f_{21},\quad c_1=c_4\neq c_2=c_3$,
- $\circ\ \big\langle\mathsf{E},\mathsf{F}^2+(f_{12}+f_{21}+f_{23})\mathsf{F}\big\rangle$ if $f_{34}=f_{12}-f_{32},\quad f_{43}=f_{21}-f_{23}$, $c_1=c_3\neq c_2=c_4$,
- $\circ\ \langle\mathsf{E}\rangle$ otherwise.

Note that for $m=2$ and $m=3$, the conditions $[\mathsf{H},\mathsf{F}]=0$ and $[\mathsf{H},\mathsf{C}]=0$ imply $[\mathsf{H},\operatorname{diag}\mathsf{F}c]=0$, but this is not the case for $m=4$.

Remark 4.31. If $c\notin\langle\bar1\rangle$, the space $\mathfrak{H}$ is larger than the minimal case $\mathfrak{H}=\langle\mathsf{E}\rangle$ only under specific additional constraints not only on the subalgebra parameter tuple c, which can be chosen arbitrarily, but also on the essential components of the matrix F, which define the model. Although for $m\in\{3,4\}$ the constraints are linear in (c,F) and decoupled with respect to c and F, after analysing the commutation relations $[\mathsf{H},\mathsf{F}]=[\mathsf{H},\mathsf{C}]=[\mathsf{H},\mathsf{B}]=0$, we conjecture that in higher dimensions m, there exist nonlinear constrains coupling c and F.

Remark 4.32. While we have exhaustively described the space $\mathfrak{H}$ for low values of m, the same problem for an arbitrary value of m remains open. It is not even clear what are justified conjectures about deeper properties of this space. Furthermore, as noted in Remark 4.31, non-maximal extensions of $\mathfrak{H}$ definitely require that the corresponding model parameters satisfy a system of algebraic equations and thus their set is of measure zero in the space of admitted model parameters. This is why for the remainder of Section 4.8, we consider the entire space $\mathfrak{H}=\mathbb{R}_{m-1}[\mathsf{F}]$ if $c\in\langle\bar1\rangle$ and its subspace $\langle\mathsf{E}\rangle$ otherwise, which correspond to the *completely decoupled* and

generically coupled reduced systems (4.44), respectively. This remark is also relevant for the analogous reduced systems (4.65). At the same time, the complete description of the nontrivial extensions of the space $\mathfrak{H}$ may be of interest for finding bifurcation values of the model parameters.

The kernel Lie invariance algebra $\mathfrak{g}^{\cap}_{\mathcal{R}}$ of the class $\mathcal{R}$ is spanned by the vector fields

$$\partial_{z_2}, \quad v^j\partial_{v^j}, \quad \partial_{v^1}, \quad \dots, \quad \partial_{v^m}, \quad \kappa(\partial_{v^1}+\dots+\partial_{v^m}),$$

where κ is an arbitrary smooth function of z_1. Among these vector fields, only $v^j\partial_{v^j}$ is not induced by a Lie-symmetry vector field of the original system $\mathcal{M}_\theta$. Any system $\mathcal{R}_\varrho$ from the class $\mathcal{R}$ is invariant with respect to the algebra $\mathfrak{g}^{\rm gen}_\varrho$ spanned by

$$\partial_{z_2}, \quad (\mathsf{H}v)^j\partial_{v^j}, \quad \zeta^j(p,q)\partial_{v^j},$$

where the tuple $(\zeta^1,\dots,\zeta^m)$ runs through the solution set of the system $\mathcal{R}_\varrho$, and H runs through a basis of the space $\mathfrak{H}$ of (constant) matrices commuting with the matrices F, C and B. The subalgebra of $\mathfrak{g}^{\rm gen}_\varrho$ constituted by induced symmetries is spanned by the vector fields ∂_{z_2} and $\zeta^j(z_1,z_2)\partial_{v^j}$ with $\zeta^j_2=0$ and $\zeta^j_1=\zeta^k_1$. All the other elements of $\mathfrak{g}^{\rm gen}_\varrho$ are hidden symmetries for the original system $\mathcal{M}_\theta$.

The classification of extensions of $\mathfrak{g}^{\rm gen}_\varrho$, i.e., the cases where $\mathfrak{g}^{\rm max}_\varrho\neq\mathfrak{g}^{\rm gen}_\varrho$, is carried out modulo the action of the equivalence transformations in the class $\mathcal{R}$ that are induced by elements of the point symmetry pseudogroup G of the system $\mathcal{M}_\theta$, which is the same for all systems from the class $\mathcal{M}$. The only essential among these equivalence transformations are the translations with respect to z_1. Since $\tau_{11}=\tau_2=0$, $\xi_1=\xi_{22}=0$ and $\tau_1=\xi_2$, we have $\tau=at+b$ and $\xi_2=a$ for some constants a and b. Then the classifying equation $\tau(\chi^2)_1(1+\chi^2)^{-1}=2\xi_2$ takes the form $(at+b)(1+\chi^2)_1=2a(1+\chi^2)$. Up to translations with respect to z_1 and multiplying the classifying equation by a nonzero constant, there are two inequivalent cases of this equation, in addition to the identity corresponding to the absence of extension, with $(a,b)=(1,0)$ and with $(a,b)=(0,1)$. In view of this argument, the inequivalent extensions of the algebra $\mathfrak{g}^{\rm gen}_\varrho$ are exhausted by two cases,

$$\begin{aligned}\chi_1=0\colon&\quad \mathfrak{g}^{\rm max}_\varrho=\mathfrak{g}^{\rm gen}_\varrho+\langle\partial_{z_1}\rangle,\\ \chi=\pm\sqrt{\alpha z_1^2-1}\colon&\quad \mathfrak{g}^{\rm max}_\varrho=\mathfrak{g}^{\rm gen}_\varrho+\langle z_1\partial_{z_1}+z_2\partial_{z_2}\rangle,\end{aligned}$$

where α is a positive constant. Recall that the extension ∂_{z_1} is induced by the Lie-symmetry vector field ∂_t of $\mathcal{M}_\theta$, whereas the extension $z_1\partial_{z_1}+z_2\partial_{z_2}$ is a hidden Lie symmetry of $\mathcal{M}_\theta$.

We split further consideration into cases depending on two criteria, the dimension of the space $\mathfrak{H}$, see Remark 4.32, and the presence of Lie-symmetry vector fields with nonzero z_1-components. According to the first criterion, we have the *decoupled* and the *coupled* cases, where $\dim\mathfrak{H}$ is minimal and maximal, respectively. Each of these cases additionally splits into the *general*, the *scale-invariant* and the *shift-invariant* cases using the second criterion and meaning shifts and scalings with respect to z_1.

Completely decoupled case

In view of Lemma 4.29, the most singular case of the group classification of the class $\mathcal{R}$ of systems (4.44) takes place if and only if $c\in\langle\bar1\rangle$, i.e., $c=0$ modulo the G-equivalence, which corresponds to the maximal possible dimension of the general algebra $\mathfrak{g}^{\rm gen}_\varrho$. Consequently, we have $\mathsf{C}=0$, which implies $\mathsf{B}=\beta\mathsf{E}$. Under these constraints, the system $\mathcal{R}_\varrho$ takes the form

$$((1+\chi^2)v_{22})_1+\beta v_2+\mathsf{F}v_1=0, \tag{4.48}$$

and thus it is decoupled. In this case, the algebra $\mathfrak{g}^{\rm gen}_\varrho$ coincides with the span

$$\langle\partial_{z_2},\ (\mathsf{F}^{i-1}v)^j\partial_{v^j},\ i=1,\dots,m,\ \zeta^j(p,q)\partial_{v^j}\rangle,$$

where the tuple $(\zeta^1,\dots,\zeta^m)$ runs through the solution set of the system $\mathcal{R}_\varrho$.

Using the change of the dependent variables $\tilde v:=\mathsf{P}^{-1}v$, we diagonalize the matrix F to its eigenvalue matrix $\Lambda=\operatorname{diag}(\lambda_1,\dots,\lambda_m)$, see the end of Section 4.2.3 and thus we map the system (4.48) to the decoupled system $((1+\chi^2)\tilde v_{22})_1+\beta\tilde v_2+\Lambda\tilde v_1=0$, or, componentwise,

$$((1+\chi^2)\tilde v^i_{22})_1+\beta\tilde v^i_2+\lambda_i\tilde v^i_1=0,\quad i=1,\dots,m. \tag{4.49}$$

Since $\lambda_m=0$, the mth equation of the system (4.49) takes the form

$$((1+\chi^2)\tilde v^m_{22})_1+\beta\tilde v^m_2=0.$$

This equation obviously integrates to

$$\big((1+\chi^2)\tilde v^m_2\big)_1+\beta\tilde v^m=\varsigma,$$

where $\varsigma=\varsigma(z_1)$ is an arbitrary smooth function. Changing the variables in the last equation according to

$$\hat z_1=\beta\int\frac{\mathrm dz_1}{1+\chi^2},\quad \hat z_2=z_2,\quad \hat v^m=\tilde v^m-\frac{\varsigma}{\beta},\tag{4.50}$$

we derive the famous Klein–Gordon equation in the light-cone variables

$$\hat v^m_{\hat z_1\hat z_2}+\hat v^m=0.\tag{4.51}$$

There are large families of known exact solutions for this equation, including invariant ones, see, [93, Sections 1.1–1.3] for separation of variables based on symmetries for the counterpart of (4.51) in the standard spacetime coordinates or the collection of results for this counterpart in [118, Section 4.1.3]. The generalized symmetries and the local conservation laws of the equation (4.51), which can be interpreted as the corresponding hidden objects of the original system (4.1), were exhaustively described in [110].

As a result, the collection of the $\mathfrak s^{\chi}_{1.2}$-invariant solutions of the original system (4.1) can be represented in terms of the general solutions of the equations (4.49),

- $$\psi=\sum_{i=1}^{m-1}\tilde v^i(z_1,z_2)e_i+\left(\hat v^m(\hat z_1,\hat z_2)-\frac{\chi_t}{2}y^2+\frac{\chi_{tt}}{\beta}z_2+\frac{\varsigma}{\beta}\right)\bar 1,\tag{4.52}$$

where $z_1:=t$, $\hat z_1:=\beta\int(1+\chi^2)^{-1}\mathrm dt$, $\hat z_2=z_2:=x-\chi(t)y$, e_i is an eigenvector of the matrix $\mathsf F$ corresponding to its eigenvalue λ_i and, moreover, $\lambda_m=0$ and $e_m=\bar 1$. The function $\tilde v^i$ is an arbitrary solution of the ith equation in (4.49), $i=1,\dots,m-1$, the function $\hat v^m$ is an arbitrary solution of the Klein–Gordon equation (4.51), $\tilde v^m(z_1,z_2)=\hat v^m(\hat z_1,\hat z_2)$, and χ and ς are arbitrary functions of t. The function ς can be neglected modulo the G-equivalence.

Since each of the equations (4.49) and (4.51) is linear and homogeneous, any linear combination of its solutions is its solution as well. This is why the representation (4.52) gives wide families of exact solutions of the original system (4.1). The remainder of this subsection is devoted to finding closed-form solutions of the equations (4.49) using

its Lie reductions (even with respect to induced symmetries) or the other methods like Shapovalov–Shirokov noncommutative integration [134], especially, its simplest variant, which can be described in terms of complexification.

Consider now the ith equation in (4.49). Using the results of Section 4.8.2, we obtain that any equation of the form (4.49) is invariant with respect to the algebra $\mathfrak{a}_{\chi\lambda_i}$ spanned by the vector fields

$$\partial_{z_2}, \quad \tilde{v}^i\partial_{\tilde{v}^i}, \quad \zeta^i(z_1,z_2)\partial_{\tilde{v}^i},$$

where the function ζ^i runs through the solution set of (4.49) with the fixed λ_i, β and χ. The inequivalent extensions of the algebra $\mathfrak{a}_{\chi\lambda_i}$ are exhausted by $\langle\partial_{z_1}\rangle$ for $\chi_1=0$ and $\langle z_1\partial_{z_1}+z_2\partial_{z_2}\rangle$ for $\chi=\pm\sqrt{\alpha z_1^2-1}$, where α is a positive constant.

General case. For general χ, a complete list of inequivalent one-dimensional subalgebras of $\mathfrak{a}_{\chi\lambda_i}$ that are appropriate for Lie reduction consists of the subalgebras of the form $\langle\partial_{z_2}+b_i\tilde{v}^i\partial_{\tilde{v}^i}\rangle$, where b_i is an arbitrary real constant. An ansatz corresponding to such a subalgebra is $\tilde{v}^i=\mathrm{e}^{b_iz_2}\varphi^i(\omega)$ with $\omega:=z_1$, and it reduces the equation (4.49) with the fixed i to the homogeneous linear first-order ordinary differential equations

$$b_i^2\big((1+\chi^2)\varphi^i\big)_\omega+\lambda_i\varphi^i_\omega+\beta b_i\varphi^i=0,$$

which can be easily integrated. Pulling its general solution back with respect to the above ansatz, we obtain an explicit solution of the equation (4.49),

- $\tilde{v}^i(z_1,z_2)=\dfrac{A_i}{b_i^2(1+\chi^2)+\lambda_i}\exp\left(b_iz_2-\displaystyle\int\frac{\beta b_i\mathrm{d}z_1}{b_i^2(1+\chi^2)+\lambda_i}\right)$ if $b_i\neq0$ or $i\neq m$,
  $\tilde{v}^m(z_1,z_2)=\varphi^m(z_1)\quad$ if $\quad b_m=0,$

where A_i and b_i are arbitrary constants, φ^m is an arbitrary function of z_1, and this function can be neglected modulo the G-equivalence. Applying the complexification trick, we can assume that the constants A_i and b_i are complex. We choose the constant b_i to be imaginary, $b_i=\gamma_i\mathrm{i}$ with $\gamma_i\neq0$, and obtain wave-like solutions of the form

- $\tilde{v}^i(z_1,z_2)=\dfrac{\hat{A}_i}{\gamma_i^2(1+\chi^2)-\lambda_i}\cos\left(\gamma_iz_2+\displaystyle\int\frac{\beta\gamma_i\mathrm{d}z_1}{\gamma_i^2(1+\chi^2)-\lambda_i}+\alpha_i\right),$

where $\hat{A}_i$, α_i and γ_i are arbitrary real constants with $\gamma_i \neq 0$. Since $\lambda_i \leqslant 0$, all such solutions are bounded.

Scale-invariant case. This case take place when $\chi = \pm\sqrt{\alpha z_1^2 - 1}$ with $\alpha > 0$, and thus we have one more family of inequivalent one-dimensional subalgebras of the algebra $\mathfrak{a}_{\chi\lambda_i}$,

$$\langle z_1\partial_{z_1} + z_2\partial_{z_2} + b_i\tilde{v}^i\partial_{\tilde{v}^i}\rangle,$$

again parameterized by an arbitrary real constant b_i. For any subalgebra from this family, an ansatz constructed for $\tilde{v}$ using it can be chosen in the form $\tilde{v}^i = |z_1|^{b_i}\varphi^i(\omega)$ with $\omega := z_2/z_1$, which reduces the equation (4.49) to

$$\alpha\omega\varphi^i_{\omega\omega\omega} - \alpha b_i\varphi^i_{\omega\omega} + (\lambda_i\omega - \beta)\varphi^i_\omega - \lambda_i b_i\varphi^i = 0. \tag{4.53}$$

Since $\lambda_m = 0$, the mth equation of the system (4.53) is

$$\big(\omega\varphi^m_{\omega\omega} - (b_m+1)\varphi^m_\omega - \beta\alpha^{-1}\varphi^m\big)_\omega = 0$$

and can be integrated in terms of the Bessel functions $J_\nu(z)$ and $Y_\nu(z)$ for $\omega < 0$ and in terms of the modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$ for $\omega > 0$,

$$\varphi^m = |\omega|^{\frac{1}{2}b_m+1}\big(A_{1m}J_{b_m+2}(\sqrt{\beta'|\omega|}) + A_{2m}Y_{b_m+2}(\sqrt{\beta'|\omega|})\big) + A_{3m} \quad \text{if} \quad \omega < 0,$$
$$\varphi^m = \omega^{\frac{1}{2}b_m+1}\big(A_{1m}I_{b_m+2}(\sqrt{\beta'\omega}) + A_{2m}K_{b_m+2}(\sqrt{\beta'\omega})\big) + A_{3m} \quad \text{if} \quad \omega > 0,$$

where A_{1m}, A_{2m} and A_{3m} are arbitrary constants, and $\beta' := 4\beta\alpha^{-1}$. When $b_i = 0$, $i = 1,\dots,m-1$, the general solution of the ith equation of the form (4.53) can be written as

$$\varphi^i = A_{1i}\mathrm{Mi}_{\mu_i,\frac{1}{2}}(\nu_i\omega) + A_{2i}\mathrm{Wi}_{\mu_i,\frac{1}{2}}(\nu_i\omega)\mathrm{d}\omega + A_{3i},$$

where $\mu_i := \frac{\beta}{2}(-\alpha\lambda_i)^{-1/2}$ and $\nu_i := 2(-\alpha^{-1}\lambda_i)^{1/2}$, and by $\mathrm{Mi}_{\mu,\kappa}(z)$ and $\mathrm{Wi}_{\mu,\kappa}(z)$ we denote antiderivatives of Whittaker functions,

$$\mathrm{Mi}_{\mu,\kappa}(z) := \int M_{\mu,\kappa}(z)\mathrm{d}z, \quad \mathrm{Wi}_{\mu,\kappa}(z) := \int W_{\mu,\kappa}(z)\mathrm{d}z.$$

Since $\alpha\lambda_i < 0$ for all $i = 1, \dots, m-1$, the constants μ_i and ν_i are real, which implies that the derived solutions are real-valued. Pulling the obtained solution back with respect to the used ansatz, we obtain the following solution of the system (4.49):

- $\chi = \pm\sqrt{\alpha z_1^2 - 1}$:

$$\tilde v^i = A_{1i}\mathrm{Mi}_{\mu_i,\frac12}(\nu_i\omega) + A_{2i}\mathrm{Wi}_{\mu_i,\frac12}(\nu_i\omega), \quad i = 1, \dots, m-1,$$
$$\tilde v^m = |z_1|^{b_m}|\omega|^{\frac12 b_m+1}\big(A_{1m}J_{b_m+2}(\sqrt{\beta'|\omega|}) + A_{2m}Y_{b_m+2}(\sqrt{\beta'|\omega|})\big) + A_{3m}|z_1|^{b_m}, \ \omega < 0,$$
$$\tilde v^m = |z_1|^{b_m}\omega^{\frac12 b_m+1}\big(A_{1m}I_{b_m+2}(\sqrt{\beta'\omega}) + A_{2m}K_{b_m+2}(\sqrt{\beta'\omega})\big) + A_{3m}|z_1|^{b_m}, \ \omega > 0,$$

where $\omega := z_2/z_1$, A_{1i}, A_{2i} and A_{3m} are arbitrary real constants, and $\beta' := 4\beta\alpha^{-1}$. Moreover, up to the G-equivalence, the constant A_{3m} can be gauged to zero.

Shift-invariant case. When $\chi_1 = 0$, each equation (4.49) with $i < m$ is a linearized Benjamin–Bona–Mahony (BBM) equation,

$$\mathcal{B}_i\colon \quad (1+\chi^2)\tilde v^i_{122} + \beta\tilde v^i_2 + \lambda_i\tilde v^i_1 = 0. \tag{4.54}$$

The dispersion relation for this equation is

$$\omega^i(k) = -\frac{\beta k}{(1+\chi^2)k^2 - \lambda_i}.$$

Since $\lambda_i < 0$ if $i < m$, the phase velocity $\omega^i(k)/k$ is bounded on the entire $\mathbb{R}_k$. Therefore, given initial condition $\tilde v^i(0, z_2) := g^i(z_2)$, the general solution of (4.54) can be written using Fourier transform as

- $\chi_t = 0\colon \quad \tilde v^i(z_1, z_2) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty}\hat g^i(k)\mathrm{e}^{\mathrm{i}(kz_2 - \omega^i(k)z_1)}\mathrm{d}k,$

where $\hat g^i(k)$ is the Fourier transform of the initial value $g^i(z_2)$. If the initial value is sufficiently regular, then the corresponding solution is bounded in the entire space $\mathbb{R}^2_{z_1,z_2}$. More specifically, if $g^i(z_2)$ belongs to the Sobolev space $H^s(\mathbb{R})$ for $s > 1/2$, the solution $\tilde v^i(z_1, \cdot)$ remains in that space for all z_1. The (one-dimensional) Sobolev embedding theorem, which states that $H^s(\mathbb{R}) \subset L^\infty(\mathbb{R})$ for $s > 1/2$, implies that the solution $\tilde v$ is bounded such that $|\tilde v(z_1, z_2)| \leqslant C\|\tilde v(z_1, \cdot)\|_{H^s(\mathbb{R})}$ for some constant C and for all z_1.

The maximal Lie invariance algebra $\mathfrak{g}_i$ of the linearized BBM equation $\mathcal{B}_i$ is spanned by the vector fields

$$\partial_{z_1}, \quad \partial_{z_2}, \quad \tilde{v}^i\partial_{\tilde{v}^i}, \quad \zeta^i(z_1,z_2)\partial_{\tilde{v}^i},$$

where the parameter-function ζ runs through the solution set of this equation. The algebra $\mathfrak{g}_i$ splits over the infinite-dimensional ideal $\mathfrak{g}_i^{\rm lin} := \{\zeta^i(z_1,z_2)\partial_{\tilde{v}^i}\}$, which is associated with the linear superposition of solutions of (4.54). More specifically, the algebra $\mathfrak{g}_i$ is the semidirect sum of the ideal $\mathfrak{g}_i^{\rm lin}$ with the complementing subalgebra $\mathfrak{g}_i^{\rm ess}$, which is three-dimensional, abelian and spanned by the vector fields ∂_{z_1}, ∂_{z_2} and $\tilde{v}^i\partial_{\tilde{v}^i}$.

The structure of the algebra $\mathfrak{g}_i^{\rm ess}$ suggests that most Lie and generalized reductions of $\mathcal{B}_i$ result in solutions that have at least polynomial growth as z_1 or z_2 tends to infinity and thus their physical relevance is under question. At the same time, one can easily construct more interesting solutions of $\mathcal{B}_i$. Using a subalgebra of the form $\langle\partial_{z_1}+\alpha_i\partial_{z_2}+\gamma_i\tilde{v}^i\partial_{\tilde{v}^i}\rangle$, where α_i and γ_i are arbitrary real constants with $\alpha_i\geqslant 0$, we construct modulated wave ansatz $\tilde{v}^i=\mathrm{e}^{\gamma_i z_1}\varphi^i(\omega)$, where $\omega=z_2-\alpha_i z_1$, and reduce (4.54) to the constant-coefficient homogeneous linear third-order ordinary differential equation

$$\alpha_i(1+\chi^2)\varphi^i_{\omega\omega\omega}-\gamma_i(1+\chi^2)\varphi^i_{\omega\omega}+(\alpha_i\lambda_i-\beta)\varphi^i_\omega-\gamma_i\lambda_i\varphi^i=0. \tag{4.55}$$

Since the parameter β is positive, the real parts of each root of the characteristic polynomial of the equation (4.55) are nonzero. Hence each solution of (4.55) is unbounded on the entire space $\mathbb{R}_\omega$. Nevertheless, we can apply the complexification trick and assume that γ_i is purely imaginary, that is, $\gamma_i=\mathrm{i}\delta_i$ for some real parameter δ_i. Substituting the ansatz $\varphi^i=\exp(\mathrm{i}r\omega)$ into the equation (4.55) gives that r is a root of the cubic polynomial $a_3r^3+a_2r^2+a_1r+a_0$ with $a_0:=\delta_i\lambda_i$, $a_1:=\beta-\alpha_i\lambda_i$, $a_2:=-\delta_i(1+\chi^2)$ and $a_3:=\alpha_i(1+\chi^2)$. Real roots of this polynomial definitely exist and each of them leads to a solution family of $\mathcal{B}_i$. In total, we have the following collection of simple-wave periodic solutions of the equation $\mathcal{B}_i$, which can be, of course, linearly superposed:

$$\circ\quad \tilde{v}^i(z_1,z_2)=C\cos\big(\delta_i z_1+r_i(z_1-\alpha_i z_2)+\theta\big), \tag{4.56}$$

where δ_i, α_i, C and θ are arbitrary real constants, and r_i is a real root of the

above polynomial, whose coefficients are parameterized by δ_i and α_i. In particular, this polynomial has three distinct real roots if (and only if) its discriminant $\Delta = 18a_0a_1a_2a_3 - 4a_0a_2^3 + a_1^2a_2^2 - 4a_1^3a_3 - 27a_0^2a_3^2$ is positive.

We illustrate the solution (4.56) using the numerical data from Section 4.2.5. Assuming that χ is constant, substituting the solution (4.56) into (4.52) leads to

$$\psi = \sum_{i=1}^{3} \psi_{i0} \cos\big(\delta_i z_1 + r_i(z_1 - \alpha_i z_2) + \theta\big) e_i, \tag{4.57}$$

where ψ_{i0} are scaling constants. The parameters for this simulation are defined as follows:

$$\begin{aligned}
&\chi = 0.7,\\
&\alpha_1 = 1.2 \cdot 10^{-4}, \quad \delta_1 = 4 \cdot 10^{-6}, \quad r_1 = 3.34 \cdot 10^{-2}, \quad \theta_1 = \pi/3, \quad \psi_{10} = 15000,\\
&\alpha_2 = 1.5 \cdot 10^{-4}, \quad \delta_2 = 3 \cdot 10^{-6}, \quad r_2 = 2.00 \cdot 10^{-2}, \quad \theta_2 = \pi/4, \quad \psi_{20} = 12000,\\
&\alpha_3 = 1.4 \cdot 10^{-4}, \quad \delta_3 = 3.5 \cdot 10^{-6}, \quad r_1 = 2.50 \cdot 10^{-2}, \quad \theta_3 = \pi/6, \quad \psi_{30} = 10000.
\end{aligned}$$

The resulting solution is plotted on Figure 4.9 at time snapshots $t = 0$ s, $t = 3600$ s, and $t = 7200$ s. Physically, these profiles represent the evolution of traveling baroclinic Rossby waves.

Generically coupled case

Recall that the completely decoupled case, which corresponds to the zero parameter tuple c, represents the most singular case in the group classification of the class $\mathcal{R}$, see Lemma 4.29. The opposite, generic case, when the components of c are pairwise distinct, is described in Lemma 4.30, which states that then $\mathfrak{H} = \langle \mathsf{E} \rangle$ and thus the algebra $\mathfrak{g}_\varrho^{\rm gen}$ coincides with the span

$$\langle \partial_{z_2},\ v^j\partial_{v^j},\ \zeta^j(p,q)\partial_{v^j} \rangle,$$

where the tuple $(\zeta^1, \dots, \zeta^m)$ runs through the solution set of the corresponding system $\mathcal{R}_\varrho$ of the form (4.44). According to Remark 4.32, for further consideration with an arbitrary $c \notin \langle \bar{1} \rangle$, we use, instead of the entire spaces $\mathfrak{H}$ depending on (c, F), the common subspace $\langle \mathsf{E} \rangle$ of all such spaces.

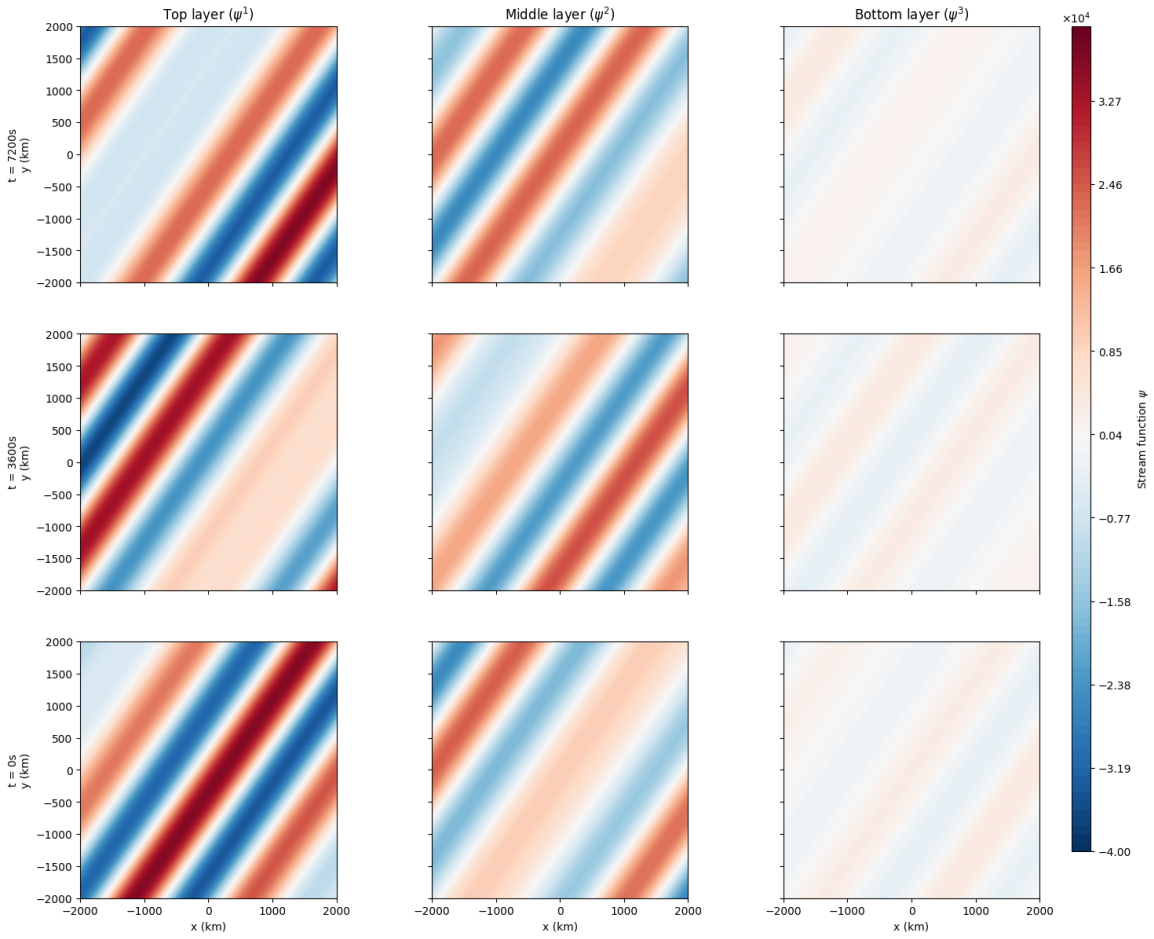


Figure 4.9: Traveling Rossby wave (4.57) in three layer model at the time snapshots $t = 0$ s, $t = 3600$ s and $t = 7200$ s from the last row to the first row.

General case. A complete list of inequivalent one-dimensional subalgebras of $\mathfrak{g}^{\max}_{\varrho}$ that are appropriate for Lie reduction of the system (4.44) contains the subalgebras $\langle \partial_{z_2} + \mu v^j \partial_{v^j} \rangle$, $\mu \in \mathbb{R}$. An ansatz associated with such a subalgebra reads $v = \mathrm{e}^{\mu z_2} \varphi(\omega)$ with $\omega = z_1$. Substituting this ansatz into the system (4.44), expanding B according to its definition, $\mathsf{B} := \operatorname{diag} \mathsf{F} c - \mathsf{CF} + \beta \mathsf{E}$, and rearranging summands, we derive the reduced system

$$(\partial_\omega - \mu \mathsf{C}) \tilde{\mathsf{F}} \varphi + \mu \tilde{\mathsf{B}} \varphi = 0, \tag{4.58}$$

where $\mathsf{C} := \operatorname{diag} c$, $\tilde{\mathsf{F}} := \mathsf{F} + \mu^2 (1 + \chi^2) \mathsf{E}$ and $\tilde{\mathsf{B}} := \operatorname{diag} \mathsf{F} c + \beta \mathsf{E}$. The modified matrix $\tilde{\mathsf{F}}$ commutes with F, and thus it is diagonalizable, its eigenvectors and corresponding eigenvalues are e_i and $\lambda_i + \mu^2 (1 + \chi^2)$, respectively, the transition matrix

to this eigenbasis coincides with P, see Section 4.2.3 for the notation. Moreover, if $\chi_\omega \neq 0$, the matrix $\tilde{\mathsf{F}}$ is invertible as a matrix-valued function. Substituting $\tilde\varphi = \tilde{\mathsf{F}}\varphi$ into (4.58) we derive a variable-coefficient linear homogeneous system of first-order ordinary differential equations in the canonical form,

$$\tilde\varphi_\omega - \mu(\mathsf{C} - \tilde{\mathsf{B}}\tilde{\mathsf{F}}^{-1})\tilde\varphi = 0. \tag{4.59}$$

The corresponding solutions of the system (4.1) can be represented in the form

- $\psi = \mathrm{e}^{\mu(x-\chi y)}\tilde{\mathsf{F}}^{-1}\tilde\varphi - \dfrac{1}{2}\chi_t y^2\bar 1 + yc + \dfrac{\chi_{tt}}{\beta}(x-\chi y)\bar 1 - \dfrac{\chi_t}{\beta}c,$

where $\tilde\varphi = \tilde\varphi(\omega)$ with $\omega = t$ is an arbitrary solution of (4.59), c is an arbitrary constant tuple that does not belong to $\langle\bar 1\rangle$, μ is an arbitrary constant, χ is an arbitrary smooth function of t, and $\tilde{\mathsf{F}} := \mathsf{F} + \mu^2(1+\chi^2)\mathsf{E}$. Complexifying these solutions, setting the parameter μ to be imaginary and splitting the result into the real and the imaginary parts, we construct solutions that are bounded and periodic with respect to $x - \chi y$.

Scale-invariant case. When $\chi = \pm\sqrt{\alpha z_1^2 - 1}$, a complete list of inequivalent subalgebras of $\mathfrak{g}^{\max}_\varrho$ that are appropriate for Lie reduction of the system (4.44) includes one more subalgebra family $\langle z_1\partial_{z_1} + z_2\partial_{z_2} + \mu v^j\partial_{v^j}\rangle$, $\mu \in \mathbb{R}$. For a fixed μ, an associated ansatz is $v = |z_1|^\mu\varphi(\omega)$ with $\omega := z_2/z_1$, which leads to the reduced system

$$\alpha(\omega^3\mathsf{E} + \mathsf{C})\varphi_{\omega\omega\omega} - \alpha(\mu-2)\omega^2\varphi_{\omega\omega} + (\omega\mathsf{F} - \mathsf{B})\varphi_\omega - \mu\mathsf{F}\varphi = 0. \tag{4.60}$$

The integration of this system does not look feasible since even the analogous system (4.53) in the decoupled case was solved in terms of Bessel functions and antiderivatives of Whittaker functions. For the sake of completeness, we write a representation for the corresponding solutions of the system (4.1):

- $\psi = |t|^\mu\varphi - \dfrac{\varepsilon\alpha t}{\sqrt{\alpha t^2-1}}(y^2\bar 1 + 2\beta^{-1}c) + yc - \dfrac{\varepsilon\alpha t\omega}{\beta(\alpha t^2-1)^{3/2}}\bar 1,$

where $\varepsilon = \pm 1$, $\varphi = \varphi(\omega)$ with $\omega = (x + \varepsilon y\sqrt{\alpha t^2-1})/t$ is an arbitrary solution of (4.60), c is an arbitrary constant m-tuple with $c \notin \langle\bar 1\rangle$, α and μ are arbitrary constants with $\alpha > 0$.

Shift-invariant case. When $\chi_1 = 0$, a complete list of inequivalent one-dimensional subalgebras of the maximal Lie invariance algebra $\mathfrak{g}^{\max}_{\varrho}$ of the system (4.44) that are appropriate for Lie reductions includes at least the subalgebra families,

$$\langle \partial_{z_1} + \nu\partial_{z_2} + \mu v^j \partial_{v^j} \rangle \quad \text{and} \quad \langle \partial_{z_2} + \mu v^j \partial_{v^j} \rangle,$$

where $\nu, \mu \in \mathbb{R}$.

Using a subalgebra from the first family, we construct the ansatz $v = \mathrm{e}^{\mu z_1}\varphi(\omega)$ with $\omega = z_2 - \nu z_1$, which reduces the system (4.44) to a constant-coefficient homogeneous linear system of ordinary differential equations that are at most of order three,

$$(1+\chi^2)\hat{\mathsf{C}}\varphi_{\omega\omega\omega} - \mu(1+\chi^2)\varphi_{\omega\omega} - \hat{\mathsf{B}}\varphi_{\omega} - \mu\mathsf{F}\varphi = 0, \tag{4.61}$$

where $\hat{\mathsf{C}} := \mathsf{C} + \nu\mathsf{E}$ and $\hat{\mathsf{B}} := \operatorname{diag}\mathsf{F}c - \hat{\mathsf{C}}\mathsf{F} + \beta\mathsf{E}$. The general solution of the reduced system (4.61) can be obtained via solving the eigenvalue problem for a certain $3m$ by $3m$ matrix constructed from the matrices $(1+\chi^2)\hat{\mathsf{C}}$, $\mu(1+\chi^2)\mathsf{E}$, $\hat{\mathsf{B}}$ and $\mu\mathsf{F}$. While finding this solution for specific values of parameters causes no issues, writing it down in a closed form in terms of the involved parameters remains a challenge even for a low number of layers m since it requires further investigation of properties of the above collection of matrices. We do not consider this problem here, but we will study an analogous inhomogeneous system for the specific case $\mu = \nu = 0$ in Section 4.9.1. Nevertheless, the corresponding solution of (4.1) is given by

- $\psi = \mathrm{e}^{\mu t}\varphi + yc,$ (4.62)

where $\varphi = \varphi(\omega)$ with $\omega = x - \chi y - \nu t$ is an arbitrary solution of (4.61), c is an arbitrary constant m-tuple with $c \notin \langle \bar{1} \rangle$ and μ, ν and χ are arbitrary constants. We can complexify this solution, assuming that μ is an imaginary number. If in this framework the eigenvalue problem associated with (4.61) has imaginary solutions, the system (4.61) in its turn has bounded periodic solutions, which results in simple-wave velocity fields.

The reduction using any subalgebra from the family $\langle \partial_{z_2} + \mu v^j \partial_{v^j} \rangle$ leads to the system (4.58) with constant χ, which can be easily integrated in terms of matrix exponentials. The precise form of the general solution of (4.59) depends on whether the modified matrix $\tilde{\mathsf{F}} = \mathsf{F} + \mu^2(1+\chi^2)\mathsf{E}$ is invertible or not.

It is invertible if and only if $\lambda_i \neq -\mu^2(1+\chi^2)$ for all $i = 1, \dots, m$. Recall that λ_i are the eigenvalues of the matrix F, see the end of Section 4.2.2. In this case, the system (4.58) is equivalent to the system (4.59), which is a linear homogeneous system of first-order ordinary differential equations with constant coefficients. Hence the corresponding solution of the original system (4.1) is given by

- $\psi = \mathrm{e}^{\mu(x-\chi y)\tilde{\mathsf{F}}^{-1}} \exp\big(\mu t(\mathsf{C} - \tilde{\mathsf{B}}\tilde{\mathsf{F}}^{-1})\big)A + yc,$ (4.63)

where $\tilde{\mathsf{F}} := \mathsf{F} + \mu^2(1+\chi^2)\mathsf{E}$, $\tilde{\mathsf{B}} := \operatorname{diag}\mathsf{F}c + \beta\mathsf{E}$, $\mathsf{C} := \operatorname{diag} c$, c and A are m-tuples of arbitrary constants with $c \notin \langle 1\rangle$, and μ and χ are arbitrary constants.

Since the eigenvalues of $\tilde{\mathsf{F}}$ are pairwise different, the matrix $\tilde{\mathsf{F}}$ is degenerate if and only if there exists i such that the eigenvector e_i of $\tilde{\mathsf{F}}$ spans the kernel of $\tilde{\mathsf{F}}$, $\ker\tilde{\mathsf{F}} = \langle e_i\rangle$. This allows us to write down the corresponding solution of the original system (4.1) as follows:

- $\psi = \mathrm{e}^{\mu(x-\chi y)\tilde{\mathsf{F}}^{+}} \exp\big(\mu t(\mathsf{C} - \tilde{\mathsf{B}}\tilde{\mathsf{F}}^{+})\big)A + g(t)e_i + yc,$

where $\tilde{\mathsf{F}} := \mathsf{F} + \mu^2(1+\chi^2)\mathsf{E}$, $\tilde{\mathsf{B}} := \operatorname{diag}\mathsf{F}c + \beta\mathsf{E}$, $\mathsf{C} := \operatorname{diag} c$, c and A are tuples of arbitrary constants with $c \notin \langle \bar{1}\rangle$, μ and χ are arbitrary constants, $g(t)$ is an arbitrary smooth function of t if $\tilde{\mathsf{B}}e_i = 0$ and $g = 0$ otherwise, and the Moore–Penrose inverse $\tilde{\mathsf{F}}^{+}$ of $\tilde{\mathsf{F}}$ is defined in the same way as that for F in (4.3).

Note that in the derived solutions, we can complexify the parameters μ and A to obtain wave-like solutions. When $\mu = \mathrm{i}\nu$ with $\nu \in \mathbb{R}_{\neq 0}$, the modified matrix $\tilde{\mathsf{F}} := \mathsf{F} - \nu^2(1+\chi^2)\mathsf{E}$ is invertible, and thus the only relevant solution family for the complexification is (4.63). This yields bounded periodic solutions of (4.1) if at least some eigenvalues of the matrix $\mathsf{C} - \tilde{\mathsf{B}}\tilde{\mathsf{F}}^{-1}$ are real.

4.8.3 Subalgebra family 1.3

In many aspects, the consideration of Lie reductions with respect to the subalgebras $\mathfrak{s}_{1.3}^{\chi c\kappa}$ is analogous to that for the subalgebras $\mathfrak{s}_{1.2}^{\chi c}$. This includes the group classification of reduced systems, partitioning into cases for two-step reductions and their naming. In the spirit of Remark 4.32, special attention is paid to the *completely decoupled* and *generically coupled* cases of the reduced systems and the further splitting of the latter

one into the *general*, the *scale-invariant* and the *shift-invariant* cases when relevant.

Reduced system

The one-dimensional subalgebra $\mathfrak{s}_{1.3}^{\chi c\kappa} := \langle \mathcal{P}^x(\chi) + c_k\mathcal{J}^k + \mathcal{Z}(\kappa)\rangle$ of $\mathfrak{g}$, where $c := (c_1,\dots,c_m)^{\mathsf{T}}$ is an arbitrary element of $\operatorname{im}\mathsf{F}$ and χ and κ are arbitrary smooth functions of t, satisfies the transversality condition and is thus appropriate for Lie reduction if and only if $\chi \neq 0$. Using this subalgebra, we construct a suitable Lie ansatz,

$$\psi^i = v^i - \frac{\chi_t(t)y - \kappa(t) - c_i}{\chi(t)}x - \frac{\beta}{6}y^3 \quad \text{with} \quad z_1 = t, \quad z_2 = \chi y - \int \kappa(t)\,\mathrm{d}t,$$

to reduce the system (4.1) to the linear system

$$\begin{aligned}
&c_i\chi^2 v^i_{222} + (\chi^2 v^i_{22})_1 \\
&\quad + f_{i,i-1}\left(c_i v_2^{i-1} - c_i v_2^i + v_1^{i-1} - v_1^i - c_{i-1}v_2^i + c_i v_2^i + \frac{\beta}{2\chi^3}(z_2 + \textstyle\int\kappa)^2(c_{i-1} - c_i)\right) \\
&\quad - f_{i,i+1}\left(c_i v_2^i - c_i v_2^{i+1} + v_1^i - v_1^{i+1} - c_i v_2^i + c_{i+1}v_2^i + \frac{\beta}{2\chi^3}(z_2 + \textstyle\int\kappa)^2(c_i - c_{i+1})\right) = 0,
\end{aligned}$$

where $\int\kappa := \int \kappa(t)\,\mathrm{d}t$ denotes an antiderivative of κ with respect to $z_1 := t$. In the matrix notation, the ansatz and the reduced system take the form

$$\begin{aligned}
&\psi = v - \left(\frac{\chi_t(t)y - \kappa(t)}{\chi(t)}x - \frac{\beta}{6}y^3\right)\bar{1} + \frac{x}{\chi(t)}c \quad \text{with } z_1 = t,\ z_2 = \chi y - \int \kappa(t)\,\mathrm{d}t,\\
&\chi^2\mathsf{C}v_{222} + (\chi^2 v_{22})_1 + \mathsf{B}v_2 + \mathsf{F}v_1 + \frac{\beta}{2\chi^3}(z_2 + \textstyle\int\kappa)^2\mathsf{F}c = 0,
\end{aligned} \tag{4.64}$$

respectively. As above, $\mathsf{C} := \operatorname{diag} c$, and we also denote $\mathsf{B} := \mathsf{CF} - \operatorname{diag}\mathsf{F}c$. The facts that $(\operatorname{diag}\mathsf{F}c)\bar{1} = \mathsf{F}c$, $\operatorname{im}\mathsf{F} \perp_{\mathsf{W}} \langle\bar{1}\rangle$, $\operatorname{im}\mathsf{B} \perp_{\mathsf{W}} \langle\bar{1}\rangle$ and $\dim\operatorname{im}\mathsf{F} = m-1$ implies $\operatorname{im}\mathsf{B} \subseteq \operatorname{im}\mathsf{F}$. We also have $c \in \operatorname{im}\mathsf{F}$. This is why a particular solution of the system (4.64) takes the form

$$\hat{v} = -\beta\left(\tfrac{1}{2}z_2^2\mathcal{X}_{11} + \mathcal{K}_1 z_2 + \mathcal{N}\right)c + \beta\chi^2\mathcal{X}_{11}\mathsf{F}^+c + \beta\left(z_2\mathcal{X}_1 + \mathcal{K}\right)\mathsf{F}^+\mathsf{B}c - \beta\mathcal{X}(\mathsf{F}^+\mathsf{B})^2c,$$

where $\mathcal{X}$, $\mathcal{K}$ and $\mathcal{N}$ denote third, the second and the first antiderivatives of χ^{-3}, $\chi^{-3}\int\kappa$ and $\chi^{-3}(\int\kappa)^2$ with respect to z_1, respectively. Recall that the Moore–Penrose

inverse F^+ of F is given by (4.3). By the transformation of the dependent variables

$$v - \hat{v} \to v,$$

we reduce the system (4.64) to its homogeneous counterpart

$$\chi^2 \mathsf{C} v_{222} + (\chi^2 v_{22})_1 + \mathsf{B} v_2 + \mathsf{F} v_1 = 0, \tag{4.65}$$

Therefore, we have the following representation for the $\mathfrak{s}_{1.3}^{\chi c\kappa}$-invariant solutions of the system (4.1):

$$\begin{aligned} \bullet\quad \psi =& v - \left(\frac{\chi_t(t) y - \kappa(t)}{\chi(t)} x - \frac{\beta}{6} y^3\right) \bar{1} + \frac{x}{\chi(t)} c - \beta \left(\tfrac{1}{2} z_2^2 \mathcal{X}_{11} + \mathcal{K}_1 z_2 + \mathcal{N}\right) c \\ &+ \beta \chi^2 \mathcal{X}_{11} \mathsf{F}^+ c + \beta \left(z_2 \mathcal{X}_1 + \mathcal{K}\right) \mathsf{F}^+ \mathsf{B} c - \beta \mathcal{X} (\mathsf{F}^+ \mathsf{B})^2 c, \end{aligned} \tag{4.66}$$

where $v = v(z_1, z_2)$ is an arbitrary solution of the homogeneous linear system of partial differential equations (4.65), which can be considered as a reduced system for $\mathfrak{s}_{1.3}^{\chi c\kappa}$-invariant solutions instead of (4.64), $z_1 := t$, $z_2 = \chi y - \int \kappa(t)\, \mathrm{d}t$, χ and κ are arbitrary functions of t, $\mathcal{X}$, $\mathcal{K}$ and $\mathcal{N}$ denote third, the second and the first antiderivatives of χ^{-3}, $\chi^{-3} \int \kappa$ and $\chi^{-3} (\int \kappa)^2$ with respect to t, respectively, $c := (c_1, \dots, c_m)^{\mathsf{T}}$ is a tuple of arbitrary constants with $c \perp_{\mathsf{W}} \bar{1}$, $\mathsf{C} := \operatorname{diag} c$ and $\mathsf{B} := \mathsf{C}\mathsf{F} - \operatorname{diag} \mathsf{F} c$.

Group classification

The study of induced and hidden symmetries of the system (4.1) associated with the reduction with respect to the subalgebra $\mathfrak{s}_{1.3}^{\chi c\kappa}$ is analogous to that carried out in Section 4.8.2. We begin with finding the induced symmetries of the reduced system (4.65). Note that the subalgebra $\mathfrak{g}_1 = \langle \mathcal{P}^x(\chi), \mathcal{J}^1, \dots, \mathcal{J}^m, \mathcal{Z}(\kappa) \rangle \subset \mathfrak{g}$ is an abelian Lie algebra. The normalizer $\mathrm{N}_{\mathfrak{g}}(\mathfrak{s}_{1.3}^{\chi c\kappa})$ of $\mathfrak{s}_{1.3}^{\chi c\kappa}$ coincides with the subalgebra $\mathfrak{g}_1$ for generic values of χ, c and κ, providing $\chi_t \neq 0$ and $\kappa_t \neq 0$. We have only the following extensions of the generic case:

1. $\mathrm{N}_{\mathfrak{g}}(\mathfrak{s}_{1.3}^{\chi c\kappa}) = \langle \mathcal{P}^t \rangle + \mathfrak{g}_1$ if and only if $\chi_t = \alpha\chi$, $\kappa_t = \alpha\kappa$ for some $\alpha \neq 0$ and $c = 0$,
2. $\mathrm{N}_{\mathfrak{g}}(\mathfrak{s}_{1.3}^{\chi c\kappa}) = \langle \mathcal{P}^y \rangle + \mathfrak{g}_1$ if and only if $\chi_t = 0$ and $\kappa_t \neq 0$,
3. $\mathrm{N}_{\mathfrak{g}}(\mathfrak{s}_{1.3}^{\chi c\kappa}) = \mathfrak{g}$ if and only if $\chi_t = 0$ and $\kappa_t = 0$.

The algebras of induced symmetries corresponding to the above cases are the following:

0. $\langle z_2\rho(z_1)(\partial_{v^1}+\dots+\partial_{v^m}),\ \partial_{v^1},\ \dots,\ \partial_{v^m},\ \kappa(z_1)(\partial_{v^1}+\dots+\partial_{v^m})\rangle$,
1. $\langle \partial_{z_1},\ z_2\rho(z_1)(\partial_{v^1}+\dots+\partial_{v^m}),\ \partial_{v^1},\ \dots,\ \partial_{v^m},\ \kappa(z_1)(\partial_{v^1}+\dots+\partial_{v^m})\rangle$,
2. $\langle \partial_{z_2},\ z_2\rho(z_1)(\partial_{v^1}+\dots+\partial_{v^m}),\ \partial_{v^1},\ \dots,\ \partial_{v^m},\ \kappa(z_1)(\partial_{v^1}+\dots+\partial_{v^m})\rangle$,
3. $\langle \partial_{z_1},\ \partial_{z_2},\ z_2\rho(z_1)(\partial_{v^1}+\dots+\partial_{v^m}),\ \partial_{v^1},\ \dots,\ \partial_{v^m},\ \kappa(z_1)(\partial_{v^1}+\dots+\partial_{v^m})\rangle$.

Here $\rho=\rho(z_1)$ and $\kappa=\kappa(z_1)$ run through the set of arbitrary smooth functions of z_1.

The homogeneous reduced system (4.65) is of the same form as (4.44), which allows us to straightforwardly apply the results of Section 4.8.2 for computing the maximal Lie invariance algebras of (4.65) depending on values of involved parameters. For this purpose, we regard the set of systems of the form (4.65) as a class of systems of differential equations parameterized by an arbitrary function $\chi=\chi(z_1)$ and the components of $\theta=(\mathsf{F},\beta)$ and c. The most general form of a Lie symmetry vector field of (4.65) is

$$\tau\partial_{z_1}+\xi\partial_{z_2}+\eta^j\partial_{v^j},$$

where the components τ, ξ and η^j are smooth functions depending on (z_1,z_2,v). The maximal Lie invariance algebras of any system from the class (4.65) with the general value of the arbitrary-element tuple is infinite-dimensional and spanned by the vector fields

$$\partial_{z_2},\quad (\mathsf{H}v)^j\partial_{v^j},\quad \zeta^j(z_1,z_2)\partial_{v^j}.$$

Here the tuple $(\zeta^1,\dots,\zeta^m)$ runs through the solution set of the system (4.65), and H runs through a basis of the space $\mathfrak{H}$ of (constant) matrices commuting with the matrices F, C and $\mathsf{CF}-\operatorname{diag}\mathsf{F}c$, where the last matrix can be replaced by $\operatorname{diag}\mathsf{F}c$,

$$[\mathsf{H},\mathsf{F}]=[\mathsf{H},\mathsf{C}]=[\mathsf{H},\operatorname{diag}\mathsf{F}c]=0.$$

Some general properties of the space $\mathfrak{H}$ have been described in Lemmas 4.29 and 4.30, and the exhaustive description of $\mathfrak{H}$ for $m=2$, 3 and 4 has been given in Section 4.8.2.

For arbitrary fixed values of the arbitrary elements $\theta=(\mathsf{F},\beta)$ and c, the corresponding general Lie invariance algebra can be further extended for specific values of

the parameter function χ. There are two inequivalent cases of such extensions,

$$\begin{aligned}
&\chi_1 = 0\colon && \langle \partial_{z_2}, (\mathsf{H}v)^j \partial_{v^j}, \zeta^j(z_1,z_2)\partial_{v^j}\rangle + \langle \partial_{z_1}\rangle,\\
&\chi = \alpha z_1\colon && \langle \partial_{z_2}, (\mathsf{H}v)^j \partial_{v^j}, \zeta^j(z_1,z_2)\partial_{v^j}\rangle + \langle z_1\partial_{z_1} + z_2\partial_{z_2}\rangle,
\end{aligned}$$

where α is an arbitrary nonzero constant. The extension ∂_{z_1} is induced by the Lie-symmetry vector field ∂_t of the original system $\mathcal{M}_\theta$, whereas the extension $z_1\partial_{z_1}+z_1\partial_{z_2}$ is a genuine hidden Lie symmetry of $\mathcal{M}_\theta$.

Completely decoupled case

If $c = 0$, the system (4.65) takes the form $(\chi^2 v_{22} + \mathsf{F}v)_1 = 0$, which trivially integrates with respect to z_1 to

$$\chi^2 v_{22} + \mathsf{F}v = \varphi, \tag{4.67}$$

where $\varphi := (\varphi^1(z_2), \dots, \varphi^m(z_2))^{\mathsf{T}}$ is an m-tuple of arbitrary smooth functions of z_2. The system (4.67) is an inhomogeneous linear system of second-order ordinary differential equations with the independent variable z_2, where z_1 plays the role of a parameter. It can be decoupled via diagonalizing the matrix F and then integrated by the method of variation of parameters. More specifically, we first make the change of the dependent variables $v := \mathsf{P}\tilde{v}$ and denote $\tilde{\varphi} := \mathsf{P}^{-1}\varphi$, where P is the transition matrix to the eigenbasis of F, see Section 4.2.3. Then the system (4.67) reduces to the decoupled system

$$\chi^2 \tilde{v}_{22} + \lambda_i \tilde{v} = \tilde{\varphi}, \quad i = 1, \dots, m. \tag{4.68}$$

Recall that λ_i are the eigenvalues of F with $\lambda_1 < \dots < \lambda_m = 0$. For $i \neq m$ and $i = m$, the general solution of the equations (4.68) is given by

$$\begin{aligned}
\circ\quad \tilde{v}^i &= \frac{\mathrm{e}^{\nu^i z_2}}{2\chi^2\nu^i}\int \mathrm{e}^{-\nu^i z_2}\tilde{\varphi}^i(z_2)\mathrm{d}z_2 - \frac{\mathrm{e}^{-\nu^i z_2}}{2\chi^2\nu^i}\int \mathrm{e}^{\nu^i z_2}\tilde{\varphi}^i(z_2)\mathrm{d}z_2\\
&\quad + h^{1i}(z_1)\mathrm{e}^{\nu^i z_2} + h^{2i}(z_1)\mathrm{e}^{-\nu^i z_2},\\
\tilde{v}^m &= \chi^{-2} g(z_2) + h^{1m}(z_1) z_2 + h^{2m}(z_1),
\end{aligned}$$

respectively, where $\nu^i := \sqrt{-\lambda_i}/\chi$, h^{1i} and h^{2i} are arbitrary smooth functions of z_1, and $g = g(z_2)$ is a second antiderivative of $\tilde\varphi^m$ with respect to z_2. Pulling this solution back with respect to the ansatz, we derive the solution of the original system (4.1),

- $$\psi = \sum_{i=1}^{m-1}\left(\frac{\mathrm{e}^{\nu^i z_2}}{2\chi^2\nu^i}\int \mathrm{e}^{-\nu^i z_2}\tilde\varphi^i\,\mathrm{d}z_2 - \frac{\mathrm{e}^{-\nu^i z_2}}{2\chi^2\nu^i}\int \mathrm{e}^{\nu^i z_2}\tilde\varphi^i\,\mathrm{d}z_2 + h^{1i}\mathrm{e}^{\nu^i z_2} + h^{2i}\mathrm{e}^{-\nu^i z_2}\right)e_i$$
  $$+\Big(\chi^{-2}g + h^{1m}z_2 + h^{2m} - \frac{\chi_t y - \kappa}{\chi}x - \frac{\beta}{6}y^3\Big)\bar 1,$$

where $z_2 = \chi y - \int \kappa \mathrm{d}t$, e_i is an eigenvector of the matrix F corresponding to its eigenvalue λ_i, $\lambda_m = 0$, $e_m = \bar 1$, $\nu^i := \sqrt{-\lambda_i}/\chi(t)$, $\chi = \chi(t)$, $\kappa = \kappa(t)$, $h^{1i} = h^{1i}(t)$, $h^{2i} = h^{2i}(t)$, $\tilde\varphi^i = \tilde\varphi^i(z_2)$ and $g = g(z_2)$ are arbitrary smooth function of their arguments with $\chi \neq 0$. Note that we can gauge $h^{2m}(t)$ to zero modulo the G-equivalence.

In view of Corollary 4.2, which states that nonzero eigenvalues λ_i are real negative, and the positivity of the parameter β, only particular simple solutions from the above family are physically relevant for the entire plane.

Generically coupled case

Analogously to Section 4.8.2, we study the case of generic coupling of the reduced system (4.65) with values the subalgebra parameter c from $\operatorname{im}\mathsf{F}\setminus\{0\}$. The consideration is divided into three subcases according to the presence of Lie-symmetry vector fields with nonzero z_1-components and their kind, that is, the general case (with no such fields) and the scale- and the shift-invariant cases.

General case. A complete list of inequivalent one-dimensional subalgebras of the maximal Lie invariance algebra of the system (4.65) that are appropriate for Lie reductions necessarily includes the subalgebra family

$$\langle \partial_{z_2} + \mu v^j \partial_{v^j}\rangle,$$

where $\mu \in \mathbb{R}$. A suitable ansatz corresponding to a subalgebra from this family with a fixed value of μ is $v = \mathrm{e}^{\mu z_2}\varphi(\omega)$ with $\omega = z_1$. It reduces the system (4.65) to

$$(\partial_\omega + \mu\mathsf{C})\tilde{\mathsf{F}}\varphi - \mu\tilde{\mathsf{B}}\varphi = 0, \tag{4.69}$$

where

$$\tilde{\mathsf{F}} := \mathsf{F} + \mu^2\chi^2\mathsf{E} \quad \text{and} \quad \tilde{\mathsf{B}} := \operatorname{diag}\mathsf{F}c.$$

The modified matrix $\tilde{\mathsf{F}}$ commutes with F, it is diagonalizable, its eigenvalues and eigenvectors are $\lambda_i + \mu^2\chi^2$ and e_i, respectively, where λ_i and e_i are eigenvalues and eigenvectors of the matrix F, see Section 4.2.3 for details. In the general case, we can assume $\chi_\omega \neq 0$, and then the matrix $\tilde{\mathsf{F}}$ is invertible as a matrix-valued function. Substituting $\varphi = \tilde{\mathsf{F}}^{-1}\tilde{\varphi}$ into (4.69), we obtain a system of homogeneous linear first-order ordinary differential equations with variable coefficients in the canonical form

$$\tilde{\varphi}_\omega + \mu(\mathsf{C} - \tilde{\mathsf{B}}\tilde{\mathsf{F}}^{-1})\tilde{\varphi} = 0. \tag{4.70}$$

The corresponding solutions of the reduced system (4.65) can be represented as

$$\circ \quad v = \mathrm{e}^{\mu z_2}\tilde{\mathsf{F}}^{-1}\tilde{\varphi}, \tag{4.71}$$

where $\tilde{\varphi} = \tilde{\varphi}(z_1)$ is an arbitrary solution of (4.70), c is an arbitrary constant m-tuple with $c \in \operatorname{im}\mathsf{F}\setminus\{0\}$, μ is an arbitrary constant, χ and κ are arbitrary smooth functions of t, and $\tilde{\mathsf{F}} := \mathsf{F} + \mu^2\chi^2\mathsf{E}$. Substituting (4.71) into (4.66), we obtain a solution family of (4.1).

Scale-invariant case. For $\chi = \alpha z_1$, a complete list of inequivalent one-dimensional subalgebras of the maximal Lie invariance algebra of (4.65) in addition includes the subalgebra family $\langle z_1\partial_{z_1} + z_2\partial_{z_2} + \mu v^j\partial_{v^j}\rangle$, where $\mu \in \mathbb{R}$. Using such a subalgebra with fixed μ, we construct an ansatz $v = |z_1|^\mu\varphi(\omega)$ with $\omega := z_2/z_1$, which leads to the reduced system

$$\alpha^2\hat{\mathsf{C}}\varphi_{\omega\omega\omega} + \mu\alpha^2\varphi_{\omega\omega} + \hat{\mathsf{B}}\varphi_\omega + \mu\mathsf{F}\varphi = 0, \tag{4.72}$$

where $\hat{\mathsf{C}} = \mathsf{C} - \omega\mathsf{E}$ and $\hat{\mathsf{B}} := \hat{\mathsf{C}}\mathsf{F} - \operatorname{diag}\mathsf{F}c$. The corresponding solutions of the reduced system (4.65) can be represented as

$$\circ \quad v = |z_1|^\mu\varphi, \tag{4.73}$$

where $\varphi = \varphi(z_2/z_1)$ is an arbitrary solution of (4.72), c is an arbitrary constant m-tuple with $c \in \operatorname{im}\mathsf{F} \setminus \{0\}$, μ is an arbitrary constant, χ and κ are arbitrary smooth functions of t. Substituting (4.73) into (4.66), we obtain the associated representation of solutions of (4.1).

Shift-invariant case. When χ is an arbitrary nonzero constant, a complete list of inequivalent one-dimensional subalgebras of the maximal Lie invariance algebra of the system (4.65) that are appropriate for Lie reductions contains at least the subalgebra families $\langle \partial_{z_1} + \nu\partial_{z_2} + \mu v^j\partial_{v^j}\rangle$ and $\langle \partial_{z_2} + \mu v^j\partial_{v^j}\rangle$, where $\nu, \mu \in \mathbb{R}$.

Any subalgebra from the first family $\langle \partial_{z_1} + \nu\partial_{z_2} + \mu v^j\partial_{v^j}\rangle$ gives rise to the ansatz $v = \mathrm{e}^{\mu z_1}\varphi(\omega)$ with $\omega = z_2 - \nu z_1$, which reduces the system (4.65) to

$$\chi^2\hat{\mathsf{C}}\varphi_{\omega\omega\omega} + \chi^2\mu\varphi_{\omega\omega} + \hat{\mathsf{B}}\varphi_\omega + \mu\mathsf{F}\varphi = 0, \tag{4.74}$$

where $\hat{\mathsf{C}} = \mathsf{C} - \nu\mathsf{E}$ and $\hat{\mathsf{B}} := \hat{\mathsf{C}}\mathsf{F} - \operatorname{diag}\mathsf{F}c$. The derived system (4.74) is a homogeneous linear system of third-order ordinary differential equations. Since finding its general solution depends on solving an eigenvalue problem for a certain $3m$ by $3m$ matrix constituted by components of $\chi^2\hat{\mathsf{C}}$, $\hat{\mathsf{B}}$, $\chi^2\mu\mathsf{E}$ and $\mu\mathsf{F}$, we will not further consider this system. However, the corresponding solution of (4.65) is given by

$$\circ\quad v = \mathrm{e}^{\mu z_1}\varphi, \tag{4.75}$$

where $\varphi = \varphi(z_2 - \nu z_1)$ is an arbitrary solution of (4.74), c is an arbitrary constant m-tuple with $c \in \operatorname{im}\mathsf{F} \setminus \{0\}$, μ, χ and ν are arbitrary constants with $\chi \neq 0$.

Reduction using the subalgebra family $\langle \partial_{z_2} + \mu v^j\partial_{v^j}\rangle$ leads to (4.69), which is now a homogeneous linear system of first-order ordinary differential equations with constant coefficients. Thus, its general solution can be presented in terms of the matrix exponential. Its explicit representation depends on whether the modified matrix $\tilde{\mathsf{F}} = \mathsf{F} + \mu^2\chi^2\mathsf{E}$ is invertible or not.

The matrix $\tilde{\mathsf{F}}$ is invertible if and only if $\lambda_i \neq -\mu^2\chi^2$ for all $i = 1, \dots, m$. In this case, the solution (4.71) takes the form

$$\circ\quad v = \mathrm{e}^{\mu z_1}\tilde{\mathsf{F}}^{-1}\exp\big(-\mu(z_1 - \nu z_2)(\mathsf{C} - \tilde{\mathsf{B}}\tilde{\mathsf{F}}^{-1})\big)A, \tag{4.76}$$

where $\tilde{\mathsf{F}} := \mathsf{F} + \mu^2\chi^2\mathsf{E}$, $\tilde{\mathsf{B}} := \operatorname{diag}\mathsf{F}c$, $\mathsf{C} := \operatorname{diag}c$, c and A are tuples of arbitrary constants with $c \in \operatorname{im}\mathsf{F} \setminus \{0\}$, μ and χ are arbitrary constants.

Since eigenvalues of $\tilde{\mathsf{F}}$ are pairwise different, the matrix $\tilde{\mathsf{F}}$ is degenerate if and only if there exists i such that the eigenvector e_i of $\tilde{\mathsf{F}}$ spans the kernel of $\tilde{\mathsf{F}}$, $\ker\tilde{\mathsf{F}} = \langle e_i\rangle$. This allows us to write down the corresponding solutions of the system (4.65) as follows:

$$\circ \quad v = \mathrm{e}^{\mu z_1\tilde{\mathsf{F}}^+}\exp\big(\mu(z_2-\nu z_1)(\mathsf{C}-\tilde{\mathsf{B}}\tilde{\mathsf{F}}^+)\big)A + g(t)e_i, \tag{4.77}$$

where $\tilde{\mathsf{F}} := \mathsf{F} + \mu^2\chi^2\mathsf{E}$, $\tilde{\mathsf{B}} := \operatorname{diag}\mathsf{F}c$, $\mathsf{C} := \operatorname{diag}c$, c and A are tuples of arbitrary constants with $c \in \operatorname{im}\mathsf{F} \setminus \{0\}$, μ and χ are arbitrary constants, $g(t)$ is an arbitrary smooth function of t if $\tilde{\mathsf{B}}e_i = 0$ and $g = 0$ otherwise, and the Moore–Penrose inverse $\tilde{\mathsf{F}}^+$ of $\tilde{\mathsf{F}}$ is defined in the same way as that for F in (4.3). Substituting the solutions (4.75), (4.76) and (4.77) into (4.66), we obtain the associated representation of solutions of the original system (4.1).

4.9 Codimension-two Lie reductions

Among the subalgebras listed in Theorem 4.24, only the first, the second, the fourth and the sixth families are appropriate for the Lie reduction procedure. The corresponding reductions remain important to consider despite the fact that they can be interpreted as two-step Lie reductions of the system (4.1) with involving induced Lie symmetries, where the intermediate codimension-one reduced systems are linear systems of partial differential equations studied in Sections 4.8.2 and 4.8.3.

More specifically, the two-dimensional abelian subalgebras $\mathfrak{s}_{2.1}^{abc}$, $\mathfrak{s}_{2.2}^{abc}$ and $\mathfrak{s}_{2.6}^{\chi bc\kappa}$ of $\mathfrak{g}$ contain one-dimensional subalgebras $\mathfrak{s}_{1.2}^{ac0}$, $\mathfrak{s}_{1.3}^{1c0}$ and $\mathfrak{s}_{1.3}^{1c\kappa}$. This means that Lie reductions with respect to these two-dimensional subalgebras are two-step reductions of the system (4.1) with the intermediate codimension-one reduced systems (4.44), (4.65) and (4.65) and the final reduced systems (4.61) with $\chi = a$, $\mu = \nu = 0$, (4.74) with $\chi = 1$, $\mu = 0$, $\nu = a$ and (4.69) with $\mu = 0$, respectively. However, since we have exhaustively integrated not all codimension-one reduced systems, the consideration of the above codimension-two reductions is relevant, leading to the derivation of new closed-form exact solutions of (4.1) supplementing those presented in Section 4.8.

The Lie reduction with respect to the subalgebra $\mathfrak{s}_{2.4}^{ab\sigma}$ is of a different nature. Although this subalgebra is nonabelian, it contains the one-dimensional ideal $\mathfrak{s}_{1.3}^{\chi 0\kappa}$ with $\chi = \mathrm{e}^{\sigma t}$ and $\kappa = at\mathrm{e}^{\sigma t}$. Therefore, the associated Lie reduction is just a two-step reduction of the system (4.1), where the intermediate codimension-one reduced system is the completely decoupled system (4.65), which has been exhaustively integrated in Section 4.8.3. Nevertheless, for the sake of completeness of the presentation, we examine this codimension-two reduction of (4.1) as well and construct the corresponding closed-form solutions of the system (4.1).

In this section, $\varphi^1, \dots, \varphi^m$ denote the new dependent variables of the new independent variable ω, $\varphi := (\varphi^1, \dots, \varphi^m)^{\mathsf{T}}$.

4.9.1 Subalgebra family 2.1

An ansatz associated with the two-dimensional subalgebra $\mathfrak{s}_{2.1}^{abc} := \langle \mathcal{P}^t + b_k\mathcal{J}^k,\ \mathcal{P}^y + \mathcal{P}^x(a) + c_k\mathcal{J}^k\rangle$ of $\mathfrak{g}$, where a, $b := (b_1, \dots, b_m)^{\mathsf{T}}$, $c := (c_1, \dots, c_m)^{\mathsf{T}}$ are arbitrary constants with $b \in \operatorname{im}\mathsf{F}$, is $\psi^i = \varphi^i(\omega) + c_i y + b_i t$ or, in the vector form,

$$\psi = \varphi(\omega) + yc + tb \quad \text{with} \quad \omega := x - ay.$$

This ansatz reduces the system (4.1) to

$$\begin{aligned} &c_i(1+a^2)\varphi^i_{\omega\omega\omega} - f_{i,i-1}(c_{i-1}\varphi^i_\omega - c_i\varphi^{i-1}_\omega - b_i + b_{i-1}) \\ &\quad - f_{i,i+1}(c_{i+1}\varphi^i_\omega - c_i\varphi^{i+1}_\omega - b_i + b_{i+1}) - \beta\varphi^i_\omega = 0, \end{aligned}$$

which can be represented using the matrix notation $\mathsf{C} := \operatorname{diag} c$ and $\hat{\mathsf{C}} := \mathsf{CF} - \operatorname{diag}(\mathsf{F}c + \beta\bar{1})$ as

$$(1+a^2)\mathsf{C}\varphi_{\omega\omega\omega} + \hat{\mathsf{C}}\varphi_\omega - \mathsf{F}b = 0. \tag{4.78}$$

Let K and I denote the kernel and the image of the matrix C viewed as a linear operator on the space $\mathbb{R}^m$, respectively, $K := \ker\mathsf{C}$ and $I := \operatorname{im}\mathsf{C}$. Since C is a diagonal matrix, the description of subspaces K and I is evident and we have the decomposition $\mathbb{R}^m = K \oplus I$. Denote by π_K and π_I the natural projections onto subspaces K and I relative to that decomposition, respectively. We have that $\hat{\mathsf{C}}I \subset I$, i.e., I is an invariant subspace of $\hat{\mathsf{C}}$. At the same time, $\hat{\mathsf{C}}K \cap I \neq \{0\}$ if $K \neq \{0\}$. By the definition

of $\hat{\mathsf{C}}$ we have $\pi_K\hat{\mathsf{C}} = -\pi_K \operatorname{diag}(\mathsf{F}c+\beta\bar{1})$. This allows us to split the system (4.78) into system of equations on components $\varphi^0 := \pi_K(\varphi)$ and $\varphi^1 := \pi_I(\varphi)$ of φ,

$$\operatorname{diag}(\mathsf{F}c+\beta\bar{1})\varphi^0_\omega = -\pi_K\mathsf{F}b, \tag{4.79}$$

$$(1+a^2)\mathsf{C}\varphi^1_{\omega\omega} + \hat{\mathsf{C}}\varphi^1_\omega = \pi_I(\mathsf{F}b - \hat{\mathsf{C}}\varphi^0_\omega). \tag{4.80}$$

By the Rouché–Capelli theorem, the system (4.79) is consistent if and only if the rank of its coefficient matrix $\operatorname{diag}(\mathsf{F}c+\beta\bar{1})$ is equal to the rank of its augmented matrix

$$\big(\operatorname{diag}(\mathsf{F}c+\beta\bar{1}) \mid \pi_K\mathsf{F}b\big).$$

In this case, the system (4.79) easily integrates,

$$\circ\quad \varphi^0(\omega) = -\omega\big(\operatorname{diag}(\mathsf{F}c+\beta\bar{1})\big)^+\pi_K\mathsf{F}b + g(\omega) + B, \tag{4.81}$$

where $g(\omega)$ is a tuple of arbitrary functions that takes values in $K\cap\ker\operatorname{diag}(\mathsf{F}c+\beta\bar{1})$, B is a tuple of integration constants that can be assumed to belong to $K\cap\operatorname{im}\operatorname{diag}(\mathsf{F}c+\beta\bar{1})$ and $\big(\operatorname{diag}(\mathsf{F}c+\beta\bar{1})\big)^+$ is the Moore–Penrose inverse of $\operatorname{diag}(\mathsf{F}c+\beta\bar{1})$.

Let C^+ be the Moore–Penrose inverse of C. The matrix of the operator

$$(1+a^2)^{-1}\mathsf{C}^+\hat{\mathsf{C}}|_I$$

in the restriction of the standard basis $\mathbb{R}^m$ to I is represented by a block-diagonal matrix, and each of its blocks is a tridiagonal matrix with positive off-diagonal entries. Therefore, this matrix is diagonalizable. Denote by ν_1, …, ν_l its eigenvalues, by $\tilde{e}_1$, …, $\tilde{e}_l$ the corresponding eigenvectors, where $l := \dim I$, and by R the transition matrix from the above standard basis of the subspace I to the eigenbasis $(\tilde{e}_1,\dots,\tilde{e}_l)$. Consider the operators

$$\mathsf{H} := (1+a^2)^{-1}\mathsf{R}^{-1}\mathsf{C}^+\pi_I \quad\text{and}\quad \mathsf{B} := \mathsf{F} + \hat{\mathsf{C}}\big(\operatorname{diag}(\mathsf{F}c+\beta\bar{1})\big)^+\pi_K\mathsf{F}$$

and define $\tilde{h} := \mathsf{H}(\mathsf{B}b + \hat{\mathsf{C}}g_\omega)$. Using the change of dependent variables $\tilde{\varphi}^1 = \mathsf{R}^{-1}\varphi^1$, we decouple the system (4.80) as follows:

$$\tilde{\varphi}^{1s}_{\omega\omega} + \nu_s\tilde{\varphi}^{1s}_\omega = \tilde{h}^s, \quad s\in\{1,\dots,l\}.$$

The general solution of each equation from this system depends on the sign of $\tilde{\nu}_i$ and it can be written as follows:

- $\nu_s = 0\colon\quad \tilde{\varphi}^{1s}(\omega) = A_{0s} + A_{1s}\omega + A_{2s}\omega^2 + \dfrac{\omega^3}{6}\mathsf{H}\mathsf{B}b + \mathsf{H}\hat{\mathsf{C}}G,$

  $\nu_s < 0\colon\quad \tilde{\varphi}^{1s}(\omega) = A_{0s} + A_{1s}\mathrm{e}^{\sqrt{-\nu_s}\omega} + A_{2s}\mathrm{e}^{-\sqrt{-\nu_s}\omega}$
  $+\dfrac{\omega}{\nu_s}\mathsf{H}\mathsf{B}b + \dfrac{1}{\nu_s}\mathsf{H}\hat{\mathsf{C}}\displaystyle\int_{\omega_0}^{\omega}\big(1-\cosh\big(\sqrt{-\nu_s}(\omega-\varsigma)\big)\big)g_\varsigma(\varsigma)\,\mathrm{d}\varsigma,$ (4.82)

  $\nu_s > 0\colon\quad \tilde{\varphi}^{1s}(\omega) = A_{0s} + A_{1s}\sin(\sqrt{\nu_s}\omega) + A_{2s}\cos(\sqrt{\nu_s}\omega)$
  $+\dfrac{\omega}{\nu_s}\mathsf{H}\mathsf{B}b + \dfrac{1}{\nu_s}\mathsf{H}\hat{\mathsf{C}}\displaystyle\int_{\omega_0}^{\omega}\big(1-\cos(\sqrt{\nu_s}(\omega-\varsigma))\big)g_\varsigma(\varsigma)\,\mathrm{d}\varsigma,$

where A_{0s}, A_{1s} and A_{2s} are arbitrary constants, and G is a tuple of second antiderivatives of the tuple g, $G_{\omega\omega} = g$. Pulling the obtained solution back, we obtain the solution of the original system (4.1)

- $\psi = \displaystyle\sum_{s=1}^{l}\tilde{\varphi}^{1s}(x-ay)\tilde{e}_s + \varphi^0(x-ay) + tb + yc,$

where the functions φ^0 and φ^{1s} are defined in (4.81) and (4.82), respectively, $\tilde{e}_1,\dots,\tilde{e}_l$ are eigenvectors of the matrix $(1+a^2)^{-1}\mathsf{C}^+\hat{\mathsf{C}}|_I$ corresponding to its eigenvalues $\nu_1,\dots,\nu_l$, C^+ is the Moore–Penrose inverse of $\mathsf{C} := \operatorname{diag} c$, $I := \operatorname{im}\mathsf{C}$, $\hat{\mathsf{C}} := \mathsf{C}\mathsf{F} - \operatorname{diag}(\mathsf{F}c+\beta\bar{1})$, $b = (b_1,\dots,b_m)^{\mathsf{T}}$ and $c = (c_1,\dots,c_m)^{\mathsf{T}}$ are arbitrary constant tuples with $b \in \operatorname{im}\mathsf{F}$.

4.9.2 Subalgebra family 2.2

As associated with the two-dimensional subalgebra $\mathfrak{s}_{2.2}^{abc} := \langle \mathcal{P}^t + a\mathcal{P}^y + b_k\mathcal{J}^k,\ \mathcal{P}^x(1) + c_k\mathcal{J}^k\rangle$ of $\mathfrak{g}$, where $b := (b_1,\dots,b_m)^{\mathsf{T}}$, $c := (c_1,\dots,c_m)^{\mathsf{T}}$ and a are arbitrary constants with $\sigma \neq 0$ and $b \in \operatorname{im}\mathsf{F}$, we choose the ansatz $\psi^i = \varphi^i(\omega) + c_i x + b_i t$ with $\omega = y - at$. The corresponding reduced system then is

$$\begin{aligned}&(c_i-a)\varphi^i_{\omega\omega\omega} + f_{i,i-1}\big((c_i-a)\varphi^{i-1}_\omega - (c_{i-1}-a)\varphi^i_\omega + b_{i-1} - b_i\big)\\ &\quad - f_{i,i+1}\big((c_{i+1}-a)\varphi^i_\omega - (c_i-a)\varphi^{i+1}_\omega + b_i - b_{i+1}\big) + c_i\beta = 0.\end{aligned}$$

In the vector notation, the ansatz and the reduced system respectively take the form

$$\begin{aligned}&\psi = \varphi(\omega) + xc + tb \quad \text{with} \quad \omega := y - at,\\ &\mathsf{C}\varphi_{\omega\omega\omega} + \hat{\mathsf{C}}\varphi_\omega + \mathsf{F}b + \beta c = 0,\end{aligned} \tag{4.83}$$

where $\mathsf{C} := \operatorname{diag}(c - a\bar{1})$ and $\hat{\mathsf{C}} := \mathsf{CF} - \operatorname{diag}\mathsf{F}c$. The form of the reduced system is similar to that of the system (4.78), therefore, to find the general solution of (4.83) we use the approach from Section 4.9.1.

Since the matrix C is diagonal, we have the direct sum decomposition $\mathbb{R}^m = K \oplus I$, where $K := \ker \mathsf{C}$ and $I := \operatorname{im}\mathsf{C}$. Denote by π_K and π_I the natural projections from $\mathbb{R}^m$ onto K and I according to this decomposition, respectively. It follows from the definition of the matrix $\hat{\mathsf{C}}$ that $\hat{\mathsf{C}}I \subset I$, $\hat{\mathsf{C}}K \cap I \neq \{0\}$ if $K \neq \{0\}$, and $\pi_K\hat{\mathsf{C}} = -\pi_K \operatorname{diag}\mathsf{F}c$, which allows us to decompose the reduced system (4.83) according to the above decomposition of $\mathbb{R}^m$,

$$\operatorname{diag}\mathsf{F}c\varphi^0_\omega = \pi_K(\mathsf{F}b + \beta c), \tag{4.84}$$

$$\mathsf{C}\varphi^1_{\omega\omega\omega} + \hat{\mathsf{C}}\varphi^1_\omega = -\pi_I(\mathsf{F}b + \beta c + \hat{\mathsf{C}}\varphi^0_\omega), \tag{4.85}$$

where $\varphi^0 := \pi_K(\varphi)$ and $\varphi^1 := \pi_I(\varphi)$. The system (4.84) is compatible if and only if $\operatorname{rank}\operatorname{diag}\mathsf{F}c = \operatorname{rank}\big(\operatorname{diag}\mathsf{F}c \mid \pi_K(\mathsf{F}b + \beta c)\big)$, where $\big(\operatorname{diag}\mathsf{F}c \mid \pi_K(\mathsf{F}b + \beta c)\big)$ is the augmented matrix of the system (4.84). In this case, the general solution of (4.84) is given by

$$\circ \quad \varphi^0 = \omega\big(\operatorname{diag}\mathsf{F}c\big)^+\pi_K(\mathsf{F}b + \beta c) + B + g(\omega), \tag{4.86}$$

where $g(\omega)$ is a tuple of arbitrary functions that takes values in $K \cap \ker\operatorname{diag}\mathsf{F}c$, B is a tuple of integration constants that can be assumed to belong to $K \cap \operatorname{im}\operatorname{diag}\mathsf{F}c$ and $\big(\operatorname{diag}\mathsf{F}c\big)^+$ is the Moore–Penrose inverse of $\operatorname{diag}\mathsf{F}c$.

It can be shown using the arguments as in Section 4.9.1 that the matrix $\mathsf{C}^+\hat{\mathsf{C}}|_I$, where C^+ is the Moore–Penrose inverse of C, is diagonalizable. Denote by $\nu_1, \ldots, \nu_l$ its eigenvalues, where $l := \dim I$, by $\tilde{e}_1, \ldots, \tilde{e}_l$ the corresponding eigenvectors and by R the transition matrix from the standard basis of the subspace I to the eigenbasis $(\tilde{e}_1, \ldots, \tilde{e}_l)$. Consider the operators $\mathsf{H} := \mathsf{C}^+\pi_I$ and $\mathsf{B} := \mathsf{F} + \hat{\mathsf{C}}\big(\operatorname{diag}\mathsf{F}c\big)^+\pi_K\mathsf{F}$ and define $\tilde{h} := -\mathsf{H}(\mathsf{B}b + \beta c + \hat{\mathsf{C}}g_\omega)$. The substitution $\varphi^1 := \mathsf{R}\tilde{\varphi}^1$ enables decoupling the

system (4.85) to the system

$$\tilde\varphi^{1s}_{\omega\omega\omega}+\nu_s\tilde\varphi^{1s}_{\omega}=\tilde h^s,\quad s\in\{1,\dots,l\}.$$

The general solution of this system, depending on the sign of ν_s, is given by

$$\begin{aligned}
&\circ\quad \nu_s=0\colon\quad \tilde\varphi^{1s}(\omega)=A_{0s}+A_{1s}\omega+A_{2s}\omega^2-\frac{\omega^3}{6}\mathsf{H}(\mathsf{B}b+\beta c)-\mathsf{H}\hat{\mathsf{C}}G,\\
&\quad \nu_s<0\colon\quad \tilde\varphi^{1s}(\omega)=A_{0s}+A_{1s}\mathrm{e}^{\sqrt{-\nu_s}\omega}+A_{2s}\mathrm{e}^{-\sqrt{-\nu_s}\omega}\\
&\qquad\qquad -\frac{\omega}{\nu_s}\mathsf{H}(\mathsf{B}b+\beta c)-\frac{1}{\nu_s}\mathsf{H}\hat{\mathsf{C}}\int_{\omega_0}^{\omega}\big(1-\cosh\big(\sqrt{-\nu_s}(\omega-\varsigma)\big)\big)g_\varsigma(\varsigma)\,\mathrm{d}\varsigma,\qquad (4.87)\\
&\quad \nu_s>0\colon\quad \tilde\varphi^{1s}(\omega)=A_{0s}+A_{1s}\sin(\sqrt{\nu_s}\omega)+A_{2s}\cos(\sqrt{\nu_s}\omega)\\
&\qquad\qquad -\frac{\omega}{\nu_s}\mathsf{H}(\mathsf{B}b+\beta c)-\frac{1}{\nu_s}\mathsf{H}\hat{\mathsf{C}}\int_{\omega_0}^{\omega}\big(1-\cos(\sqrt{\nu_s}(\omega-\varsigma))\big)g_\varsigma(\varsigma)\,\mathrm{d}\varsigma,
\end{aligned}$$

where A_{0s}, A_{1s} and A_{2s} are arbitrary constants, and G is a tuple of second antiderivatives of the tuple g, $G_{\omega\omega}=g$. Pulling the obtained solution back, we construct the solution of the original system (4.1)

- $\psi=\sum\limits_{s=1}^{l}\tilde\varphi^{1s}(y-at)\tilde e_s+\varphi^0(y-at)+xc+tb,$

where the functions φ^0 and φ^{1s} are defined in (4.86) and (4.87), respectively, $\tilde e_1,\dots,\tilde e_l$ are eigenvectors of the matrix $\mathsf{C}^+\hat{\mathsf{C}}|_I$ corresponding to its eigenvalues $\nu_1,\dots,\nu_l$, C^+ is the Moore–Penrose inverse of $\mathsf{C}:=\operatorname{diag}(c-a\bar1)$, $I:=\operatorname{im}\mathsf{C}$, $\hat{\mathsf{C}}:=\mathsf{CF}-\operatorname{diag}\mathsf{F}c$, $b=(b_1,\dots,b_m)^{\mathsf{T}}$ and $c=(c_1,\dots,c_m)^{\mathsf{T}}$ are arbitrary constant tuples with $b\in\operatorname{im}\mathsf{F}$.

4.9.3 Subalgebra family 2.4

Consider the two-dimensional subalgebra $\mathfrak{s}^{ab\sigma}_{2.4}:=\langle\mathcal{P}^t+a\mathcal{P}^y+b_k\mathcal{J}^k,\ \mathcal{P}^x(\mathrm{e}^{\sigma t})+\mathcal{Z}(a\sigma t\mathrm{e}^{\sigma t})\rangle$ of $\mathfrak{g}$, where $b:=(b_1,\dots,b_m)^{\mathsf{T}}$, a and σ are arbitrary constants with $b\in\operatorname{im}\mathsf{F}$ and $\sigma\neq0$. With this subalgebra, we construct the ansatz $\psi^i=\varphi^i(\omega)+b_it-\sigma(y-at)x$ or, in the vector form,

$$\psi=\varphi(\omega)+tb-\sigma(y-at)x\bar1,\quad\text{where}\quad\omega:=y-at,$$

and reduce the original system (4.1) to

$$\begin{aligned}&(\sigma\omega+a)\varphi^i_{\omega\omega\omega}+f_{i,i-1}((\sigma\omega+a)\varphi^{i-1}_\omega-(\sigma\omega+a)\varphi^i_\omega-b_{i-1}+b_i)\\&\quad-f_{i,i+1}((\sigma\omega+a)\varphi^i_\omega-(\sigma\omega+a)\varphi^{i+1}_\omega-b_i+b_{i+1})+\beta\sigma\omega=0,\end{aligned}$$

which can also be conveniently represented in the matrix form

$$\varphi_{\omega\omega\omega}+\mathsf{F}\varphi_\omega=(\sigma\omega+a)^{-1}(-\sigma\beta\omega\bar{1}+\mathsf{F}b). \tag{4.88}$$

This is an inhomogeneous linear system of constant-coefficient third-order differential equations, which can be completely decoupled. More specifically, denote

$$\breve{\varphi}(\omega):=\beta\left(-\frac{\omega^3}{6}+\frac{1}{2}\frac{a}{\sigma}\left(\omega+\frac{a}{\sigma}\right)^2\left(\ln\left(\omega+\frac{a}{\sigma}\right)-\frac{3}{2}\right)\right)\bar{1}. \tag{4.89}$$

By the change of the dependent variables $\tilde{\varphi}=\mathsf{P}^{-1}(\varphi-\breve{\varphi})$, see the end of Section 4.2.3 for the notation, we reduce the system (4.88) to the decoupled system of linear inhomogeneous constant-coefficient third-order differential equations

$$\tilde{\varphi}_{\omega\omega\omega}+\Lambda\tilde{\varphi}_\omega=(\sigma\omega+a)^{-1}\Lambda\tilde{b},$$

where $\tilde{b}:=\mathsf{P}^{-1}b$. Recall that $\Lambda:=\mathsf{P}^{-1}\mathsf{F}\mathsf{P}=\operatorname{diag}(\lambda_1,\dots,\lambda_m)$, λ_1, …, λ_m are the eigenvalues of F with $\lambda_1<\dots<\lambda_m$, $\lambda_m=0$ and thus all other λ_i are negative. The general solution of the decoupled system is given by

- $$\begin{aligned}\tilde{\varphi}^i(\omega)&=\frac{\lambda_i\tilde{b}_i}{2}\mathrm{e}^{-\kappa_i\omega}\int\frac{\mathrm{e}^{\kappa_i\omega}}{\sigma\omega+a}\mathrm{d}\omega-\frac{\lambda_i\tilde{b}_i}{2}\mathrm{e}^{\kappa_i\omega}\int\frac{\mathrm{e}^{-\kappa_i\omega}}{\sigma\omega+a}\mathrm{d}\omega+\frac{\lambda_i\tilde{b}_i}{\sigma}\ln|\sigma\omega+a|\\&\quad+A_i\mathrm{e}^{-\kappa_i\omega}+B_i\mathrm{e}^{\kappa_i\omega}+C_i,\quad\kappa_i:=\sqrt{-\lambda_i},\ i=1,\dots,m-1,\end{aligned} \tag{4.90}$$

  $$\tilde{\varphi}^m(\omega)=A_m\omega^2+B_m\omega+C_m. \tag{4.91}$$

The corresponding solutions of the original system (4.1) are of the form

- $$\psi=\sum_{i=1}^{m-1}\tilde{\varphi}^i(y-at)e_i+\tilde{\varphi}^m(y-at)\bar{1}+\breve{\varphi}(y-at)+tb-\sigma(y-at)x\bar{1},$$

where e_1, …, e_{m-1}, $e_m:=\bar{1}$ are eigenvectors of the matrix F for its eigenvalues λ_1, …, λ_{m-1}, $\lambda_m=0$, which are the columns of the matrix P, the functions $\breve{\varphi}$, $\tilde{\varphi}^i$,

$i=1,\dots,m-1$, and $\tilde\varphi^m$ are defined in (4.89), (4.90) and (4.91), respectively, $b=(b_1,\dots,b_m)$, σ and a are arbitrary constants with $\sigma\neq0$, and $\tilde b:=\mathsf{P}^{-1}b$.

4.9.4 Subalgebra family 2.6

An ansatz constructed with the subalgebra $\mathfrak{s}_{2.6}^{\chi bc\kappa}:=\langle\mathcal{P}^y+\mathcal{P}^x(\chi)+b_k\mathcal{J}^k,\ \mathcal{P}^x(1)+c_k\mathcal{J}^k+\mathcal{Z}(\kappa)\rangle$ with arbitrary constant tuples $b:=(b_1,\dots,b_m)^{\mathsf T}$ and $c:=(c_1,\dots,c_m)^{\mathsf T}$ from $\operatorname{im}\mathsf{F}$ and arbitrary smooth functions χ and κ of t, is $\psi^i=\varphi^i(\omega)+yb_i+(x-\chi y)c_i+\kappa(x-\chi y)-\frac12\chi_t y^2$ or, in the vector notation,

$$\psi=\varphi(\omega)+yb+(x-\chi y)c+\big(\kappa(x-\chi y)-\tfrac12\chi_t y^2\big)\bar1,\quad\text{where}\quad\omega:=t.$$

The corresponding reduced system is

$$\begin{aligned}&f_{i,i-1}(\varphi^{i-1}_\omega-\varphi^i_\omega+\kappa(b_{i-1}-b_i)+b_{i-1}c_i-b_ic_{i+1})\\&-f_{i,i+1}(\varphi^i_\omega-\varphi^{i+1}_\omega+\kappa(b_i-b_{i+1})+b_ic_{i+1}-b_{i+1}c_i)+\beta\kappa+\beta c_i-\chi_{tt}=0.\end{aligned}\tag{4.92}$$

It can also be more conveniently represented in the matrix form

$$\mathsf{F}\varphi_\omega+(\mathsf{C}+\kappa\mathsf{E})\mathsf{F}b-\mathsf{B}\mathsf{F}c+\beta c+(\beta\kappa-\chi_{\omega\omega})\bar1=0,$$

where $\mathsf{B}:=\operatorname{diag}b$ and $\mathsf{C}:=\operatorname{diag}c$. This is a system of linear first-order ordinary differential equations with the degenerate coefficient matrix F of the first derivative tuple φ_ω. Taking the W-weighted inner product of the system with $\bar1$, we derive $(\beta\kappa-\chi_{\omega\omega})(\bar1,\bar1)_{\mathsf W}=0$ since

$$\operatorname{im}\mathsf{F}\perp_{\mathsf W}\langle\bar1\rangle\quad\text{and}\quad(\bar1,\mathsf{C}\mathsf{F}b-\mathsf{B}\mathsf{F}c+\beta c)_{\mathsf W}=0.$$

Hence the reduced system is consistent if and only if $\beta\kappa-\chi_{\omega\omega}=0$, i.e., $\kappa=\beta^{-1}\chi_{\omega\omega}$, and its general solution is

$$\varphi=-\beta^{-1}\chi_\omega b-\omega\mathsf{F}^+(\mathsf{C}\mathsf{F}b-\mathsf{B}\mathsf{F}c+\beta c)+\varsigma+g\bar1,$$

where ς is an arbitrary constant tuple from $\operatorname{im}\mathsf{F}$, g is an arbitrary smooth function of ω and the Moore–Penrose inverse F^+ of F is given by (4.3). Pulling this solution

back to a solution of the original system (4.1) by the above ansatz, we obtain

$$\bullet\quad \psi = (x-\chi y)c + (y-\beta^{-1}\chi_t)b - t\mathsf{F}^+(\mathsf{C}\mathsf{F}b - \mathsf{B}\mathsf{F}c + \beta c) + \varsigma \\ \qquad + \left(\frac{\chi_{tt}}{\beta}(x-\chi y) - \frac{\chi_t}{2}y^2 + g\right)\bar{1}, \tag{4.93}$$

where b, c and ς are arbitrary constant tuples from $\operatorname{im}\mathsf{F}$, $\mathsf{B} := \operatorname{diag} b$ and $\mathsf{C} := \operatorname{diag} c$, χ and g are arbitrary smooth functions of t and the Moore–Penrose inverse F^+ of F is given by (4.3). The presence of the arbitrary function g and the arbitrary constant tuple ς from $\operatorname{im}\mathsf{F}$ is explained by the invariance of the system (4.1) with respect to the point transformations $\mathcal{Z}(g)$ and $\mathcal{J}^i(\varsigma_i)$, and thus these function and constant tuple can be set to zero up to the G-equivalence.

Remark 4.33. A solution of the multi-layer problem (4.1) gives rise to the velocity fields that depend only on t in all layers if and only if it is of the form

$$\psi = (x-\zeta^2)c + (y+\zeta^1)b + (\zeta^1_t x + \zeta^2_t y + g)\bar{1} - t\mathsf{F}^+(\mathsf{C}\mathsf{F}b - \mathsf{B}\mathsf{F}c + \beta c), \tag{4.94}$$

where the constant tuples b and c are arbitrary elements of $\operatorname{im}\mathsf{F}$, $\mathsf{B} := \operatorname{diag} b$ and $\mathsf{C} := \operatorname{diag} c$, the Moore–Penrose inverse F^+ of F is given by (4.3), and ζ^1, ζ^2 and g are arbitrary functions of t. The functions ζ^2 and g can be set to be equal to zero up to the G-equivalence, i.e., more specifically, using the pushforwards by the transformations $\mathcal{P}^x(\zeta^2)$ and $\mathcal{Z}(-g)$. The vector fields associated with the baroclinic components of each solution of the form (4.94) are constant in all layers. The intersection of the families (4.93) and (4.94) is singled out from them by the constraints $\chi_t = 0$ and $\zeta^1_t = \zeta^2_t = 0$, respectively, and it consists of the solutions of the multi-layer problem (4.1) with constant velocity fields in all layers.

4.10 Conclusion

In the present paper, we have systematically carried out symmetry analysis of the multi-layer quasi-geostrophic problem (4.1), significantly extending and generalizing results of [15, 94]. In the course of this study, we have successfully addressed two primary challenges. Firstly, the system (4.1) consists of an arbitrary number $m \in \mathbb{N}$ of equations, which makes it impossible to apply specialized computer algebra packages to find the maximal Lie invariance algebra $\mathfrak{g}$ of this system. Secondly, the

system (4.1) is nontrivially coupled, and this coupling is described by the tridiagonal vertical coupling matrix F. This in turn complicates studying Lie reductions and constructing closed-form exact solutions of (4.1). Below, we summarize the results of this paper as well as our approaches to resolving the above challenges.

The multi-layer quasi-geostrophic model can be represented in the componentwise form (4.1) and the vector form (4.2). Each representation has its own application scope: the componentwise form is more convenient for computing the algebra $\mathfrak{g}$ and the point-symmetry pseudogroup G of this model, while the vector form facilitates the study of conservation laws, Hamiltonian structure and Lie reductions of (4.1). Spectral properties of the coupling matrix F play an important role in the course of this study. It turns out that the matrix F is diagonalizable and is of rank $m-1$, its nonzero eigenvalues are negative real and the eigenbasis of F allows us to distinguish barotropic and baroclinic modes.

In Section 4.3 we have extended the well-known results on local conservation laws and a Hamiltonian structure of the β-plane vorticity equation [135] to the general case of the system (4.1), which results in the first correct presentation of families of its local conservation laws and its Hamiltonian structure, including a Hamiltonian, a Hamiltonian operator and the Casimir (distinguished) functionals for this operator. All the presented conserved functionals have a natural physical or mathematical interpretation. Details on the construction of the above object will be presented in another paper jointly with the proof that the listed conservation-law characteristics span the entire space of conservation-law characteristics of (4.1) up to order two. Note that this construction is essentially based on using the weighted inner product with the weight matrix W related to the vertical coupling matrix F; in this inner product, the matrix F is symmetric. We also conjecture that the system (4.1) admits no other independent local conservation laws.

It was evident that deriving the determining equations for the Lie-symmetry vector fields of the system (4.1) via the direct application of the Lie invariance criteria would involve tedious and cumbersome computations. This is why we have used two tricks to overcome this challenge. The first trick consists in the formal replacement of the spatial independent variables (x,y) by the complex conjugate variables $z=x+\mathrm{i}y$ and $\bar z=x-\mathrm{i}y$, which maps the Laplacian $\psi^i_{xx}+\psi^i_{yy}$ to $4\psi^i_{z\bar z}$ and the original system (4.1) to the system (4.6). It has allowed us to significantly reduce the size of expressions in the

course of the computation and to canonically distinguish the jet variables $\psi^i_{tz\bar z}$ as the leading ones in the latter system. The second trick is to introduce the superclasses $\mathcal{V}$ and $\hat{\mathcal{V}}$ of systems of the form (4.7) and (4.8), respectively, $\mathcal{V}\supset\hat{\mathcal{V}}\supset\mathcal{M}^{\rm c}$, where $\mathcal{M}^{\rm c}$ is the class of systems of the form (4.6). Applying the Lie invariance criterion to systems from the widest class $\mathcal{V}$, we have derived the general form of Lie-symmetry vector fields for systems from this class in Lemma 4.6. We have successively specified this form for the systems from the classes $\hat{\mathcal{V}}$ and $\mathcal{M}^{\rm c}$, which has led to Lemmas 4.7 and 4.8, respectively. Note that the maximal Lie invariance algebra of any system from the class $\mathcal{M}^{\rm c}$ coincides with the algebra $\mathfrak{g}^{\rm c}$ obtained in the latter lemma. Finally, by pulling back the algebra $\mathfrak{g}^{\rm c}$ with respect to the transformation $(x,y)\mapsto(z,\bar z)$, we have found the Lie invariance algebra $\mathfrak{g}$ of the multi-layer quasi-geostrophic problem (4.1) in Theorem 4.9. In addition, we have singled out the subalgebra of Hamiltonian Lie symmetries in $\mathfrak{g}$.

After studying the structure of the Lie invariance algebra $\mathfrak{g}$ and constructing its megaideals, we have applied the megaideal-based version of the algebraic method to construct the complete point-symmetry pseudogroup G of (4.1) in Theorem 4.13 and singled out its canonical discrete elements. Based on this result, we have computed the equivalence groupoid, the usual equivalence group and the equivalence algebra of the class $\mathcal{M}$ of systems of the form (4.1) and carried out its group classification in Theorems 4.24 and 4.16, Corollary 4.21 and Theorem 4.22, respectively. The performed analysis has revealed that the kernel Lie invariance algebra of the class $\mathcal{M}$ coincides with the algebra $\mathfrak{g}$ and that there is no Lie symmetry extension within this class. We have also described the generalized equivalence group of $\mathcal{M}$ and its effective counterpart in Remark 4.20.

In Theorems 4.23 and 4.24, we have shown that there are three and eight families of G-inequivalent one- and two-dimensional subalgebras of the algebra $\mathfrak{g}$, respectively. While each family of one-dimensional subalgebras is appropriate for Lie reduction, only four out of eight two-dimensional subalgebra families have this property. The rest of the paper has been devoted to the systematic and comprehensive study of G-inequivalent Lie reductions of the multi-layer quasi-geostrophic problem (4.1). As mentioned above, for the purpose of studying Lie reductions, the vector form (4.2) is more convenient than the original system (4.1).

In Section 4.8.1, using the first subalgebra family from Theorem 4.23, we have

reduced the system (4.2) to the nonlinear systems of the form (4.27). Their maximal Lie invariance algebras coincide with each other for all admitted values of the involved parameters and are entirely induced by the Lie invariance algebra $\mathfrak{g}$ of the original system (4.1), see Theorem 4.26. As a particular case of the corresponding invariant solutions, which is especially convenient for the study and the richest in the sense of possible physical interpretation, one naturally singles out the stationary solutions such that on each, ith, layer, the potential vorticity q^i affinely depends on the stream function ψ^i. Fixing this affine dependence leads to a coupled linear system of the form (4.29). Finding a particular solution of this system and diagonalizing its coupling matrix $\mathsf{F}-\mathsf{B}$, where the matrix B is diagonal, we reduce the system (4.29) to the decoupled homogeneous system (4.30) of the modified Helmholtz, Laplace or Helmholtz equations. Among the solutions of the original system (4.1) that have been constructed in the above way, we single out those with velocity fields that are defined on the entire (x,y)-plane and bounded and write down their representation (4.33) in terms of generalized Herglotz wave functions. This representation has allowed us to recover baroclinic Rossby waves, coherent baroclinic eddies and hetons as the simplest typical flow patterns. We emphasize that the exact solutions of (4.1) obtained under the same fixed affine dependence can be linearly superposed, thus generating even larger solution families with more complex behavior, see, e.g., Figure 4.6. Merging solutions associated with different systems of the form (4.29) on subdomains by imposing interface boundary conditions between these subdomains, we have also extended the construction of dipolar vortices (modons) from [81] to the multi-layer case. The above physically relevant solutions have been illustrated using real-world geophysical data given in Section 4.2.5 for a three-layered ocean model.

The codimension-one Lie reductions of (4.2) with respect to subalgebras from the second family presented in Theorem 4.23 have led to the homogeneous linear systems of partial differential equations of the form (4.44); see Section 4.8.2. We have carried out the group classification of these reduced systems and described associated hidden symmetries of the original (nonlinear) system (4.2). It turns out that besides the most valuable transformations of linear superposition of solutions, there are other nontrivial symmetries of this kind. On top of that, the group classification has also allowed us to distinguish two important cases for values of parameters appearing in (4.44): the completely decoupled case (4.48) and the generically coupled case. These two cases are further split into the general, scale- and shift-invariant subcases according to the

presence and the kind of submodel Lie symmetries involving t. In the completely decoupled case, we were able to find wide families of solutions in terms of elementary functions, integrals of Whittaker functions, Bessel functions and the general solutions of Klein–Gordon and linearized Benjamin–Bona–Mahony equations. The integration of the generically coupled case is more involved, but we have constructed wide families of exact solutions (4.63) for the shift-invariant subcase.

The analysis of Lie reductions with respect to the subalgebras from the third family given in Theorem 4.23 follows the pattern of those for the second one since the structure of the obtained reduced systems (4.65) is similar to the structure of the systems (4.44). The difference worth mentioning has arisen only for the completely decoupled systems of the form (4.65). By integrating with respect to the first invariant independent variable z_1, each of them has been reduced to a system of ordinary differential equations with z_2 as its independent variable, whereas z_1 plays the role of a parameter, and has thus been completely integrated in quadratures.

The main benefit of Lie reductions with respect to one-dimensional subalgebras of $\mathfrak{g}$ is that most of the associated reduced systems of partial differential equations in two independent variables are linear or at least reduce to linear systems under certain differential constraints. In other words, these systems admit the linear superposition of solutions in total or for large subsets of their solutions. However, integrating them may be a challenging problem. Codimension-two Lie reductions of (4.2) lead to systems of ordinary differential equations of order at most three with constant coefficients, whose integration reduces to solving certain eigenvalue problems. We have shown that each codimension-two Lie reduction of (4.2) is a two-step reduction with a codimension-one Lie reduction as the first step. At the same time, three out of four families of codimension-two Lie reductions are more convenient for deriving exact solutions of (4.2) than their two-step counterparts from Section 4.8.

We would like to emphasize that the presented families of exact solutions can be further extended by acting with point symmetries of the original model (4.2), with point symmetries of related submodels and, in case of linear submodels, with their linear recursion operators or, more specifically, with Lie-symmetry operators, see, e.g., [72, 75, 120].

The results of the present paper open several avenues for further research. In particular, while we have constructed families of conservation laws and a Hamiltonian

structure of the multi-layer quasi-geostrophic problem (4.1) in Section 4.3, their comprehensive description with all required proofs remains an open problem. A related direction for the study is to extend results of [94] on Lax pairs of the two-layer model to an arbitrary number of layers. One can also consider generalized and potential symmetries and recursion operators of (4.1). It is also of interest to apply non-Lie methods of finding exact solutions to (4.1), including conditional symmetries [41, Chapter 5], the method of differential constraints [104] (or "side conditions" [106]) and the ansatz method [40].

Appendix 4.A Eigenvectors of the matrix of vertical coupling via orthogonal polynomials

While Section 4.2.3 establishes a number of spectral properties of the vertical coupling matrix F for the multi-layer quasi-geostrophic problem (4.1), several aspects remain unresolved. For example, although we know that the matrix F is diagonalizable with real, non-positive and distinct eigenvalues, explicitly determining its spectrum and the transition matrix R to its eigenbasis remains a challenging problem. In the generic case with $m > 5$, finding exact algebraic expressions is impossible due to the Abel–Ruffini theorem.

Nevertheless, the theory of orthogonal polynomials allows us to express the components of eigenvectors of F as specific orthogonal polynomials evaluated at the eigenvalues. Furthermore, when the parameters $f_{i,i+1}$ and $f_{i,i-1}$ are at most quadratic polynomials with respect to i or q^i, where $0 < q < 1$, we obtain the orthogonal polynomials from the Askey scheme and their q-analogues, respectively, [66, Chapters 9, 14].

The matrix F is famous in the theory of continuous-time Markov processes as the rate matrix of a birth-death process, see, e.g., [61]. Consider the spectral problem for F written in the form

$$\mathsf{F}v^s = \Lambda(s)v^s,$$

where $v^s \in \mathbb{R}^m$, $s \in \{1, \ldots, m\}$. This spectral problem gives rise to the three-term recurrence relation on the components of the eigenvector $v^s := (v_1^s, v_2^s, \ldots, v_m^s)^\mathsf{T}$:

$$f_{i,i+1}(v_i^s - v_{i+1}^s) + f_{i,i-1}(v_i^s - v_{i-1}^s) = -\Lambda(s)v_i. \tag{4.95}$$

This discrete problem was systematically addressed for birth–death processes in [133] by expressing the sequence $\{v^s\}$ as orthogonal polynomials in a discrete variable (i or q^i) with respect to the orthogonality measure $\phi_0(i)^2$ [133, eq. (2.7)]. This approach yields 18 exactly solvable systems of the form (4.95) within the discrete (q-)Askey scheme: 10 for finite m and 8 for $m = \infty$.

Given the finite stratification in (4.1), we focus exclusively on the 10 cases with finite m. Below, we summarize and adapt the relevant results from [133, Section IV.A]. More specifically, we provide explicit expressions for the parameters $f_{i,i+1}$ and $f_{i,i-1}$ as functions of i or q^i, the eigenvalues $\Lambda(s)$ and the components of the eigenvectors v^s, while omitting the explicit details on the discrete measure $\phi_0(i)^2$, which can be found in the original source.

To make the presentation self-contained, we recall the definitions of (q-)Pochhammer symbols and (q-)hypergeometric functions.

- Pochhammer symbol:

$$(a)_n := \prod_{k=1}^{n}(a + k - 1) = a(a + 1)\cdots(a + n - 1) = \frac{\Gamma(a + n)}{\Gamma(a)}.$$

- q-Pochhammer symbol:

$$(a; q)_n := \prod_{k=1}^{n}(1 - aq^{k-1}) = (1 - a)(1 - aq)\cdots(1 - aq^{n-1}).$$

- Hypergeometric series:

$${}_rF_s(a_1, \ldots, a_r; b_1, \ldots, b_s; z) := \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r)_n}{(b_1, \ldots, b_s)_n} \frac{z^n}{n!},$$

  where $(a_1, \ldots, a_r)_n := \prod_{j=1}^{r}(a_j)_n$.

- q-hypergeometric series:

$$ {}_r\phi_s(a_1,\dots,a_r;b_1,\dots,b_s;q,z) := \sum_{n=0}^{\infty} \frac{(a_1,\dots,a_r;q)_n}{(b_1,\dots,b_s;q)_n} \Big((-1)^n q^{\binom{n}{2}}\Big)^{1+s-r} \frac{z^n}{(q;q)_n}, $$

where $(a_1,\dots,a_r;q)_n := \prod_{j=1}^{r}(a_j;q)_n$.

Racah. The Racah polynomial is the most generic hypergeometric orthogonal polynomial of a discrete variable. The functions $f_{i,i+1}$ and $f_{i,i-1}$ depend on four real parameters a, b, c and d,

$$ f_{i,i+1} = -\frac{(i+a)(i+b)(i+c)(i+d)}{(2i+d)(2i+1+d)}, $$
$$ f_{i,i-1} = -\frac{(i+d-a)(i+d-b)(i+d-c)i}{(2i-1+d)(2i+d)}. $$

The parameters satisfy the following constraints:

$$ a \geqslant b, \quad d > 0, \quad a > m+d, \quad 0 < b < 1+d, \quad c := -m. $$

This leads to the spectrum

$$ \Lambda(s) = -s(s+\tilde{d}), \quad \tilde{d} := a+b+c-d-1, $$
$$ v_i^s = {}_4F_3(-s, s+\tilde{d}, -i, i+d; a, b, c; 1). $$

The dual polynomial is again a Racah polynomial under the parameter correspondence $(a,b,c,d) \to (a,b,c,\tilde{d})$.

Hahn. The functions $f_{i,i+1}$ and $f_{i,i-1}$ are quadratic in i with two real parameters a and b:

$$ f_{i,i+1} = (i+a)(m-i), \quad f_{i,i-1} = i(b+m-i). $$

The spectrum is quadratic with respect to s:

$$ \Lambda(s) = -s(s+a+b-1), \quad v_i^s = {}_3F_2(-s, s+a+b-1, -i; a, -m; 1). $$

Dual Hahn. The set of parameters is the same as for Hahn polynomials. The

functions $f_{i,i+1}$ and $f_{i,i-1}$ are rational with respect to i:

$$f_{i,i+1} = \frac{(i+a)(i+a+b-1)(m-i)}{(2i-1+a+b)(2i+a+b)},$$
$$f_{i,i-1} = \frac{i(i+b-1)(i+a+b+m-1)}{(2i-2+a+b)(2i-1+a+b)},$$

yielding linear spectrum in s:

$$\Lambda(s) = -s, \quad v_i^s = {}_3F_2(-s, i+a+b-1, -i; a, -m; 1).$$

Krawtchouk (self-dual). The functions $f_{i,i+1}$ and $f_{i,i-1}$ are linear with respect to i,

$$f_{i,i+1} = p(m-i), \quad f_{i,i-1} = (1-p)i, \quad 0 < p < 1$$

giving rise to linear spectrum

$$\Lambda(s) = -s, \quad v_i^s = {}_2F_1(-s, -i; -m; p^{-1}).$$

q-Racah. The q-Racah polynomials are the most generic ones within the q-scheme of orthogonal polynomials. They are parameterized by four real parameters a, b, c and d satisfying

$$c = q^{-m}, \quad a \leqslant b, \quad 0 < d < 1, \quad 0 < a < q^m d,$$
$$qd < b < 1, \quad \tilde{d} < q^{-1}, \quad \tilde{d} := abcd^{-1}q^{-1}.$$

The functions $f_{i,i+1}$ and $f_{i,i-1}$ are rational with respect to q^i:

$$f_{i,i+1} = -\frac{(1-aq^i)(1-bq^i)(1-cq^i)(1-dq^i)}{(1-dq^{2i})(1-dq^{2i+1})},$$
$$f_{i,i-1} = -\tilde{d}\frac{(1-a^{-1}dq^i)(1-b^{-1}dq^i)(1-c^{-1}dq^i)(1-q^i)}{(1-dq^{2i-1})(1-dq^{2i})},$$

which yields the spectrum

$$\Lambda(s) = (1-q^{-s})(1-\tilde{d}q^s), \quad v_i^s = {}_4\phi_3(q^{-s}, \tilde{d}q^s, q^{-i}, dq^i; a, b, c; q, q).$$

q-Hahn. The q-Hahn polynomials depend on the real parameters $a > 0$ and $0 < b < 1$. The functions $f_{i,i+1}$ and $f_{i,i-1}$ are quadratic polynomials in q^i:

$$f_{i,i+1} = (1 - aq^i)(q^{i-m} - 1), \quad f_{i,i-1} = aq^{-1}(1 - q^i)(q^{i-m} - b)$$
$$\Lambda(s) = (1 - q^{-s})(1 - abq^{s-1}), \quad v_i^s = {}_3\phi_2(q^{-s}, abq^{s-1}, q^{-i}; a, q^{-m}; q, q).$$

Dual q-Hahn. The parameters are the same as for q-Hahn polynomials. The functions $f_{i,i+1}$ and $f_{i,i-1}$ are rational in q^i:

$$f_{i,i+1} = \frac{(q^{i-m} - 1)(1 - aq^i)(1 - abq^{i-1})}{(1 - abq^{2i-1})(1 - abq^{2i})},$$
$$f_{i,i-1} = aq^{i-m-1}\frac{(1 - q^i)(1 - abq^{i+m-1})(1 - bq^{i-1})}{(1 - abq^{2i-2})(1 - abq^{2i-1})},$$
$$\Lambda(s) = 1 - q^{-s}, \quad v_i^s = {}_3\phi_2(q^{-s}, abq^{i-1}, q^{-i}; a, q^{-m}; q, q).$$

Quantum q-Krawtchouk. These polynomials depend on one positive parameter $p > q^{-m}$. The functions $f_{i,i+1}$ and $f_{i,i-1}$ are quadratic polynomials in q^i:

$$f_{i,i+1} = p^{-1}q^i(q^{i-m} - 1), \quad f_{i,i-1} = (1 - q^i)(1 - p^{-1}q^{i-m-1}).$$
$$\Lambda(s) = q^{-s} - 1, \quad v_i^s = {}_2\phi_1(q^{-s}, q^{-i}; q^{-m}; q, pq^{s+1}).$$

q-Krawtchouk. This depends on one positive parameter $p > 0$ and the functions $f_{i,i+1}$ and $f_{i,i-1}$ are linear with respect to q^i:

$$f_{i,i+1} = q^{i-m} - 1, \quad f_{i,i-1} = p(1 - q^i),$$
$$\Lambda(s) = (1 - q^{-s})(1 + pq^s), \quad v_i^s = {}_3\phi_2(q^{-s}, q^{-i}, -pq^s; q^{-m}, 0; q, q).$$

Affine q-Krawtchouk. These polynomials have one positive parameter p and the functions $f_{i,i+1}$ and $f_{i,i-1}$ are quadratic with respect to q^i:

$$f_{i,i+1} = (q^{i-m} - 1)(1 - pq^{i+1}), \quad f_{i,i-1} = pq^{i-m}(1 - q^i), \quad 0 < p < q^{-1},$$
$$\Lambda(s) = 1 - q^{-s}, \quad v_i^s = {}_3\phi_2(q^{-s}, q^{-i}, 0; pq, q^{-m}; q, q).$$

Bibliography

[1] I. S. Akhatov, R. K. Gazizov, and N. K. Ibragimov. Nonlocal symmetries. A heuristic approach. In *Current problems in mathematics. Newest results, Vol. 34*, Itogi Nauki i Tekhniki, pages 3–83, 195. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1989.

[2] W. F. Ames, R. L. Anderson, V. A. Dorodnitsyn, E. V. Ferapontov, R. K. Gazizov, N. H. Ibragimov, and S. R. Svirshchevskiĭ. *CRC handbook of Lie group analysis of differential equations. Vol. 1.* CRC Press, Boca Raton, FL, 1994. Symmetries, exact solutions and conservation laws.

[3] S. C. Anco and G. Bluman. Nonlocal symmetries and nonlocal conservation laws of Maxwell's equations. *J. Math. Phys.*, 38:3508–3532, 1997.

[4] S. C. Anco and G. Bluman. Direct construction method for conservation laws of partial differential equations. I. Examples of conservation law classifications. *European J. Appl. Math.*, 13:545–566, 2002.

[5] S. C. Anco and G. Bluman. Direct construction method for conservation laws of partial differential equations. II. General treatment. *European J. Appl. Math.*, 13:567–585, 2002.

[6] S. C. Anco and J. Pohjanpelto. Classification of local conservation laws of Maxwell's equations. *Acta Appl. Math.*, 69(3):285–327, 2001.

[7] S. C. Anco and J. Pohjanpelto. Symmetries and currents of massless neutrino fields, electromagnetic and graviton fields. In *Symmetry in physics*, volume 34 of *CRM Proc. Lecture Notes*, pages 1–12. Amer. Math. Soc., Providence, RI, 2004.

[8] V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov. *Applications of group-theoretical methods in hydrodynamics*, volume 450 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 1998.

[9] H. Baran and M. Marvan. Jets. a software for differential calculus on jet spaces and diffieties. `http://jets.math.slu.cz`.

[10] E. Barucci, S. Polidoro, and V. Vespri. Some results on partial differential equations and Asian options. *Math. Models Methods Appl. Sci.*, 11:475–497, 2001.

[11] G. M. Bergman. The diamond lemma for ring theory. *Adv. in Math.*, 29:178–218, 1978.

[12] A. Bihlo, E. Dos Santos Cardoso-Bihlo, and R. O. Popovych. Complete group classification of a class of nonlinear wave equations. *J. Math. Phys.*, 53:123515, 32, 2012.

[13] A. Bihlo, E. Dos Santos Cardoso-Bihlo, and R. O. Popovych. Algebraic method for finding equivalence groups. *J. Phys. Conf. Ser.*, 621:012001, 2015.

[14] A. Bihlo, N. Poltavets, and R. O. Popovych. Lie symmetries of two-dimensional shallow water equations with variable bottom topography. *Chaos*, 30:073132, 17, 2020.

[15] A. Bihlo and R. O. Popovych. Lie symmetry analysis and exact solutions of the quasigeostrophic two-layer problem. *J. Math. Phys.*, 52:033103, 24, 2011.

[16] A. Bihlo and R. O. Popovych. Point symmetry group of the barotropic vorticity equation. In *Group analysis of differential equations and integrable systems*, pages 15–27. Department of Mathematics and Statistics, University of Cyprus, Nicosia, 2011.

[17] A. Bihlo and R. O. Popovych. Group classification of linear evolution equations. *J. Math. Anal. Appl.*, 448:982–1005, 2017.

[18] A. Bihlo and R. O. Popovych. Zeroth-order conservation laws of two-dimensional shallow water equations with variable bottom topography. *Stud. Appl. Math.*, 145:291–321, 2020.

[19] G. W. Bluman and S. C. Anco. *Symmetry and integration methods for differential equations*, volume 154 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2002.

[20] G. W. Bluman, A. F. Cheviakov, and S. C. Anco. *Applications of symmetry methods to partial differential equations*, volume 168 of *Applied Mathematical Sciences*. Springer, New York, 2010.

[21] G. W. Bluman and S. Kumei. *Symmetries and differential equations*, volume 81 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1989.

[22] A. V. Bocharov, V. N. Chetverikov, S. V. Duzhin, N. G. Khor'kova, I. S. Krasil'shchik, A. V. Samokhin, Y. N. Torkhov, A. M. Verbovetsky, and A. M. Vinogradov. *Symmetries and conservation laws for differential equations of mathematical physics*. American Mathematical Society, Providence, 1999.

[23] A. Borel. *Essays in the History of Lie Groups and Algebraic Groups*, volume 21 of *History of Mathematics*. American Mathematical Society, Providence, RI, 2001.

[24] C. P. Boyer, E. G. Kalnins, and W. Miller, Jr. Symmetry and separation of variables for the Helmholtz and Laplace equations. *Nagoya Math. J.*, 60:35–80, 1976.

[25] V. M. Boyko, M. Kunzinger, and R. O. Popovych. Singular reduction modules of differential equations. *J. Math. Phys.*, 57:101503, 2016.

[26] V. M. Boyko, R. O. Popovych, and O. O. Vinnichenko. Point- and contact-symmetry pseudogroups of dispersionless Nizhnik equation. *Commun. Nonlinear Sci. Numer. Simul.*, 132:07915, 2024.

[27] J. Carminati and K. Vu. Symbolic computation and differential equations: Lie symmetries. *J. Symbolic Comput.*, 29:95–116, 2000.

[28] Y. Y. Chapovskyi, S. D. Koval, and O. Zhur. Subalgebras of Lie algebras. Example of $\mathfrak{sl}_3(\mathbb{R})$ revisited. *J. Algebra Appl.*, pages Paper No. 2850012, 43, 2026.

[29] A. F. Cheviakov. GeM software package for computation of symmetries and conservation laws of differential equations. *Comput. Phys. Comm.*, 176:48–61, 2007.

[30] A. F. Cheviakov. Conservation properties and potential systems of vorticity-type equations. *J. Math. Phys.*, 55(3):033508, 16, 2014.

[31] T. Christodoulakis, G. O. Papadopoulos, and A. Dimakis. Automorphisms of real four-dimensional Lie algebras and the invariant characterization of homogeneous 4-spaces. *J. Phys. A*, 36:427–441, 2003.

[32] D. Colton and R. Kress. *Inverse acoustic and electromagnetic scattering theory*, volume 93 of *Applied Mathematical Sciences*. Springer, Cham, 2019.

[33] C. Cotter, D. Crisan, D. Holm, W. Pan, and I. Shevchenko. Data assimilation for a quasi-geostrophic model with circulation-preserving stochastic transport noise. *J. Stat. Phys.*, 179:1186–1221, 2020.

[34] M. N. Crowe and E. R. Johnson. Modon solutions in an N-layer quasi-geostrophic model. *J. Fluid Mech.*, 994:R1, 2024.

[35] V. V. Davydovych. Preliminary group classification of (2+1)-dimensional linear ultraparabolic Kolmogorov-Fokker-Planck equations. *Ufa Math. J.*, 7:38–46, 2015.

[36] E. Dos Santos Cardoso-Bihlo, A. Bihlo, and R. O. Popovych. Enhanced preliminary group classification of a class of generalized diffusion equations. *Commun. Nonlinear Sci. Numer. Simul.*, 16:3622–3638, 2011.

[37] E. Dos Santos Cardoso-Bihlo and R. O. Popovych. Complete point symmetry group of the barotropic vorticity equation on a rotating sphere. *J. Engrg. Math.*, 82:31–38, 2013.

[38] E. Dos Santos Cardoso-Bihlo and R. O. Popovych. On the ineffectiveness of constant rotation in the primitive equations and their symmetry analysis. *Commun. Nonlinear Sci. Numer. Simul.*, 101:Paper No. 105885, 15, 2021.

[39] G. R. Flierl, V. D. Larichev, J. C. McWilliams, and G. M. Reznik. The dynamics of baroclinic and barotropic solitary eddies. *Dyn. Atmos. Oceans*, 5:1–41, 1980.

[40] W. Fushchych. Ansatz'95. *J. Nonlinear Math. Phys.*, 2:216–235, 1995.

[41] W. I. Fushchych, V. M. Shtelen, and M. I. Serov. *Symmetry analysis and exact solutions of equations of nonlinear mathematical physics*. Kluwer Academic Publishers, Dordrecht, 1993.

[42] C. W. Gardiner. *Handbook of stochastic methods*, volume 13 of *Springer Series in Synergetics*. Springer-Verlag, Berlin, second edition, 1985. For physics, chemistry and the natural sciences.

[43] A. González-López, N. Kamran, and P. J. Olver. Lie algebras of vector fields in the real plane. *Proc. London Math. Soc. (3)*, 64:339–368, 1992.

[44] F. Güngör. Equivalence and symmetries for variable coefficient linear heat type equations. I. *J. Math. Phys.*, 59:051507, 2018.

[45] F. Güngör. Equivalence and symmetries for variable coefficient linear heat type equations. II. Fundamental solutions. *J. Math. Phys.*, 59:061507, 2018.

[46] P. Hartman and C. Wilcox. On solutions of the Helmholz equation in exterior domains. *Math. Z.*, 75:228–255, 1960/61.

[47] J. Hilgert and K.-H. Neeb. *Structure and geometry of Lie groups*. Springer Monographs in Mathematics. Springer, New York, 2012.

[48] P. E. Hydon. Discrete point symmetries of ordinary differential equations. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 454:1961–1972, 1998.

[49] P. E. Hydon. How to find discrete contact symmetries. *J. Nonlinear Math. Phys.*, 5:405–416, 1998.

[50] P. E. Hydon. How to construct the discrete symmetries of partial differential equations. *European J. Appl. Math.*, 11:515–527, 2000.

[51] P. E. Hydon. *Symmetry methods for differential equations*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2000. A beginner's guide.

[52] N. H. Ibragimov. *Transformation groups applied to mathematical physics*. Mathematics and its Applications (Soviet Series). D. Reidel Publishing Co., Dordrecht, 1985.

[53] N. H. Ibragimov, A. V. Aksenov, V. A. Baikov, V. A. Chugunov, R. K. Gazizov, and A. G. Meshkov. *CRC Handbook of Lie Group Analysis of Differential Equations. Vol. 2: Applications in Engineering and Physical Sciences*. CRC Press, Boca Raton, FL, 1995.

[54] N. H. Ibragimov and R. N. Ibragimov. *Applications of Lie group analysis in geophysical fluid dynamics*, volume 2 of *Series on Complexity, Nonlinearity and Chaos*. World Scientific Publishing Co. Pte. Ltd., Singapore; Higher Education Press, Beijing, 2011.

[55] N. H. Ibragimov, M. Torrisi, and A. Valenti. Preliminary group classification of equations $v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)$. *J. Math. Phys.*, 32:2988–2995, 1991.

[56] N. M. Ivanova, R. O. Popovych, and C. Sophocleous. Group analysis of variable coefficient diffusion-convection equations. I. Enhanced group classification. *Lobachevskii J. Math.*, 31(2):100–122, 2010.

[57] N. Jacobson. *Lie algebras*. Interscience Tracts in Pure and Applied Mathematics, No. 10. Interscience Publishers (a division of John Wiley & Sons, Inc.), New York-London, 1962.

[58] V. M. Kamenkovich, M. N. Koshlyakov, and A. S. Monin. *Synoptic Eddies in the Ocean*. D. Reidel, Dordrecht, Holland, 1986.

[59] V. M. Kamenkovich, V. D. Larichev, and B. V. Kharkov. A quasigeostrophic baroclinic model for the analysis of synoptic eddies in the open ocean. *Oceanology*, 21:673–679, 1981.

[60] E. Kamke. *Differentialgleichungen*. B.G. Teubner, Stuttgart, 1977.

[61] S. Karlin and J. McGregor. The classification of birth and death processes. *Trans. Amer. Math. Soc.*, 86:366–400, 1957.

[62] O. Kelbin, A. F. Cheviakov, and M. Oberlack. New conservation laws of helically symmetric, plane and rotationally symmetric viscous and inviscid flows. *J. Fluid Mech.*, 721:340–366, 2013.

[63] J. G. Kingston and C. Sophocleous. On form-preserving point transformations of partial differential equations. *J. Phys. A*, 31:1597–1619, 1998.

[64] Z. Kizner and D. Berson. Emergence of modons from collapsing vortex structures on the β-plane. *J. Mar. Res.*, 58:375–403, 2000.

[65] M. V. Kochetov and S. D. Koval. Generic graded contractions of Lie algebras. *J. Algebra*, 696:245–277, 2026.

[66] R. Koekoek, P. A. Lesky, and R. F. Swarttouw. *Hypergeometric orthogonal polynomials and their q-analogues*. Springer, Berlin, 2010.

[67] A. Kolmogoroff. Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung. *Math. Ann.*, 104:415–458, 1931.

[68] A. Kolmogoroff. Zufällige Bewegungen (zur Theorie der Brownschen Bewegung). *Ann. of Math. (2)*, 35:116–117, 1934.

[69] S. D. Koval. Extended symmetry analysis of remarkable (1+2)-dimensional Fokker–Planck equation. Master's thesis, Memorial University of Newfoundland, St. John's, Canada, 2022.

[70] S. D. Koval, A. Bihlo, and R. O. Popovych. Extended symmetry analysis of remarkable (1 + 2)-dimensional Fokker-Planck equation. *European J. Appl. Math.*, 34:1067–1098, 2023.

[71] S. D. Koval, A. Bihlo, and R. O. Popovych. Symmetry analysis and exact solutions of multi-layer quasi-geostrophic problem. *J. Nonlinear Sci.*, 2026. (accepted for publication), arXiv:2603.26959, 65 pp.

[72] S. D. Koval, E. Dos Santos Cardoso-Bihlo, and R. O. Popovych. Surprising symmetry properties and exact solutions of Kolmogorov backward equations with power diffusivity. *Stud. Appl. Math.*, 155(2):e70105, 2025.

[73] S. D. Koval and R. O. Popovych. Linear generalized symmetries of linear systems of differential equations. in preparation.

[74] S. D. Koval and R. O. Popovych. Point and generalized symmetries of the heat equation revisited. *J. Math. Anal. Appl.*, 527:127430, 2023.

[75] S. D. Koval and R. O. Popovych. Extended symmetry analysis of (1+2)-dimensional fine Kolmogorov backward equation. *Stud. Appl. Math.*, 153(1):e12695, 2024.

[76] S. S. Kovalenko, I. M. Kopas, and V. I. Stogniy. Preliminary group classification of a class of generalized linear kolmogorov equations. *Res. Bull. Natl. Tech. Univ. Ukraine "Kyiv Polytechnic Institute"*, 4:67–72, 2013. in Ukrainian.

[77] I. S. Krasil'shchik, V. V. Lychagin, and A. M. Vinogradov. *Geometry of jet spaces and nonlinear partial differential equations*, volume 1 of *Advanced Studies in Contemporary Mathematics*. Gordon and Breach Science Publishers, New York, 1986.

[78] M. Kunzinger. *Lie transformation groups: an introduction to symmetry group analysis of differential equations*. University of Vienna, Vienna, Austria, 2015. Lecture Notes.

[79] C. Kurujyibwami, D. R. Popovych, and R. O. Popovych. Algebraic method of group classification for semi-normalized classes of differential equations. arXiv:2408.16897 [math-ph], 2024.

[80] V. Lahno, R. Zhdanov, and O. Magda. Group classification and exact solutions of nonlinear wave equations. *Acta Appl. Math.*, 91:253–313, 2006.

[81] V. D. Larichev and G. M. Reznik. Strongly nonlinear two-dimensional rossby solitons: An exact solution. *Dokl. Akad. Nauk SSSR*, 231(5):1077–1079, 1976.

[82] M. V. Lawson. *Inverse semigroups*. World Scientific Publishing Co., Inc., River Edge, NJ, 1998. The theory of partial symmetries.

[83] M. V. Lawson. The Möbius inverse monoid. *J. Algebra*, 200:428–438, 1998.

[84] D. Levi, M. C. Nucci, C. Rogers, and P. Winternitz. Group theoretical analysis of a rotating shallow liquid in a rigid container. *J. Phys. A*, 22:4743–4767, 1989.

[85] S. Lie. Diskussion der Differentialgleichung $\frac{d^2z}{dxdy} = f(z)$. *Arch. for Math.*, 6:112–124, 1881. Reprinted in: Lie, S., Gesammelte Abhandlungen, Vol. 3, B.G. Teubner, Leipzig and H. Aschehoug & Co, Kristiania, 469–478.

[86] S. Lie. Über die Integration durch bestimmte Integrale von einer Klasse linear partieller Differentialgleichungen. *Archiv for Mathematik og Naturvidenskab*, 6:328–368, 1881. English translation by N. H. Ibragimov: "On integration of a class of linear partial differential equations by means of definite integrals," in *CRC Handbook of Lie Group Analysis of Differential Equations*, Vol. 2, CRC Press, Boca Raton, FL, 1995, pp. 473–508.

[87] S. Lie. *Theorie der Transformationsgruppen*. B.G. Teubner, Leipzig, 1888–1893. (in collaboration with Engel F., 3 volumes).

[88] E. N. Lorenz. Deterministic nonperiodic flow. *J. Atmospheric Sci.*, 20:130–141, 1963.

[89] D. S. Maltseva and R. O. Popovych. Point-symmetry pseudogroup, Lie reductions and exact solutions of Boiti-Leon-Pempinelli system. *Phys. D*, 460:134081, 2024.

[90] M. Marvan. Sufficient set of integrability conditions of an orthonomic system. *Found. Comput. Math.*, 9:651–674, 2009.

[91] V. B. Matveev and M. A. Salle. *Darboux transformations and solitons*. Springer Series in Nonlinear Dynamics. Springer-Verlag, Berlin, 1991.

[92] S. V. Meleshko. Generalization of the equivalence transformations. *J. Nonlinear Math. Phys.*, 3:170–174, 1996. Symmetry in nonlinear mathematical physics, Vol. 2 (Kiev, 1995).

[93] W. Miller, Jr. *Symmetry and separation of variables*, volume Vol. 4 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1977. With a foreword by Richard Askey.

[94] O. I. Morozov. The quasigeostrophic two-layer model: Lax representations and conservation laws. *J. Geom. Phys.*, 192:104954, 2023.

[95] O. I. Morozov and J.-H. Chang. The dispersionless Veselov-Novikov equation: symmetries, exact solutions, and conservation laws. *Anal. Math. Phys.*, 11:126, 2021.

[96] V. V. Morozov. Classification of nilpotent Lie algebras of sixth order. *Izv. Vyssš. Učebn. Zaved. Matematika*, 1958:161–171, 1958.

[97] G. M. Mubarakzjanov. Classification of real structures of Lie algebras of fifth order. *Izv. Vyssš. Učebn. Zaved. Matematika*, 3:99–106, 1963.

[98] G. M. Mubarakzjanov. Classification of solvable Lie algebras of sixth order with a non-nilpotent basis element. *Izv. Vyssš. Učebn. Zaved. Matematika*, 4:104–116, 1963.

[99] G. M. Mubarakzjanov. On solvable Lie algebras. *Izv. Vyssš. Učebn. Zaved. Matematika*, 1:114–123, 1963.

[100] J. J. Nauw, H. A. Dijkstra, and E. Simonnet. Regimes of low-frequency variability in a three-layer quasi-geostrophic ocean model. *Journal of Marine Research*, 62(5):685–720, 2004.

[101] U. Niederer. The maximal kinematical invariance group of the free Schrödinger equation. *Helv. Phys. Acta*, 45:802–810, 1972/73.

[102] E. Noether. Invariant variation problems. *Transport Theory Statist. Phys.*, 1(3):186–207, 1971.

[103] P. J. Olver. *Applications of Lie groups to differential equations*, volume 107 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1993.

[104] P. J. Olver. Direct reduction and differential constraints. *Proc. Roy. Soc. London Ser. A*, 444:509–523, 1994.

[105] P. J. Olver. *Classical invariant theory*, volume 44 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1999.

[106] P. J. Olver and P. Rosenau. The construction of special solutions to partial differential equations. *Phys. Lett. A*, 114:107–112, 1986.

[107] S. Opanasenko, A. Bihlo, and R. O. Popovych. Group analysis of general Burgers-Korteweg–de Vries equations. *J. Math. Phys.*, 58:081511, 37, 2017.

[108] S. Opanasenko, A. Bihlo, R. O. Popovych, and A. Sergyeyev. Extended symmetry analysis of an isothermal no-slip drift flux model. *Phys. D*, 402:132188, 2020.

[109] S. Opanasenko, V. Boyko, and R. O. Popovych. Enhanced group classification of nonlinear diffusion-reaction equations with gradient-dependent diffusivity. *J. Math. Anal. Appl.*, 484:123739, 30, 2020.

[110] S. Opanasenko and R. O. Popovych. Generalized symmetries and conservation laws of $(1 + 1)$-dimensional Klein-Gordon equation. *J. Math. Phys.*, 61:101515, 13, 2020.

[111] S. Opanasenko and R. O. Popovych. Mapping method of group classification. *J. Math. Anal. Appl.*, 513:126209, 2022.

[112] L. V. Ovsiannikov. *Group Analysis of Differential Equations*. Academic Press, New York, 1892.

[113] L. V. Ovsjannikov. Group relations of the equation of non-linear heat conductivity. *Dokl. Akad. Nauk SSSR*, 125:492–495, 1959.

[114] J. Patera and P. Winternitz. Subalgebras of real three- and four-dimensional Lie algebras. *J. Mathematical Phys.*, 18:1449–1455, 1977.

[115] J. Patera, P. Winternitz, and H. Zassenhaus. Continuous subgroups of the fundamental groups of physics. I. General method and the Poincaré group. *J. Math. Phys.*, 16:1597–1614, 1975.

[116] J. Pedlosky. *Geophysical Fluid Dynamics*. Springer-Verlag, New York, 2nd edition, 1987.

[117] J. Pohjanpelto and S. C. Anco. Generalized symmetries of massless free fields on Minkowski space. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 4:Paper 004, 17, 2008.

[118] A. D. Polyanin. *Handbook of linear partial differential equations for engineers and scientists*. Chapman & Hall/CRC, Boca Raton, FL, 2002.

[119] D. R. Popovych, A. Bihlo, and R. O. Popovych. Generalized symmetries of Burgers equation. arXiv:2406.02809, 2024.

[120] D. R. Popovych, S. D. Koval, and R. O. Popovych. Generalized symmetries of remarkable (1+2)-dimensional Fokker–Planck equation. *European Journal of Applied Mathematics*, 2025. doi:10.1017/S0956792525100107,arXiv:2409.10348.

[121] H. V. Popovych. Lie, partially invariant, and nonclassical submodels of the euler equations. *Proc. Inst. Math. NAS Ukraine*, 43(1):178–183, 2002.

[122] R. O. Popovych. Classification of admissible transformations of differential equations. In *Collection of Works of Institute of Mathematics, Kyiv*, volume 3, pages 239–254. Institute of Mathematics of NAS of Ukraine, 2006.

[123] R. O. Popovych. Reduction operators of linear second-order parabolic equations. *J. Phys. A*, 41:185202, 2008.

[124] R. O. Popovych and A. Bihlo. Symmetry preserving parameterization schemes. *J. Math. Phys.*, 53:073102, 2012.

[125] R. O. Popovych, V. M. Boyko, M. O. Nesterenko, and M. W. Lutfullin. Realizations of real low-dimensional Lie algebras. *J. Phys. A*, 36:7337–7360, 2003.

[126] R. O. Popovych, M. Kunzinger, and H. Eshraghi. Admissible transformations and normalized classes of nonlinear Schrödinger equations. *Acta Appl. Math.*, 109:315–359, 2010.

[127] R. O. Popovych, M. Kunzinger, and N. M. Ivanova. Conservation laws and potential symmetries of linear parabolic equations. *Acta Appl. Math.*, 100:113–185, 2008.

[128] J. N. Reinaud, M. A. Sokolovskiy, and X. Carton. Hetonic quartets in a two-layer quasi-geostrophic flow: V-states and stability. *Phys. Fluids*, 30:056602, 2018.

[129] B. L. Reinhart. *Differential Geometry of Foliations: The Fundamental Integrability Problem*, volume 99 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin, 1893.

[130] H. Risken. *The Fokker–Planck Equation: Methods of Solution and Applications*, volume 18 of *Springer Series in Synergetics*. Springer-Verlag, Berlin, 2nd edition, 1989.

[131] M. Rotenberg. Transport theory for growing cell population. *J. Theor. Biol.*, 103(2):181–199, 1983.

[132] W. E. Roth. The equations $AX - YB = C$ and $AX - XB = C$ in matrices. *Proc. Amer. Math. Soc.*, 3:392–396, 1952.

[133] R. Sasaki. Exactly solvable birth and death processes. *J. Math. Phys.*, 50:103509, 18, 2009.

[134] A. V. Shapovalov and I. V. Shirokov. Noncommutative integration of linear differential equations. *Theoret. and Math. Phys.*, 104:921–934, 1995.

[135] T. G. Shepherd. Symmetries, conservation laws, and hamiltonian structure in geophysical fluid dynamics. *Advances in Geophysics*, 32:287–338, 1990.

[136] W. M. Shtelen. On a method of constructing exact solutions of multidimensional linear differential equations. In *Symmetry and exact solutions of nonlinear equations of mathematical physics*, pages 31–35. Institute of Mathematics, Kyiv, 1987.

[137] W. M. Shtelen and V. I. Stogny. Symmetry properties of one- and two-dimensional Fokker–Planck equations. *J. Phys. A*, 22:L539–L543, 1989.

[138] S. Spichak and V. Stognii. Symmetry classification and exact solutions of the one-dimensional Fokker–Planck equation with arbitrary coefficients of drift and diffusion. *J. Phys. A*, 32:8341–8353, 1999.

[139] S. Spichak, V. Stogniy, and I. Kopas. Symmetry properties and exact solutions of the linear Kolmogorov equation. *Res. Bull. Natl. Tech. Univ. Ukraine "Kyiv Polytechnic Institute"*, 4:93–97, 2011. (in Ukrainian).

[140] S. Spichak and V. Stogny. Symmetry analysis of the Kramers equation. *Rep. Math. Phys.*, 40:125–130, 1997.

[141] M. E. Stern. Minimal properties of planetary eddies. *J. Mar. Res.*, 33:1–13, 1975.

[142] S. Szatmari and A. Bihlo. Symmetry analysis of a system of modified shallow-water equations. *Commun. Nonlinear Sci. Numer. Simul.*, 19:530–537, 2014.

[143] G. Teschl. Partial differential equations. Lecture Notes, 2025. Available at https://www.mat.univie.ac.at/~gerald/ftp/book-pde/pde.pdf.

[144] P. Turkowski. Low-dimensional real Lie algebras. *J. Math. Phys.*, 29:2139–2144, 1988.

[145] G. K. Vallis. *Atmospheric and Oceanic Fluid Dynamics: Fundamentals and Large-Scale Circulation*. Cambridge University Press, Cambridge, 2006.

[146] O. O. Vaneeva, A. Bihlo, and R. O. Popovych. Generalization of the algebraic method of group classification with application to nonlinear wave and elliptic equations. *Commun. Nonlinear Sci. Numer. Simul.*, 91:105419, 2020.

[147] O. O. Vaneeva, R. O. Popovych, and C. Sophocleous. Extended symmetry analysis of two-dimensional degenerate Burgers equation. *J. Geom. Phys.*, 169:104336, 2021.

[148] O. O. Vinnichenko, V. M. Boyko, and R. O. Popovych. Lie reductions and exact solutions of dispersionless Nizhnik equation. *Anal. Math. Phys.*, 14:82, 2024.

[149] A. M. Vinogradov. *Cohomological analysis of partial differential equations and secondary calculus*, volume 204 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2001.

[150] P. Winternitz. Subalgebras of Lie algebras. Example of $\mathfrak{sl}(3,\mathbb{R})$. In P. Winternitz, J. Campbell, F. C. Khanna, and L. Vinet, editors, *Symmetry in Physics: In Memory of Robert T. Sharp*, volume 34 of *CRM Proceedings & Lecture Notes*, pages 213–227. American Mathematical Society, Providence, RI, 2004.

[151] Z.-Y. Zhang, J. Zheng, L. L. Guo, and H. F. Wu. Lie symmetries and conservation laws of the Fokker–Planck equation with power diffusion. *Math. Methods Appl. Sci.*, 43:8894–8905, 2020.