Selecta Mathematica Sovietica
Vol. 11, No. 3 (1992)

0272-9903/92/030241-51 $1.50 + 0.20/0

Criteria for Irreducibility and Equivalence of Regular Gaussian Representations of Groups of Finite Upper-Triangular Matrices of Infinite Order*

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Introduction

Regular representations play an important role in the theory of representations of locally compact groups. The decomposition of a regular representation into irreducible representations contains all the irreducible representations for finite and compact groups and many irreducible representations of locally compact Lie groups. In the case of locally compact groups a regular representation itself is always reducible, since along with a right regular representation there exists a left regular one commuting with it. It is known (see Dixmier [5], 1969) that the following theorem holds for unimodular groups.

Theorem A. *The commutant of a right regular representation is generated by operators of a left representation, and the commutant of a left representation by operators of a right representation.*

Therefore it is natural to wish to construct an analogue of the regular representation in the case of infinite-dimensional groups and to investigate its properties. By an analogue of a regular representation (right or left) of an infinite-dimensional group G we mean homomorphisms

$$U^R, U^L\colon G \mapsto U(\mathscr{H} = L_2(\tilde{G}, G, d\mu)\colon$$

$$\mathscr{H} \ni f(x) \mapsto (U^R(t)f)(x) = (d\mu(xt)/d\mu(x))^{1/2} f(xt) \in \mathscr{H},$$

$$\mathscr{H} \ni f(x) \mapsto (U^L(t)f)(x) = (d\mu(t^{-1}x)/d\mu(x))^{1/2} f(t^{-1}x) \in \mathscr{H},$$

* Previously unpublished. Manuscript received 27 September 1990. Translated by J. Danskin.

where $\tilde{G}$ is a topological group or a topological G-space containing G as a dense subgroup: $G \subset \tilde{G}$, and μ is a quasi-invariant measure on $\tilde{G}$.

It seems that the first analogue to a regular representation $\phi \to U(L_2(\phi', \phi, d\mu))$ of an infinite-dimensional commutative group of a kernel space ϕ, where ϕ' is the space conjugate to ϕ, appeared in the 1961 monograph [7], by Gel'fand and Vilenkin.

Regular representations $\mathbf{R}_0^\infty \to U(L_2(\mathbf{R}^\infty, \mathbf{R}_0^\infty, d\omega))$ of the commutative group $\mathbf{R}_0^\infty$ of finite sequences of real numbers, connected with various $\mathbf{R}_0^\infty$-quasi-invariant measures on the group $\mathbf{R}^\infty = \mathbf{R}^1 \times \mathbf{R}^1 \times \cdots \supset \mathbf{R}_0^\infty$, were studied by Samoilenko in the monograph [18].

The so-called energy representation E of the group $C_0^\infty(X, G)$ of smooth mappings with compact support of a Riemannian manifold X into a compact semisimple Lie group G was studied in the papers [8, 1, 21, 9, 2, 3]. Ismagilov introduced that representation in [8] for $G = \mathrm{SU}_2$ and X a domain in $\mathbf{R}^n$. In the general case it was introduced in [1] and [21]. The irreducibility and mutual nonequivalence of such representations for various metrics were first proved in [8] in the case $d = \dim X \geq 5$ and $G = \mathrm{SU}_2$. In [21] Vershik, Gel'fand and Graev proved the irreducibility and nonequivalence in the case $d \geq 4$ and G a compact semisimple Lie group. In [2] Albeverio, Høegh-Krohn, and Testard proved irreducibility for $d \geq 3$, and, under additional conditions, for $d = 2$. Reducibility for $d = 1$ was proved in [2] and [9].

The connection with the regular representation in the case $d = 1$ was noted in the papers [1], [2], [3], [9]. In [1] Albeverio and Høegh-Krohn proved that, in the case $x = [0, t)$, the energy representation E is unitarily equivalent to the right regular representation

$$U^R\colon C_0^1([0, t), G) \to U(L_2(C([0, t), G), C_0^1([0, t), G), dW)),$$

where $C([0, t), G)$ is the space of smooth paths and dW is the Wiener measure on $C([0, t), G)$, defined by the left Brownian motion on G. In [9] Ismagilov proved, for the group $C([0, 1], G$, that along with a right representation U^R, equivalent to the energy representation E, there exists a left representation U^L. This proved the reducibility of the energy representation in the case $d = 1$. We have already noted that the authors of [2] had proved this fact as well. There they also studied the right and left regular representations U^R, U^L of the groups $C_0^\infty(\mathbf{R}^1, G)$ and $C^\infty(S^1, G)$. Together with Vershik, they proved in [3] that the representations of U^R and U^L, constructed in [2], are factor-representations, and that Theorem A holds for them. They also presented expansions of the representations U^R and U^L into direct integrals of irreducible representations.

Okamato and Sakurai in [14] constructed left and right representations U^L and U^R by the formula

$$U^L, U^R\colon O(E) \to U(\mathscr{H} = L_2((E \hat{\otimes} E)^*, O(E), d\nu))$$

for the group $O(E) = \varinjlim_m O(m)$, where $E \simeq \mathbf{R}_0^\infty$, $O(m)$ is the orthogonal group in $\mathbf{R}^m$, $(E \hat{\otimes} E)^* \simeq M$ is the space of real matrices of infinite order, and ν is the $O(E)$-bi-invariant standard Gaussian measure on $(E \hat{\otimes} E)^*$, and showed that the representation

$$O(E) \times O(E) \ni (g_1, g_2) \mapsto \omega_*(g_1, g_2) = U^L(g_1)U^R(g_2) \in U(\mathscr{H})$$

decomposes into a countable direct sum of irreducible representations. In [15] they carried the results of [14] over to the unitary group $U(E)$, where $E = C^\infty(X, \mathbf{R})$ is the space of real C^∞-functions on the compact Riemannian manifold X, and $U(E)$ the goup of invertible operators on E which are isometries in the space $L_2(X)$. In [16], Pickrell constructed a left regular representation

$$U^L\colon U(\infty) \to U(L_2(M_{\mathbf{C}}, U(\infty), d\nu))$$

of the group $U(\infty) = \varinjlim_m U(m)$, where $M_{\mathbf{C}}$ is the space of all complex matrices of infinite order and ν is the standard Gaussian measure on $M_{\mathbf{C}}$, and showed that U^L may be decomposed into a direct sum of irreducible representations. In [13] Nessonov constructed a right regular representation

$$U^R\colon \bar{B}_0^\infty \to U(L_2(\bar{B}^\infty, \bar{B}_0^\infty, d\nu))$$

of the group $\bar{B}_0^\infty$ of matrices of the form $x = \exp t + s$, where t is a diagonal matrix with a finite number of nonzero real elements. s is a finite complex strictly upper-triangular matrix, $\bar{B}^\infty$ is a group of arbitrary matrices of the form $x = \exp t + s$, ν is the standard Gaussian measure on $\bar{B}^\infty$, and proved the irreducibility of U^R.

In [10] we proved the existence of a family of Gaussian measures μ_b^2 on the group B^∞ of upper-triangular real matrices of infinite order with units on the principal diagonal, having the property (B): *A right action of the group B_0^∞ is admissible and ergodic, and a left action is inadmissible.* We constructed a family of right regular representations

$$T^{R,b}\colon B_0^\infty \to U(L_2(B^\infty, B_0^\infty, d\mu_b^2))$$

of the group B_0^∞ of finite upper-triangular matrices: $B_0^\infty \subset B^\infty$. R. S. Ismagilov stated the following conjecture: *For these representations, property (B) is equivalent to irreducibility.* G. I. Ol'shanskiĭ proposed the following: *To nonequivalent measures there correspond nonequivalent representations.* The objective of this paper is the proof of these conjectures for the group B_0^∞ and Gaussian product-measures (see also [11]).

It is likely that these conjectures are valid for other infinite-dimensional groups, and for measures which are not necessarily Gaussian. The question as to the decomposition of a reducible regular representation of the group B_0^∞ remains open.

In §1, we construct on the group B^∞ a family of Gaussian measures μ_b^ρ that have property (B) and a family of right regular representations $T^{R,b}$ of the group B_0^∞. In §2 we show that property (B) is equivalent to the irreducibility of $T^{R,b}$. The proof of that irreducibility is based on the B_0^∞-ergodicity of the measure μ_b^ρ and on the fact that the operators of multiplication by an independent variable may be approximated by the generators of one-parameter groups. In §3 we prove that to nonequivalent measures there correspond nonequivalent representations. The proof is based on the calculation, using partial Fourier transforms on the group B^∞, of the spectral measures of a family of commutative subgroups $B_0^{\square_m} \subset B_0^\infty$, $m \in \mathbf{N}$, and on a comparison of those spectral measures using Hellinger integrals. In §4 we carry out the proofs of some technical lemmas.

We would like to express our deep gratitude to R. S. Ismagilov for turning his attention to this hypothesis, and for his constant encouragement and observations, which essentially simplified some of the proofs. Also we thank G. I. Ol'shanskiĭ for his interest in the work and useful discussions of these questions.

§1. Regular representations

Suppose that B_0^∞ is the group of finite upper-triangular real matrices of infinite order with units on the principal diagonal, B^∞ the group of all upper-triangular matrices with units on the principal diagonal, and $\mathfrak{b}^\infty$ its Lie algebra, i.e. the set of all strictly upper-triangular matrices. If one denotes by E_{kn}, $k, n \in \mathbf{N}$, the matrix units of infinite order, then the elements of the group B_0^∞ (resp. B^∞) are matrices $I + x$, $x = \sum_{k<n} x_{kn} E_{kn}$, where only a finite number of elements x_{kn} are nonzero (the x_{kn} are arbitrary),

$$\mathfrak{b}^\infty = \left\{ x = \sum_{k<n} x_{kn} E_{kn} \right\}.$$

Suppose that $B(m, \mathbf{R})$ is the subgroup of B_0^∞ of matrices of the form

$$B(m, \mathbf{R}) = \left\{ t = I + \sum_{k<n\le m} x_{kn} E_{kn} \right\}.$$

Obviously $B_0^\infty = \varinjlim_m B(m, \mathbf{R})$. We will equip B_0^∞ with the inductive limi topology.

Since the group $G = B_0^\infty$ is not locally compact, there is no G-invarian measure on it (A. Weil [22]), nor any G-quasi-invariant measure either (Da Xing Xia [23]). Accordingly, some kind of analogue has to be constructed o some completion $\tilde{G}$ of the group G. If one chooses the group B^∞ to serv in the role of such a completion $\tilde{G}$, then on the group B^∞ there already exis many different B_0^∞-quasi-invariant measures, for instance Gaussian measures There is no basis whatever for giving preference to any of those measures Therefore it makes sense to consider all measures or all measures in a certai class.

It is convenient to construct the measure first on the corresponding Li algebra $\mathfrak{b}^\infty$, and then to transfer it to the group B^∞ using the exponentia mapping.

We will be dealing with matrices $b = (b_{kn})_{k<n}$ of positive numbers. We wil denote the set of such matrices by $\mathscr{B}$. We define a Gaussian measure μ_b on th space $\mathfrak{b}^\infty$ as follows:

$$d\mu_b(x) = \bigotimes_{k<n} d\mu_{kn}(x_{kn}) = \bigotimes_{k<n} \sqrt{\frac{b_{kn}}{\pi}} \exp(-b_{kn} x_{kn}^2)\, dx_{kn}.$$

Let μ_b^ρ be the measure on B^∞ which is the image of the measure μ_b under th mapping ρ:

$$\mathfrak{b}^\infty \ni x \mapsto \rho(x) = I + x \in B^\infty, \qquad \mu_b^\rho(A) = \mu_b(\rho^{-1}(A)).$$

In fact $x = \sum_{k<n} x_{kn} E_{kn}$ are the canonical coordinates of the second kind fo $\rho(x) = I + x$. Indeed, write $x_m = \sum_{k=1}^{m-1} x_{km} E_{km}$. Then obviously

$$\begin{aligned} \rho(x) = I + x &= \cdots (I + x_m) \cdots (I + x_3)(I + x_2) \\ &= \cdots \exp(x_m) \cdots \exp(x_3) \exp(x_2). \end{aligned}$$

Consider the right and left actions R_t, L_t of the group B_0^∞ on B^∞:

$$R_t s = st, \qquad L_t s = ts, \qquad t \in B_0^\infty, \qquad s \in B^\infty.$$

Denote by $(\mu_b^\rho)^{R_t}$, $(\mu_b^\rho)^{L_t}$ the images of the measure μ_b^ρ under the mapping $R_t, L_t \colon B^\infty \to B^\infty$. It turns out that the measure μ_b^ρ is always B_0^∞-right-quasi-in variant (Lemma 1.1), but it is not always B_0^∞-left-quasi-invariant (Lemma 1.2) Therefore we can construct a family of analogues of the right $T^{R,b}$ and left $T^{L,}$

(if they exist) regular representations of the group B_0^∞ in the space

$$\mathscr{H}(b) = L_2(B^\infty, d\mu_b^\varrho), \quad b \in \mathscr{B}.$$

They are

$$\mathscr{H}(b) \ni f(x) \mapsto (T_t^{R,b} f)(x) = \left(\frac{d\mu_b^\varrho(xt)}{d\mu_b^\varrho(x)}\right)^{1/2} f(xt) \in \mathscr{H}(b) \tag{1.1}$$

and

$$\mathscr{H}(b) \ni f(x) \mapsto (T_t^{L,b} f)(x) = \left(\frac{d\mu_b^\varrho(t^{-1}x)}{d\mu_b^\varrho(x)}\right)^{1/2} f(t^{-1}x) \in \mathscr{H}(b). \tag{1.2}$$

Theorem 1.1. *The right regular representation $T^{R,b}$ of the group B_0^∞ is irreducible if and only if no left shifts L_t, $t \in B_0^\infty$, are admissible for the measure μ_b^ϱ, $b \in \mathscr{B}$.*

The proof of this theorem will be given at the beginning of §2 below.

We constructed the respresentations (1.1) and (1.2) in [10], but we did not consider the question of their irreducibility. The analogue to the representation $T^{R,I}$, for the standard Gaussian measure μ_I, $I = (b_{kn})_{k<n}$, $b_{kn} \equiv 1$, was constructed independently by N. I. Nessonov [13], who proved its irreducibility. However, Nessonov's method, based on the Fourier transform and the law of large numbers, did not include the case of an arbitrary $b \in \mathscr{B}$.

Lemma 1.1. *For $t \in B_0^\infty$ the measures $(\mu_b^\varrho)^{R_t}$ and μ_b^ϱ are always equivalent.*

Proof. Under the transformation $R_t : B^\infty \to B^\infty$ only a finite number of coordinates change:

$$B^\infty \ni x = I + \sum_{k<n} x_{kn} E_{kn} \mapsto R_t(x) = I + \sum_{k<n} \tilde{x}_{kn} E_{kn},$$

where

$$\tilde{x}_{kn} = x_{kn} + \sum_{r=k+1}^{n-1} x_{kr} t_{rn} + t_{kn} \quad \text{if } k < n \le N = N(t),$$

$$\tilde{x}_{kn} = x_{kn} \quad \text{if } n > N.$$

It follows that the question reduces to the equivalence of two nondegenerate Gaussian measures in finite-dimensional space. But then they are obviously equivalent because each of them is equivalent to Lebesgue measure.

Lemma 1.2. *Suppose that $t \in B_0^\infty$. Then the measures $(\mu_b^\varrho)^{L_t}$ and μ_b^ϱ are equivalent if and only if*

$$S_{kk+1}^L(b) = \sum_{m=k+2}^{\infty} \frac{b_{km}}{b_{k+1m}} < \infty, \quad k \in \mathbf{N}. \tag{1.3}$$

Proof. We write

$$t_{kk+1} = I + tE_{kk+1}, \quad k \in \mathbf{N}, \quad t \in \mathbf{R}^1.$$

We will show that the condition $(\mu_b^\varrho)^{L_t} \sim \mu_b^\varrho$ is equivalent to (1.3). Indeed, since

$$L_{t_{kk+1}}(x) =$$

$$\begin{pmatrix} & & & \cdots & & & & \\ 0 & \cdots & 1 & t & 0 & \cdots & 0 & \cdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots \\ & & & \cdots & & & & \end{pmatrix} \begin{pmatrix} & & & & \cdots & & & \\ 0 & \cdots & 1 & x_{kk+1} & x_{kk+2} & \cdots & x_{km} & \cdots \\ 0 & \cdots & 0 & 1 & x_{k+1k+2} & \cdots & x_{k+1m} & \cdots \\ & & & & \cdots & & & \end{pmatrix}$$

$$= \begin{pmatrix} & & & & \cdots & & & & \\ 0 & \cdots & 1 & x_{kk+1} + t & x_{kk+2} + t & x_{k+1k+2} & \cdots & x_{km} + t & x_{k+1m} & \cdots \\ 0 & \cdots & 0 & 1 & x_{k+1k+2} & & \cdots & x_{k+1m} & & \cdots \\ & & & & \cdots & & & & \end{pmatrix},$$

then $\mu_b^{L_{t_{kk+1}}}$ is a product measure,

$$\mu_b^{L_t} = ((\mu_b^\varrho)^{L_t})^{\rho-1}: \mu_b^{L_{t_{kk+1}}}(x) = \left(\bigotimes_{\substack{n<m \\ n \ne k,k+1}} \mu_{nm}(x_{nm})\right) \otimes \mu_{kk+1}^{L_{t_{kk+1}}}(x_{kk+1})$$

$$\otimes \left(\bigotimes_{m=k+2}^{\infty} (\mu_{km} \otimes \mu_{k+1m})^{L_{t_{kk+1}}}(x_{km}, x_{k+1m})\right).$$

The densities of its factors relative to the factors of the measure μ_b are equal to

$$\left(\frac{d\mu_{kk+1}^{L_{t_{kk+1}}}}{d\mu_{kk+1}}\right)(x_{kk+1}) = \exp(-b_{kk+1}(x_{kk+1} + t)^2 + b_{kk+1} x_{kk+1}^2),$$

$$\frac{d(\mu_{km} \otimes \mu_{k+1m})^{L_{t_{kk+1}}}}{d(\mu_{km} \otimes \mu_{k+1m})}(x_{km}, x_{k+1m}) = \exp(-b_{km}(x_{km} + t x_{k+1m})^2 + b_{km} x_{km}^2).$$

In view of the criterion for equivalence of product measures ([20], §16, Theorem 1), the condition $\mu_b^{L_{t_{kk+1}}} \sim \mu_b$ is equivalent to the convergence of the following product $\prod$:

$$\prod = \int_{\mathbf{R}^1} \left(\frac{d\mu_{kk+1}^{L_{tkk+1}}}{d\mu_{kk+1}}\right)^{1/2} (x_{kk+1})\, d\mu_{kk+1}(x_{kk+1})$$
$$\times \prod_{m=k+2}^{\infty} \int_{\mathbf{R}^2} \left(\frac{d\mu_{km} \otimes \mu_{k+1m})^{L_{tkk+1}}}{d(\mu_{km} \otimes \mu_{k+1m})}\right)^{1/2} (x_{km}, x_{k+1m})$$
$$\times d(\mu_{km} \otimes \mu_{k+1m})(x_{km}, x_{k+1m})$$
$$= \left(\frac{b_{kk+1}}{\pi}\right)^{1/2} \int_{\mathbf{R}^1} \exp(-\tfrac{1}{2}(b_{kk+1}(x_{kk+1}+t)^2 - b_{kk+1}x_{kk+1}^2)$$
$$\times \exp(-b_{kk+1}x_{kk+1}^2)\, dx_{kk+1} \prod_{m=k+2}^{\infty} \left(\frac{b_{km}b_{k+1m}}{\pi^2}\right)^{1/2}$$
$$\times \int_{\mathbf{R}^2} \exp(-\tfrac{1}{2}(b_{km}(x_{km}+tx_{k+1m})^2 - b_{km}x_{k+1m}^2))$$
$$\times \exp(-b_{km}x_{km}^2 - b_{k+1m}x_{k+1m}^2)\, dx_{km}\, dx_{k+1m}$$
$$= \exp\left(-\frac{b_{kk+1}t^2}{4}\right) \prod_{m=k+2}^{\infty} \left(\frac{b_{k+1m}}{\pi}\right)^{1/2}$$
$$\times \int_{\mathbf{R}^1} \exp(-\tfrac{1}{4}b_{km}(tx_{k+1m})^2 - b_{k+1m}x_{k+1m}^2);$$

$$dx_{k+1m} = \exp\left(-\frac{b_{kk+1}t^2}{4}\right) \prod_{m=k+2}^{\infty} \left(\frac{b_{k+1m}}{b_{k+1m} + t^2\dfrac{b_{km}}{4}}\right)^{1/2}$$
$$= \exp\left(-\frac{b_{kk+1}t^2}{4}\right)\left(\prod_{m=k+2}^{\infty}\left(1+\frac{t^2}{4}\frac{b_{km}}{b_{k+1m}}\right)\right)^{-1/2}.$$

Thus the, convergence of the product $\prod$ is equivalent to the convergence of the series

$$S_{kk+1}^L(b) = \sum_{m=k+2}^{\infty} b_{km}b_{k+1m}^{-1}.$$

Since the one-parameter groups

$$G_{kk+1} = \{t_{kk+1} \in B_0^{\infty} \mid t_{kk+1} = I + tE_{kk+1},\, t \in \mathbf{R}^1\}, \qquad k \in \mathbf{N},$$

generate the group B_0^{∞}, then the condition that $\mu_b^{L_t} \sim \mu_b$, $t \in B_0^{\infty}$, is equivalent to the condition that $\mu_b^{L_{tkk+1}} \sim \mu_b$, $k \in \mathbf{N}$, which proves the lemma.

Remark 1.1. From what we have proved it follows in particular that the conditions $S_{kk+1}^L(b) < \infty$, $k \in \mathbf{N}$, and

$$S_{kn}^L(b) = \sum_{m=n+1}^{\infty} b_{km}b_{nm}^{-1} < \infty, \qquad k, n \in \mathbf{N}, \quad k < n,$$

are equivalent.

Lemma 1.3. *The measure μ_b^g, given on B^∞, is B_0^∞-ergodic relative to the right action.*

Proof. It is well known that any measurable function on $\mathbf{R}^\infty$ with the standard Gaussian measure, invariant under any change of the first coordinates, coincides almost everywhere with a constant function ([19], §3, Corollary 1). Therefore the proof follows from the fact that the measure μ_b is a tensor product of measures, and the fact that the subgroup $B(m, \mathbf{R})$ of the group B_0^∞ acts transitively on the subgroup $B(m, \mathbf{R}) \subset B^\infty$.

§2. Irreducibility of representations

The proof of the irreducibility of a right regular representation is based on the ergodicity of the measure μ_b^g relative to right shifts by elements of the group B_0^∞, and on the fact that the operators of multiplication by the independent variables may be approximated by the generators of one-parameter groups.

Proof of Theorem 1.1. The necessity is obvious. We will prove the sufficiency. Suppose that $(\mu_b^g)^{L_t} \perp \mu_b^g$, $t \in B_0^\infty$. Then, by Lemma 1.2,

$$S_{kn}^L(b) = \infty, \qquad k, n \in \mathbf{N}, \quad k < n.$$

We denote by $\tilde{W}(b)$ the set of selfadjoint or skew-selfadjoint operators in $\mathscr{H}(b)$ adjoined to the algebra $W(b) = (T_t^{R,b} \mid t \in B_0^\infty)''$, and show that

$$\tilde{W}(b) \supset \{x_{kn}, \partial_{pq} - b_{pq}x_{pq} \mid k < n, p < q, k, n, p, q \in \mathbf{N}\}.$$

We give the notation for the generators of the right shift $A_{kn}^R = A_{kn}^{R,b}$:

$$A_{kn}^{R,b} = \frac{d}{dt} T^{R,b}(I + tE_{k,n})\Big|_{t=0}, \qquad k, n \in \mathbf{N}, \quad k < n.$$

We calculate directly that

$$A_{kn}^{R,b} = \sum_{m=1}^{k} x_{mk}(\partial_{mn} - b_{mn}x_{mn}), \qquad x_{kk} = 1, \quad k < n, \quad \partial_{kn} = \partial/\partial k_{kn}. \quad (2.1)$$

Lemma 2.1.

$$\{x_{mn}, \partial_{pq} - b_{pq}x_{pq} \mid m < n, p < q; m, n, p, q \in \mathbf{N}\} \subset \tilde{W}(b).$$

We will carry out the proof by induction.

Basis of the induction. We will prove that

$$\{x_{12}, \partial_{1k} - b_{1k}x_{1k}, \partial_{2k+1} + b_{2k+1}x_{2k+1}, k = 2, 3, \ldots\} \subset \tilde{W}(b).$$

Indeed, the operator x_{12} may be approximated by a linear combination of operators $A^R_{1n}A^R_{2n}$, $n > 2$. For the proof of this we use a method of calculation due to R. S. Ismagilov (Lemmas 2.2–2.4). The original proof of Lemmas 2.2 and 2.4 was more complicated (see [11]).

Lemma 2.2. *The operator x_{12} may be approximated by a linear combination of operators $A^R_{1n}A^R_{2n}$ if and only if*

$$\sigma_{12}(b) = \sum_{n=3}^{\infty} \frac{b_{1n}}{b_{2n}} = \infty.$$

Proof. We calculate the deviation of $x_{12}\mathbf{1}$ from the linear span of the vectors $A^R_{1n}A^R_{2n}\mathbf{1}$, $N_1 \le n \leqslant N_2$. Since

$$A^R_{1n} = \partial_{1n} - b_{1n}x_{1n}, \qquad A^R_{2n} = x_{12}(\partial_{1n} - b_{1n}x_{1n}) + (\partial_{2n} - b_{2n}x_{2n})$$

(see (2.1)), then

$$\begin{aligned} A^R_{1n}A^R_{2n}\mathbf{1} &= x_{12}(b^2_{1n}x^2_{1n} - b_{1n}) + b_{1n}b_{2n}x_{1n}x_{2n} \\ &= -\tfrac{1}{2}b_{1n}x_{12} + b_{1n}x_{1n}y_{1n} + b_{1n}b_{2n}x_{1n}x_{2n}. \end{aligned} \tag{2.2}$$

We have made of a change of variables:

$$x^2_{1n} = \left(x^2_{1n} - \frac{1}{2b_{1n}}\right) + \frac{1}{2b_{1n}} = y_{1n} + \frac{1}{2b_{1n}};$$

then

$$\int y_{1n}\, d\mu_b = 0 \quad \text{and} \quad \int y^2_{1n}\, d\mu_b = \frac{1}{2b^2_{1n}}.$$

We multiply both sides of (2.2) by t_n, $N_1 \le n \le N_2$, such that $-\frac{1}{2}\sum_{n=N_1}^{N_2} b_{1n}t_n = 1$, and sum on n:

$$\sum_{n=N_1}^{N_2} t_n A^R_{1n}A^R_{2n}\mathbf{1} = x_{12} + \sum_{n=N_1}^{N_2} t_n b^2_{1n}x_{12}y_{1n} + \sum_{n=N_1}^{N_2} t_n b_{1n}b_{2n}x_{1n}x_{2n}.$$

We write

$$\omega_{12}(b) = \left(\sum_{n=N_1}^{N_2} t_n A^R_{1n}A^R_{2n} - x_{12}\right)\mathbf{1} = \sum_{n=N_1}^{N_2} t_n b^2_{1n}x_{12}y_{1n} + \sum_{n=N_1}^{N_2} t_n b_{1n}b_{2n}x_{1n}x_{2n}.$$

Since all the terms are uncorrelated, then

$$\begin{aligned} \|\omega_{12}(b)\|^2 &= \sum_{n=N_1}^{N_2} t^2_n \left[\frac{b^4_{1n}}{4b_{12}b^2_{1n}} + \frac{b^2_{1n}b^2_{2n}}{2b_{1n}2b_{2n}}\right] \\ &\asymp \sum_{n=N_1}^{N_2} t^2_n[b^2_{1n} + b_{1n}b_{2n}] = \sum_{n=N_1}^{N_2} t^2_n\gamma_n. \end{aligned}$$

Now we choose the t_n so as to minimize $\omega_{12}(b)$. It is easy to see that

$$\min\left\{\sum_{n=N_1}^{N_2} t^2_n\gamma_n \,\middle|\, \sum_{n=N_1}^{N_2} t_n b_{1n} = -2\right\} = 4\left(\sum_{n=N_1}^{N_2} \frac{b^2_{1n}}{\gamma_n}\right)^{-1}, \tag{2.3}$$

the minimum being taken on at

$$t_n = -\frac{2b_{1n}}{\gamma_n}\left(\sum_{k=N_1}^{N_2} \frac{b^2_{1k}}{\gamma_k}\right)^{-1}.$$

Hence, with the optimal choice for t_n, we get

$$\|\omega_{12}(b)\|^2 = 4\left(\sum_{n=N_1}^{N_2} \frac{b^2_{1n}}{\gamma_n}\right)^{-1}.$$

We will require that $\sum_{n=3}^{\infty} b^2_{1n}\gamma_n^{-1} = \infty$, i.e.

$$\sum_{n=3}^{\infty} \frac{b^2_{1n}}{b^2_{1n} + b_{1n}b_{2n}} = \sum_{n=3}^{\infty} \frac{1}{1 + b_{2n}\cdot b^{-1}_{1n}} = \infty \Leftrightarrow \sum_{n=3}^{\infty} \frac{b_{1n}}{b_{2n}} = \infty. \qquad \square$$

Thus

$$x_{12} \in \tilde{W}(b), \qquad \partial_{1k} - b_{1k}x_{1k} = A^R_{1k} \in \tilde{W}(b), \quad k > 1,$$
$$\partial_{2k} - b_{2k}x_{2k} = A^R_{2k} - x_{12}(\partial_{1k} - b_{1k}x_{1k}) \in \tilde{W}(b), \quad k > 2.$$

Now we will show that the convergence $\sum_{n=N_1}^{N_2} t_n A^R_{1n}A^R_{2n} \to x_{12}$ of the selfadjoint operators $A_{N_1,N_2} = \sum_{n=N_1}^{N_2} t_n A^R_{1n}A^R_{2n}$ (the selfadjointedness of the operators A_{N_1,N_2} follows from the commutation relations $[A^R_{1n}, A^R_{2q}] = 0$, $n, q \ge 3$, the skew-selfadjointness A^R_{kn}: $(A^R_{kn})^* = -A^R_{kn}$, and the fact that the t_n are real) to the selfadjoint operator $A = x_{12}$ holds in the strong resolvent sense. By Theorem VIII.25 of [17], it suffices to show the convergence $A_{N_1,N_2}f \to Af$ for any $f \in D$, where D is a common essential domain for all the operators A_{N_1,N_2}

and A. For the role of D we choose a dense set consisting of finite linear combinations of arbitrary monomials

$$x^{\alpha}=x_{12}^{\alpha_{12}}x_{13}^{\alpha_{13}}x_{23}^{\alpha_{23}}\dots x_{1k}^{\alpha_{1k}}x_{2k}^{\alpha_{2k}}\dots, \quad \alpha_{ij}=0,1,\dots,$$

$i<j$. Obviously D is a common essential domain for the operators A_{N_1,N_2} and A, since D consists of analytic vectors for the operators A_{N_1,N_2} and A. Suppose that $f\in D$. Since f is cylindrical, then there exists an $n_0\in\mathbf{N}$ such that f does not depend on the variables x_{1n}, x_{2n} for $n>n_0$.

Suppose that $N_1>n_0$. Then

$$\left\|\left(x_{12}-\sum_{n=N_1}^{N_2} t_nA_{1n}^RA_{2n}^R\right)f\right\|^2=\left\|\left(x_{12}-\sum_{n=N_1}^{N_2} t_n(x_{12}(b_{1n}^2x_{1n}^2-b_{1n})+b_{1n}b_{2n}x_{1n}x_{2n})\right)f\right\|^2$$
$$=\left\|\left(x_{12}\left(1-\sum_{n=N_1}^{N_2} t_n(b_{1n}^2x_{1n}^2-b_{1n})\right)f\right\|^2+\|f\|^2\cdot\left\|\left(\sum_{n=N_1}^{N_2} t_nb_{1n}b_{2n}x_{1n}x_{2n}\right)\right\|^2$$
$$=\|x_{12}f\|^2\cdot\left\|\left(1-\sum_{n=N_1}^{N_2} t_n(b_{1n}^2x_{1n}^2-b_{1n})\right)\mathbf{1}\right\|^2$$
$$+\|f\|^2\cdot\left\|\left(\sum_{n=N_1}^{N_2} t_nb_{1n}b_{2n}x_{1n}x_{2n}\right)\mathbf{1}\right\|^2\to 0,$$

since, by what has been proved,

$$\|\omega_{12}(b)\|^2=\left\|\left(x_{12}-\sum_{n=N_1}^{N_2} t_nA_{1n}^RA_{2n}^R\right)\mathbf{1}\right\|^2$$
$$=\|x_{12}\|^2\cdot\left\|\left(1-\sum_{n=N_1}^{N_2} t_n(b_{1n}^2x_{1n}^2-b_{1n})\right)\mathbf{1}\right\|^2+\left\|\left(\sum_{n=N_1}^{N_2} t_nb_{1n}b_{2n}x_{1n}x_{2n}\right)\mathbf{1}\right\|^2$$

will be small for appropriate N_1, N_2.

The induction step. Suppose that the inclusion

$$\{x_{nm},\, n<m\le p,\, \partial_{nm}-b_{nm}x_{nm},\, 1\le n\le p,\, m>n\}\subset\widetilde{W}(b)$$

holds. We will show that then

$$\{x_{lp+1},\, \partial_{p+1m}-b_{p+1m}x_{p+1m}\mid l<p+1<m\}\subset\widetilde{W}(b).$$

We will present the proof of this assertion in the form of some lemmas.

It may happen that the operators x_{lk}, $l<k$, can be approximated, in analogy with x_{12}, by operators $A_{ln}^RA_{kn}^R$, $k<n$. However, the following considerations show that this is not always possible for

$$S_{kn}^L(b)=\infty, \quad k<n.$$

Lemma 2.3. *The operator x_{lk}, $l<k$, may be approximated by operators $A_{ln}^RA_{kn}^R$, $k<n$, if and only if*

$$\tilde\sigma_{lk}(b)=\sum_{n=k+1}^{\infty} b_{ln}^2\left(\sum_{j=1}^{l}\sum_{m=1}^{k} b_{jn}b_{mn}\right)^{-1}=\infty, \quad l<k.$$

Proof. By formula (2.1),

$$A_{ln}^R=\sum_{j=1}^{l} x_{jl}(\partial_{jn}-b_{jn}x_{jn}), \qquad A_{kn}^R=\sum_{m=1}^{k} x_{mk}(\partial_{mn}-b_{mn}x_{mn}).$$

Therefore

$$A_{ln}^RA_{kn}^R\mathbf{1}=\sum_{j=1}^{l} x_{jl}(\partial_{jn}-b_{jn}x_{jn})\sum_{m=1}^{k} x_{mk}(\partial_{mn}-b_{mn}x_{mn})\mathbf{1}$$
$$=\sum_{j=1}^{l} x_{jl}(\partial_{jn}-b_{jn}x_{jn})\left(-\sum_{m=1}^{k} x_{mk}b_{mn}x_{mn}\right)\mathbf{1}$$
$$=\sum_{j=1}^{l}\sum_{m=1}^{k} b_{jn}b_{mn}x_{jl}x_{jn}x_{mk}x_{mn}-\sum_{m=1}^{l} b_{mn}x_{ml}x_{mk}$$
$$=\sum_{j=1}^{l}\sum_{\substack{m=1\\ j\neq m}}^{k} b_{jn}b_{mn}x_{jl}x_{jn}x_{mk}x_{mn}+\sum_{m=1}^{l} b_{mn}^2x_{mn}^2x_{ml}x_{mk}-\sum_{m=1}^{l} b_{mn}x_{ml}x_{mk}.$$

On making the substitution

$$x_{mn}^2=y_{mn}+\frac{1}{2b_{mn}}, \qquad \int y_{mn}\,d\mu_b=0,$$

we get

$$A_{ln}^RA_{kn}^R\mathbf{1}=\sum_{j=1}^{l}\sum_{\substack{m=1\\ j\neq m}}^{k} b_{jn}b_{mn}x_{jl}x_{jn}x_{mk}x_{mn}+\sum_{m=1}^{l} b_{mn}^2y_{mn}x_{ml}x_{mk}-\tfrac12\sum_{m=1}^{l} b_{mn}x_{ml}x_{mk}.$$

We multiply both sides of the equation through by numbers t_n, $N_1\le n\le N_2$, such that $-\frac12\sum_{n=N_1}^{N_2} b_{ln}t_n=1$. Then

$$\tilde\omega_{lk}(b):=\left(\sum_{n=N_1}^{N_2} t_nA_{ln}^RA_{kn}^R-x_{lk}\right)\mathbf{1}=\left(\sum_{n=N_1}^{N_2} t_nA_{ln}^RA_{kn}^R+\tfrac12\sum_{n=N_1}^{N_2} t_nb_{ln}x_{ln}\right)\mathbf{1}$$
$$=\sum_{n=N_1}^{N_2} t_n\left[\sum_{j=1}^{l}\sum_{\substack{m=1\\ j\neq m}}^{k} b_{jn}b_{mn}x_{jl}x_{jn}x_{mk}x_{mn}-\tfrac12\sum_{m=1}^{l-1} b_{mn}x_{ml}x_{mk}\right.$$
$$\left.+\sum_{m=1}^{l-1} b_{mn}^2y_{mn}x_{ml}x_{mk}+b_{ln}^2y_{ln}x_{lk}\right].$$

Therefore

$$\|\tilde{\omega}_{lk}(b)\|^2 = \sum_{n=N_1}^{N_2} t_n^2 \left[\sum_{m=1}^{k} \sum_{\substack{j=1\\ j\neq m}}^{l} \frac{b_{jn}b_{mn}}{16 b_{jl} b_{mk}} + \tfrac{1}{4} \sum_{m=1}^{l-1} \frac{b_{mn}^2}{4 b_{ml} b_{mk}} \right.$$

$$\left. + \sum_{m=1}^{l-1} \frac{b_{mn}^4}{2b_{mn}^2\, 4b_{ml} b_{mk}} + \frac{-b_{ln}^4}{2b_{ln}^2\, 2b_{lk}} \right],$$

$$\|\tilde{\omega}_{lk}(b)\|^2 \asymp \sum_{n=N_1}^{N_2} t_n^2 \left[\sum_{m=1}^{k} \sum_{\substack{j=1\\ j\neq m}}^{l} b_{jn} b_{mn} + \sum_{m=1}^{l} b_{mn}^2 \right]$$

$$= \sum_{n=N_1}^{N_2} t_n^2 \sum_{m=1}^{k} \sum_{j=1}^{l} b_{jn} b_{mn}.$$

Using (2.3), we get

$$\|\tilde{\omega}_{lk}(b)\|^2 = \left(\sum_{n=N_1}^{N_2} b_{ln}^2 \left(\sum_{m=1}^{k} \sum_{j=1}^{l} b_{mn} b_{jn} \right)^{-1} \right)^{-1}. \qquad \square$$

Example 2.1. Suppose that weight $b^{(1)} = (b_{kn}^{(1)})_{k<n}$ has the form

$$\begin{pmatrix} \cdots & b_{2n}^{(1)} & b_{2n+1}^{(1)} & b_{2n+2}^{(1)} & \cdots \\ \cdots & b_{3n}^{(1)} & b_{3n+1}^{(1)} & b_{3n+2}^{(1)} & \cdots \end{pmatrix} = \begin{pmatrix} \cdots & n^2 & 1 & (n+2)^2 & \cdots \\ \cdots & 1 & (n+1)^2 & 1 & \cdots \end{pmatrix},$$

$$b_{kn}^{(1)} \equiv 1, \quad k \neq 2, 3.$$

Obviously, $S_{kn}^{L}(b^{(1)}) = \infty$, $k < n$, but

$$\tilde{\sigma}_{1k}(b^{(1)}) = \sum_{n=k+1}^{\infty} \frac{b_{1n}^{(1)}}{b_{2n}^{(1)} + \cdots + b_{kn}^{(1)}} = \sum_{n=k+1}^{\infty} \frac{1}{n^{(2)} + k - 2} < \infty, \qquad k \geqslant 3.$$

It follows from Lemma 2.3 that for the weight $b^{(1)}$ it is not possible to approximate any operator x_{lk}, $l<k$, $f \to x_{lk} f$, $k \geq 3$, by operators $A_{1n}^{R} A_{kn}^{R}$. The operator x_{12} may however be so approximated. It is better to approximate with operators of the type

$$(\partial_{ln} - b_{ln} x_{ln}) A_{kn}^{R}, \qquad k < n.$$

Lemma 2.4. *For the approximation of the variables x_{lk}, $l<k$, by the operators $(\partial_{ln} - b_{ln} x_{ln}) A_{kn}^{R}$, $k<n$, it is necessary and sufficient that*

$$\sigma_{lk}(b) = \sum_{n=k+1}^{\infty} b_{ln} \left(\sum_{m=1, m\neq l}^{k} b_{mn} \right)^{-1} = \infty, \qquad l<k.$$

Proof. We have

$$(\partial_{ln} - b_{ln} x_{ln}) A_{kn}^{R} \mathbf{1} = (\partial_{ln} - b_{ln} x_{ln}) \sum_{m=1}^{k} x_{mk} (\partial_{mn} - b_{mn} x_{mn}) \mathbf{1}$$

$$= (\partial_{ln} - b_{ln} x_{ln}) \sum_{m=1}^{k} x_{mk} (-b_{mn} x_{mn}) \mathbf{1}$$

$$= \sum_{m=1}^{k} b_{ln} b_{mn} x_{mk} x_{ln} x_{mn} - b_{ln} x_{lk}.$$

On making the substitution

$$x_{ln}^2 = y_{ln} + \frac{1}{2b_{ln}},$$

we get finally

$$(\partial_{ln} - b_{ln} x_{ln}) A_{kn}^{R} \mathbf{1} = \sum_{m=1, m\neq l}^{k} b_{ln} b_{mn} x_{ln} x_{mk} x_{mn} + b_{ln}^2 y_{ln} x_{ln} - \tfrac{1}{2} b_{ln} x_{lk}. \quad (2.4)$$

Now we multiply both sides of equation (2.4) through by t_n, $N_1 \leq n \leq N_2$, choosing the t_n so that

$$\sum_{n=N_1}^{N_2} b_{ln} t_n = -2.$$

Then

$$\omega_{lk}(b) := \left(\sum_{n=N_1}^{N_2} t_n (\partial_{ln} - b_{ln} x_{ln}) A_{kn}^{R} - x_{lk} \right) \mathbf{1}$$

$$= \sum_{n=N_1}^{N_2} t_n \left[\sum_{m=1, m\neq l}^{k} b_{ln} b_{mn} x_{ln} x_{mk} x_{mn} + b_{ln}^2 y_{ln} x_{lk} \right],$$

$$\|\omega_{lk}(b)\|^2 \asymp \sum_{n=N_1}^{N_2} t_n^2 \left[\sum_{m=1, m\neq l}^{k} \frac{b_{ln} b_{mn}}{b_{mk}} + \frac{b_{ln}^2}{b_{lk}} \right] \asymp \sum_{n=N_1}^{N_2} t_n^2 \sum_{m=1}^{k} b_{ln} b_{mn}.$$

Accordingly, by formula (2.3),

$$\min\left\{ \|\omega_{lk}(b)\|^2 \,\middle|\, \sum_{n=N_1}^{N_2} b_{ln} t_n = -2 \right\} = 4 \left(\sum_{n=N_1}^{N_2} b_{ln}^2 \left(\sum_{m=1}^{k} b_{ln} b_{mn} \right)^{-1} \right)^{-1}.$$

We require that

$$\sum_{n=k+1}^{\infty} b_{ln}^2 \left(\sum_{m=1}^{k} b_{ln} b_{mn} \right)^{-1} = \sum_{n=k+1}^{\infty} b_{ln} \left(\sum_{m=1, m\neq l}^{k} b_{mn} + b_{ln} \right)^{-1} = \infty, \qquad l<k,$$

which is equivalent to

$$\sigma_{lk}(b) = \sum_{n=k+1}^{\infty} b_{ln}\left(\sum_{m=1, m\neq l}^{k} b_{mn}\right)^{-1} = \infty, \qquad l < k. \qquad \square$$

However, Example 2.1 shows that, for the weight $b^{(1)}$, it is, as before, not possible to approximate any operator $f \to x_{1k}f$, $k \geq 3$, by operators $(\partial_{1n} - b_{1n}x_{1n})A^R_{kn}$. As before, x_{12} can be.

It turns out that for any $q = 2, 3, \ldots$ there exists a $p < q$ such that the variables x_{pq} can be approximated by operators $(\partial_{pn} - b_{pn}x_{pn})A^R_{qn}$, $n > q$.

Lemma 2.5. *Suppose that $S^L_{kq}(b) = \infty$, $k = 1, 2, \ldots, q-1$. Then there exists a $p < q$ such that $\sigma_{pq}(b) = \infty$.*

Proof. We will carry out an induction. Suppose that $q = 3$ and

$$S^L_{13}(b) = \sum_{k=4}^{\infty} \frac{b_{1k}}{b_{3k}} = \infty, \qquad S^L_{23}(b) = \sum_{k=4}^{\infty} \frac{b_{2k}}{b_{3k}} = \infty.$$

Suppose the contrary, i.e. that

$$\sigma_{13}(b) = \sum_{k=4}^{\infty} \frac{b_{1k}}{b_{2k} + b_{3k}} < \infty \quad \text{and} \quad \sigma_{23}(b) = \sum_{k=4}^{\infty} \frac{b_{2k}}{b_{1k} + b_{3k}} < \infty.$$

From the fact that $\sigma_{13}(b) < \infty$ it follows that $b_{1k} < b_{2k} + b_{3k}$, $k \geq k_0$, so that

$$\sigma_{23}(b) > \sum_{k=k_0}^{\infty} \frac{b_{2k}}{b_{1k} + b_{3k}} > \sum_{k=k_0}^{\infty} \frac{b_{2k}}{b_{2k} + 2b_{3k}} = \infty,$$

since $S^L_{23}(b) = \infty$. The resulting contradiction proves the assertion for $q = 3$.

Suppose that $S^L_{kq}(b) = \infty$, $k = 1, 2, \ldots, q-1$, implies $\sigma_{pq}(b) = \infty$ for some $p < q$. We will prove this for $q+1$. Suppose the contrary, i.e. that $\sigma_{rq+1}(b) < \infty$, $r = 1, 2, \ldots, q$. From

$$\sigma_{1q+1}(b) = \sum_{n=q+2}^{\infty} \frac{b_{1n}}{\sum_{m=2}^{q+1} b_{mn}} < \infty$$

it follows that $b_{1n} < \sum_{m=2}^{q+1} b_{mn}$, $n \geq n_0$. We substitute this into $\sigma_{rq+1}(b) < \infty$, $r = 2, \ldots, q$. We have

$$\infty > \sigma_{rq+1}(b) > \sum_{n=n_0}^{\infty} b_{rn}\left(b_{rn} + 2\sum_{m=2, m\neq r}^{q+1} b_{mn}\right)^{-1}, \qquad r = 2, 3, \ldots, q.$$

This last is equivalent to

$$\sigma^{(1)}_{rq+1}(b) = \sum_{n=q+2}^{\infty} b_{rn}\left(\sum_{m=2, m\neq r}^{q+1} b_{mn}\right)^{-1} < \infty, \qquad 2 \leq r \leq q,$$

while $S^L_{kq+1}(b) = \infty$, $2 \leq k \leq q$, which contradicts the induction hypothesis, after the notations $\tilde{b}_{kn} = b_{k+1, n+1}$, $k < n$, are adjusted. $\square$

The following example shows that, for any q, there might be only one $p < q$ with $\sigma_{pq}(b) = \infty$.

Example 2.2. Suppose that $b^{(2)}_{kn} = n^2$, $1 = k < n$, $b^{(2)}_{kn} = 1$, $1 < k < n$. Then

$$S^L_{kq}(b^{21}) = \infty, \quad k < q, \quad \sigma_{1q}(b^{(2)}) = \infty, \quad \sigma_{pq}(b^{(2)}) < \infty, \quad 1 < p < q.$$

There remains some "adjustment" of the operators $(\partial_{ln} - b_{ln}x_{ln})A^R_{kn}$, $k < n$, in order that the variables x_{lk}, $l < k$, might be approximated.

Lemma 2.6. *Suppose that $S^L_{kn}(b) = \infty$, $k < n$. Then the variables x_{lk}, $l < k$, are approximated by the operators*

$$_{ln}A^{(n_1, \ldots, n_r)}_{kn} = (\partial_{ln} - b_{ln}x_{ln})A^{(n_1, \ldots, n_r)}_{kn},$$

where

$$A^{(n_1, \ldots, n_r)}_{kn} = \sum_{m=1, m\notin\{n_1, \ldots, n_r\}}^{k} x_{mk}(\partial_{mn} - b_{mn}x_{mn}), \qquad x_{kk} \equiv 1,$$

under an appropriate choice of $n_i \in \mathbf{N}$, $1 \leqslant n_i < k$, $n_i \neq n_j$, $i \neq j$, $1 \in i, j \leq r \leq k-2$.

Proof. This amounts to a proof of the inductive step. Suppose that the inclusion

$$\{x_{nm}, n < m \leq p; \partial_{nm} - b_{nm}x_{nm} \mid 1 \leq n \leq p, m > n\} \subset \tilde{W}(b)$$

is satisfied. We will prove that then

$$\{x_{lp+1}, \partial_{p+1m} - b_{p+1m}x_{p+1m} \mid l < p+1 < m\} \subset \tilde{W}(b).$$

For the approximation of x_{lp+1} by the operators $(\partial_{ln} - b_{ln}x_{ln})A^R_{p+1n}$, in view of Lemma 2.4, it is necessary and sufficient that

$$\sigma_{lp+1}(b) = \sum_{n=p+2}^{\infty} b_{ln}\left(\sum_{m=1, m\neq l}^{p+1} b_{mn}\right)^{-1} = \infty, \qquad l < p+1.$$

In view of Lemma 2.5, one of the series $\sigma_{lp+1}(b)$, $l < p+1$, diverges.

Hence, for some $n_1 < p+1$, $\sigma_{n_1 p+1}(b) = \infty$, i.e. we may approximate $x_{n_1 p+1}$ by the operators $(\partial_{n_1 n} - b_{n_1 n} x_{n_1 n}) A^R_{p+1n}$. In view of Lemma 2.5, x_{rp+1} may be approximated by the operators ${}_{rn}A^{(n_1)}_{p+1n}$ if and only if

$$\sigma^{(n_1)}_{rp+1}(b) = \sum_{n=p+2}^{\infty} b_{rn} \left(\sum_{m=1, m \neq r, n_1}^{p+1} b_{mn} \right)^{-1} = \infty, \qquad 1 \leqslant k \leq p, \quad r \neq n_1.$$

By Lemma 2.5, one of the series $\sigma^{(n_1)}_{rp+1}(b)$ diverges. Suppose that $\sigma^{(n_1)}_{n_2 p+1}(b) = \infty$. Then we may approximate $x_{n_2 p+1}$, where $n_2 \neq n_1$, by operators ${}_{n_2 n}A^{(n_1)}_{p+1n}$, and so forth. As a result we obtain a sequence $n_1, \dots, n_p$ such that the variables $x_{n_k p+1}$, $k = 1, \dots, p$, may be approximated by operators ${}_{n_k n}A^{(n_1, \dots, n_{k-1})}_{p+1n}$, and $\{n_1, \dots, n_p\}$ is a permutation of $\{1, \dots, p\}$. From

$$\partial_{p+1m} - b_{p+1m} x_{p+1m} = A^R_{p+1m} - \sum_{r=1}^{p} x_{rp+1}(\partial_{rm} - b_{rm} x_{rm})$$

it follows that the inclusion

$$\partial_{p+1m} - b_{p+1m} x_{p+1m} \in \tilde{W}(b), \qquad m > p+1,$$

holds; this completes the proof of Lemma 2.1.

Thus we have adjoined to the von Neumann algebra $W(b) = (T^{R,b}_t \mid t \in B^\infty_0)''$ the operators of multiplication by the independent variables x_{kn}, $k < n$, $k, n \in \mathbf{N}$. Therefore the von Neumann algebra contains the operators

$$\{U_{kn}(t) = \exp(itx_{kn}) \mid t \in \mathbf{R}^1, k, n \in \mathbf{N}, k < n\}.$$

Now suppose that the bounded operator $A \in L(L_2(B^\infty, d\mu^\varrho_b))$ commutes with all the operators $(T^{R,b}_t \mid t \in B^\infty_0)$. We will show that then it is a multiple of the identity: $A = \lambda I$, $\lambda \in \mathbf{C}^1$. Indeed, in this case A commutes with the operators $U_{kn}(t)$. Hence A is the operator of multiplication by an essentially bounded function: $A = f_A(x)$. In view of the commutation relations $[f_A(x), T^{R,b}_t] = 0$, we conclude that the function $f_A(x)$ is invariant relative to the action of the group B^∞_0: $f_A(x) = f_A(xt)$ for almost all $x \in B^\infty$, $t \in B^\infty_0$. In view of the ergodicity of the measure μ_0, $f_A(X) = \text{const}$, i.e. $A = \lambda I$, as we were required to prove. □

Thus we have constructed a family $T^{R,b}$, $b \in \mathscr{B}$, of analogues to regular representations of the group B^∞_0. Among these the irreducible representations are distinguished by the condition $b \in \mathscr{B}^L$:

$$\mathscr{B}^L = \left\{ b \in \mathscr{B} \mid S^L_{kn}(b) = \sum_{m=n+1}^{\infty} b_{km} b^{-1}_{nm} = \infty, k, n \in \mathbf{N}, k < n \right\}.$$

§3. Equivalence of representations

The question naturally arises as to which of the irreducible representations $T^{R,b}$, $b \in \mathscr{B}^L$, are equivalent.

Theorem 3.1. *The irreducible representations $T^{R,b^{(1)}}$ and $T^{R,b^{(2)}}$ are equivalent if and only if the measures $\mu_{b^{(1)}}$ and $\mu_{b^{(2)}}$ are equivalent.*

It is well known (see [17], Chap. II) that two product measures $\mu_{b^{(1)}}$ and $\mu_{b^{(2)}}$ are equivalent if and only if

$$H(b^{(1)}, b^{(2)}) = H(\mu_{b^{(1)}}, \mu_{b^{(2)}}) = \left(\prod_{k<n} \frac{(b^{(1)}_{kn} + b^{(2)}_{kn})^2}{4 b^{(1)}_{kn} b^{(2)}_{kn}} \right)^{-1} > 0. \tag{3.1}$$

Theorems 2.1 and 3.1 give a final description of the regular representations $T^{R,b}$, $b \in \mathscr{B}$, of the group B^∞_0.

Theorem 3.2. *Among the right regular representations $T^{R,b}$, $b \in \mathscr{B}$, the irreducible ones are distinguished by the condition $b \in \mathscr{B}^L$. Among the irreducible representations, equivalent ones are distinguished by the condition $H(\mu_{b^{(1)}}, \mu_{b^{(2)}}) > 0$.*

The proof of Theorem 3.1 is based on an explicit computation of the spectral measures $\sigma^{\square_m}_b$ of the restrictions of the representations $T^{R,b}$ to the infinite-dimensional commutative subgroups

$$B^{\square_m}_0 \subset B^\infty_0 : B^\infty_0 = \cdots = B^{\square_m}_0 \cdots B^{\square_2}_0 B^{\square_1}_0,$$

and on the comparison of these spectral measures using Hellinger integrals. The calculation of the spectral measure $\sigma^{\square_m}_b$ makes use of a partial Fourier transform, due to N. I. Nessonov [**13**], carrying the generators of one-parameter groups of $B^{\square_m}_0$ into operators of multiplication by a function.

The sufficiency is obvious. Indeed, suppose that $\mu_{b^{(1)}} \sim \mu_{b^{(2)}}$. Then $\mu^\varrho_{b^{(1)}} \sim \mu^\varrho_{b^{(2)}}$, and the unitary operator $U: \mathscr{H}(b^{(1)}) \to \mathscr{H}(b^{(2)})$ of multiplication by the function $(d\mu^\varrho_{b^{(1)}}/d\mu^\varrho_{b^{(2)}})(x)$, where $\mathscr{H}(b) = L_2(B^\infty, d\mu^\varrho_b)$, will intertwine the representations $T^{R,b^{(1)}}$ and $T^{R,b^{(2)}}$, i.e.

$$UT^{R,b^{(1)}}_{(t)} = T^{R,b^{(2)}}_{(t)} U, \qquad t \in B^\infty_0.$$

Necessity. We will prove that $T^{R,b^{(1)}} \sim T^{R,b^{(2)}}$ implies that $\mu_{b^{(1)}} \sim \mu_{b^{(2)}}$. Denote by $W(b) = (T^{R,b}_t \mid t \in B^\infty_0)''$ the von Neumann algebra generated by the operators $(T^{R,b}_t \mid t \in B^\infty_0)$, by $\tilde{W}(b)$ the set of selfadjoint or skew-selfadjoint operators $A = \int \lambda \, dE(\lambda)$ adjoined to the algebra $W(b)$, i.e. such that their spectral projectors $E_A(\Delta)$ lie in $W(b)$, and by $\Delta \in \mathscr{B}(\mathbf{R}^1)$ the σ-algebra of Borel sets on the axis. Suppose that $W_1(b) = \{x_{kn} \mid k < n\}$ is the set of operators of multiplication by the independent variables $f \to x_{kn} f$ in the space $\mathscr{H}(b)$. Then, as we proved in Theorem 2.1, $W_1(b) \subset \tilde{W}(b)$. Suppose that

$$\nabla_m = \{(k, n) \mid k < n \le m\}, \qquad \Box_m = \{(k, n) \mid k \le m < n\}, \qquad m \in \mathbf{N}, \quad \nabla_1 = \varnothing,$$

$$\mu_b^{\nabla_m} = \bigotimes_{(k,n) \in \nabla_m} \mu_{kn}, \qquad \mu_b^{\Box_m} = \bigotimes_{(k,n) \in \Box_m} \mu_{kn},$$

$$\mathscr{H}^{\nabla_m}(b) = L_2(B^{\nabla_m}, (\mu_b^{\nabla_m})^\rho), \qquad \mathscr{H}^{\Box_m}(b) = L_2(B^{\Box_m}, (\mu_b^{\Box_m})^\rho),$$

where

$$B^{\nabla_m} = \left\{x \in B_0^\infty \mid x = I + \sum_{(k,n) \in \nabla_m} x_{kn} E_{k,n}\right\},$$

$$B^{\Box_m} = \left\{x \in B^\infty \mid x = I + \sum_{(k,n) \in \Box_m} x_{kn} E_{k,n}\right\},$$

$$B_0^{\Box_m} = B^{\Box_m} \cap B_0^\infty.$$

We note that $B^{\Box_m}$ is a commutative subgroup of the group B^∞. Then

$$\tilde{W}(b)^{\Box_m} = \{iA_{kn}^{R,b} \mid (k, n) \in \Box_m\}$$

is a commuting family of operators of the set $\tilde{W}(b)$. We recall that the *spectral measure* $\sigma(\mathbf{A})$ of a family $\mathbf{A} = (A_k)_{k \in \mathbf{N}}$ of selfadjoint operators A_k, $k \in \mathbf{N}$, commutative in the sense of resolution of identity, is any scalar measure $\sigma(\mathbf{A}, \Delta)$ on the σ-algebra $\mathscr{B}(\mathbf{R}^\infty) \ni \Delta$, generated by cylindrical sets with Borel bases

$$C(\mathbf{R}^\infty) = \{C(k_1, \dots, k_n, \Delta_1, \dots, \Delta_n) = \{x \in \mathbf{R}^\infty \mid x_{k_1} \in \Delta_1, \dots, x_{k_n} \in \Delta_n\} \mid \Delta_i \in \mathscr{B}(\mathbf{R}^1), i = 1, \dots, n, k_1, \dots, k_n, n \in \mathbf{N}\},$$

equivalent to the joint resolution of identity $\mathbf{E}_\mathbf{A}$ of the family $\mathbf{A}$ of operators, defined on the cylindrical sets by the formula

$$\mathbf{E}_\mathbf{A}(C(k_1, \dots, k_n, \Delta_1, \dots, \Delta_n)) = E_{k_1}(\Delta_1) \cdots E_{k_n}(\Delta_n).$$

Suppose that $\sigma_b^{\Box_m} = \sigma(\tilde{W}_{(b)}^{\Box_m})$ is the spectral measure of the family of operators $\tilde{W}(b)^{\Box_m}$, $m \in \mathbf{N}$.

Assume that the representations $T^{R,b^{(1)}}$ and $T^{R,b^{(2)}}$ are equivalent, i.e. that there exists a unitary operator $U\colon \mathscr{H}(b^{(1)}) \to \mathscr{H}(b^{(2)})$ such that $UT_t^{R,b^{(1)}} = T_t^{R,b^{(2)}} U$, $t \in B_0^\infty$. We will write for short $T^{R,b^{(1)}} \overset{U}{\sim} T^{R,b^{(2)}}$.

The proof of necessity rests on two lemmas.

Lemma 3.1. *Suppose that* $\sigma_{b^{(1)}}^{\Box_m} \sim \sigma_{b^{(2)}}^{\Box_m}$. *Then*

$$\mu_{b^{(1)}}^{\Box_m} \sim \mu_{b^{(2)}}^{\Box_m}, \qquad m \in \mathbf{N}.$$

Lemma 3.2. *Assume that* $T^{R,b^{(1)}}$ *and* $T^{R,b^{(2)}}$ *are equivalent irreducible unitary representations*: $T^{R,b^{(1)}} \overset{U}{\sim} T^{R,b^{(2)}}$. *Then* $W_1(b^{(1)}) \sim W_1(b^{(2)})$ *with the same intertwining operator* U, *and* $Ux_{kn} = x_{kn} U$, $k < n$.

The necessity of Theorem 3.1 follows from Lemma 3.2. Indeed,

$$T^{R,b^{(1)}} \sim T^{R,b^{(2)}} \Rightarrow W_1(b^{(1)}) \sim W_1(b^{(2)}) \Rightarrow \sigma(W_1(b^{(1)})) \sim \sigma(W_1(b^{(2)})).$$

But the spectral measure $\sigma(W_1(b))$ of the family of operators of multiplication by independent variables in the space $\mathscr{H}(b)$ is obviously equivalent to μ_b^ρ. Hence we have

$$\sigma(W_1(b^{(1)})) \sim \sigma(W_1(b^{(2)})) \Rightarrow \mu_{b^{(1)}}^\rho \sim \mu_{b^{(2)}}^\rho \Rightarrow \mu_{b^{(1)}} \sim \mu_{b^{(2)}}. \qquad \Box$$

The proof of Lemma 3.1 comes down to the explicit calculation of the spectral measures $\sigma_b^{\Box_m}$, $m \in \mathbf{N}$, and the calculation of the Hellinger integrals $H(\sigma_{b^{(1)}}^{\Box_m}, \sigma_{b^{(2)}}^{\Box_m})$.

We recall the definition and properties of the Hellinger integral ([12], Chap. 2, §2).

Suppose that μ and ν are two probability measures on the measure space $(X, \mathscr{B})$. Assume that λ is a probability measure such that $\mu < \lambda$, $\nu < \lambda$, for example $\lambda = (\mu + \nu)/2$. The Hellinger integral for μ and ν is defined as follows:

$$H(\mu, \nu) = \int_X \sqrt{\frac{d\mu}{d\lambda}} \sqrt{\frac{d\nu}{d\lambda}}\, d\lambda.$$

It does not depend on λ, and has the following properties:

(H1) $0 \le H(\mu, \nu) \le 1$ (the Schwarz inequality);

(H2) $H(\mu, \nu) = 1 \Leftrightarrow \mu = \nu$;

(H3) $H(\mu, \nu) = 0 \Leftrightarrow \mu \perp \nu$;

(H4) $\mu \sim \nu \Rightarrow H(\mu, \nu) > 0$.

The converse to (H4) does not hold in general.

We fix on a number $m \in \mathbf{N}$ and do a Fourier transform $\mathscr{F}_m$ of the space $\mathscr{H}^{\nabla_m}(b) \otimes \mathscr{H}^{\Box_m}(b)$, in which the operators $A_{kn}^{R,b}$ of the family $\tilde{W}(b)^{\Box_m}$ act.

Write $\mathfrak{b}^{\nabla_m} = \rho^{-1}(B^{\nabla_m})$, $\mathfrak{b}^{\Box_m} = \rho^{-1}(B^{\Box_m})$. Suppose that

$$t = \sum_{k < n \le m} t_{kn} E_{kn} \in \mathfrak{b}^{\nabla_m}, \qquad y = \sum_{k \le m < n} y_{kn} E_{kn}, \qquad x = \sum_{k \le m < n} x_{kn} E_{kn} \in \mathfrak{b}^{\Box_m}.$$

Obviously $B^{\square_m}$ is a commutative normal subgroup in B^∞. Consider the semi-direct product $B^{\nabla_m} \rtimes B^{\square_m}$. Obviously

$$1 + t + y = (I + y)(I + t) = \rho(y)\rho(t) \in B^{\nabla_m} \rtimes B^{\square_m}.$$

For the function

$$f(\rho(y)\rho(t)) \in \mathcal{H}^{\nabla_m}(b) \otimes \mathcal{H}^{\square_m}(b) = L_2(B^{\nabla_m} \rtimes B^{\square_m}, (\mu_b^{\nabla_m} \otimes \mu_b^{\square_m})^\rho)$$

we define the partial Fourier–Weiner transform (see [4], or [5], Chap. II, §6)

$$(\mathcal{F}_m f)(\rho(x)\rho(t)) = \tilde{f}_m(\rho(x)\rho(t)) = \exp\left(\tfrac{1}{2} \sum_{(k,n) \in \square_m} x_{kn}^2 b_{kn}^{-1}\right)$$

$$\times \int_{B^{\square_m}} \exp\left(i \sum_{(k,n) \in \square_m} x_{kn} y_{kn}\right) f(\rho(y)\rho(t))\, d(\mu_{b/2}^{\square_m})^\rho(\rho(y))$$

$$= \exp(\tfrac{1}{2}(B^{-1}x, x)) \int_{\mathfrak{b}^{\square_m}} \exp(i(x, y)) f(\rho(y)\rho(t))\, d\mu_{b/2}^{\square_m}(y),$$

where $(b/2)_{kn} = b_{kn}/2$, $k < n$, and B is the diagonal operator $(Bx)_{kn} = b_{kn} x_{kn}$, $(k, n) \in \square_m$. $\mathcal{F}_m$ is a unitary operator from $\mathcal{H}^{\nabla_m}(b) \otimes \mathcal{H}^{\square_m}(b)$ into

$$\mathcal{H}^{\nabla_m}(b) \otimes \mathcal{H}^{\square_m}(b^{-1}), (b^{-1})_{kn} = b_{kn}^{-1}, \qquad (k, n) \in \square_m.$$

Suppose that

$$z = \sum_{k \le m < n} z_{kn} E_{kn} \in \mathfrak{b}^{\square_m}.$$

We will calculate $\rho(y)\rho(t)\rho(z)$. We have

$$\rho(y)\rho(t)\rho(z) = \rho(y)\rho(t)\rho(z)\rho(t)^{-1}\rho(t).$$

Since $B^{\square_m}$ is a normal subgroup in $B^{\nabla_m} \rtimes B^{\square_m}$, then

$$\rho(t)\rho(z)\rho(t)^{-1} = \mathrm{Ad}_{\rho(t)}\rho(z) \in B^{\square_m}.$$

Therefore the mapping $\varphi_{\rho(t)}\colon \rho^{-1} \circ \mathrm{Ad}_{\rho(t)} \circ \rho$ operates from $\mathfrak{b}^{\square_m}$ into $\mathfrak{b}^{\square_m}$: $\varphi_{\rho(t)}\colon \mathfrak{b}^{\square_m} \to \mathfrak{b}^{\square_m}$, $t \in \mathfrak{b}^{\nabla_m}$. A direct calculation yields

$$z' := \varphi_{\rho(t)} z = \rho(t) z = L_{\rho(t)} z, \qquad (\rho(t)z)_{kn} = z_{kn} + \sum_{r=k+1}^{n-1} t_{kr} z_{rn}, \quad k < n.$$

Since, for $x, z \in \mathfrak{b}^{\square_m}$, $\rho(x)\rho(z) = \rho(x + z)$, then

$$\rho(y)\rho(t)\rho(z) = \rho(y)\rho(z')\rho(t) = \rho(y + z')\rho(t).$$

In the Fourier images the operators $T_{\rho(z)}^{R,b}$, $\rho(z) \in B_0^{\square_m}$, take the form

$$\tilde{f}_m(\rho(x)\rho(t)) \mapsto \exp(\tfrac{1}{2}(B^{-1}x, x)) \int_{B^{\square_m}} \exp i(x, y) f(\rho(y)\rho(t)\rho(z))$$

$$\times \left(\frac{d(\mu_b^{\nabla_m} \otimes \mu_b^{\square_m})^\rho(\rho(y)\rho(t)\rho(z))}{d(\mu_b^{\nabla_m} \otimes \mu_b^{\square_m})^\rho(\rho(y)\rho(t))}\right)^{1/2} d(\mu_{b/2}^{\square_m})^\rho(\rho(y)) = \exp(\tfrac{1}{2})(B^{-1}x, x))$$

$$\times \int_{B^{\square_m}} \exp i(x, y) f(\rho(y + z')\rho(t)) \left(\frac{d(\mu_b^{\nabla_m} \otimes \mu_b^{\square_m})^\rho(\rho(y + z')\rho(t))}{d(\mu_b^{\nabla_m} \otimes \mu_b^{\square_m})^\rho(\rho(y)\rho(t))}\right)^{1/2}$$

$$\times d(\mu_{b/2}^{\square_m})^\rho(\rho(y))$$

$$= \exp(\tfrac{1}{2}(B^{-1}x, x)) \int_{B^{\square_m}} \exp i(x, y) f(\rho(y + z')\rho(t))\, d(\mu_{b/2}^{\square_m})^\rho(\rho(y + z'))$$

$$= \exp(-i(x, z'))\tilde{f}_m(\rho(x)\rho(t)).$$

Here we have made the substitutions

$$w = y + z', \qquad y = w - z', \quad \exp i(x, y) = \exp i(k, w) \exp(-i(x, z')).$$

Accordingly,

$$(\mathcal{F}_m T_{\rho(z)}^{R,b} \mathcal{F}_m^{-1} f)(\rho(x)\rho(t)) = \exp(-i(x, \rho(t)z)) f(\rho(x)\rho(t)).$$

The generators $iA_{k,n}^{R,b}$ now go over into $i\tilde{A}_{kn}^{R,b}$, the operators of multiplication by the following functions:

$$i\tilde{A}_{kn}^{R,b} = \frac{d}{dz_{kn}} \exp(-i(x, \rho(t)z))\, \big|_{z=0}$$

$$= i \frac{d}{dz_{kn}} \exp\left(-i \sum_{(p,q) \in \square_m} x_{pq}\left(z_{pq} + \sum_{z=p+1}^{q-1} t_{pr} z_{rq}\right)\right)\bigg|_{z=0} = \left(x_{kn} + \sum_{r=1}^{n-1} t_{rk} x_{rn}\right)$$

$$= (\rho(t^T)x)_{kn} = (L_{\rho(t^T)}(x))_{kn}, \qquad (k, n) \in \square_n. \tag{3.2}$$

Since

$$\mathcal{F}_m\colon \mathcal{H}^{\nabla_m}(b) \otimes \mathcal{H}^{\square_m}(b) \to \mathcal{H}^{\nabla_m}(b) \otimes \mathcal{H}^{\square_m}(b^{-1})$$

is a unitary operator ([5], Chap. II, Theorem 5.1), then the spectral measure of the family $iA_{kn}^{R,n}$, $(k, n) \in \square_m$, is equivalent to the spectral measure $\sigma_b^{\square_m}$ of the family of operators $i\tilde{A}_{kn}^{R,b}$, $(k, n) \in \square_m$. The measure $\sigma_b^{\square_m}$ on the group $B^{\square_m}$ and its image $\sigma_b^{\square_m, \rho^{-1}}(A) = \sigma_b^{\square_m}(\rho(A))$ on the algebra $\mathfrak{b}^{\square_m}$ will be denoted by the same symbol: $\sigma_b^{\square_m}$.

We will show that the measure $\sigma_b^{\square_m}$ on the algebra $\mathfrak{b}^{\square_m}$ has the following form for $A \in \mathscr{B}(\mathfrak{b}^{\square_m})$, where $\mathscr{B}(\mathfrak{b}^{\square_m})$ is the σ-algebra of Borel sets on $\mathfrak{b}^{\square_m}$:

$$\sigma_b^{\square_m}(A) = \int_{\mathfrak{b}^{\square_m}} (\mu_{b^{-1}}^{\square_m})^{L^{-1}_{\rho(t^T)}}(A)\, d\mu_b^{\square_m}(t). \tag{3.3}$$

Indeed, by the definition of $\sigma_b^{\square_m}$ we have

$$\begin{aligned}\sigma_b^{\square_m}(A) &= (\mu_b^{\nabla_m} \otimes \mu_{b^{-1}}^{\square_m})((t, x) \in \mathfrak{b}^{\nabla_m} \times \mathfrak{b}^{\square_m} \mid L_{\rho(t^T)}(x) = \rho(t^T)x \in A)\\ &= \int_{\mathfrak{b}^{\nabla_m}} (\mu_{b^{-1}}^{\square_m})^{L^{-1}_{\rho(t^T)}}(A)\, d\mu_b^{\nabla_m}(t)\\ &= \int_{B^{\nabla_m}} (\mu_{b^{-1}}^{\square_m})^{\rho \circ L^{-1}_{\rho(t^T)}}(\rho(A))\, d(\mu_b^{\nabla_m})^{\rho}(\rho(t)).\end{aligned}$$

Since the measure

$$d\mu_b^{\nabla_m} = \bigotimes_{(k,n) \in \nabla_m} d\mu_{kn}$$

on $\mathfrak{b}^{\nabla_m}$ is equivalent to the standard Gaussian measure

$$d\mu_1^{\nabla_m}(t) = \bigotimes_{(p,q) \in \nabla_m} \frac{1}{\sqrt{\pi}} \exp(-t_{pq}^2)\, dt_{pq},$$

then the measure (3.3) is equivalent ([5], §18, Theorem 1) to the following measure:

$$\sigma_b^{\square_m}(A) = \int_{\mathfrak{b}^{\nabla_m}} (\mu_{b^{-1}}^{\square_m})^{L^{-1}_{\rho(t^T)}}(A)\, d\mu_1^{\nabla_m}(t). \tag{3.4}$$

In order to calculate the Hellinger integrals $H^m = H(\sigma_{b^1}^{\square_m}, \sigma_{b^2}^{\square_m})$ of the measures $\sigma_{b^1}^{\square_m}$ and $\sigma_{b^2}^{\square_m}$ on the space $\mathfrak{b}^{\square_m}$, we calculate the Hellinger integrals $H^{m,n} = H(\sigma_{b(1)}^{\square_{m,n}}, \sigma_{b(2)}^{\square_{m,n}})$ of the projections $\sigma_b^{\square_{m,n}}$ of the measure $\sigma_b^{\square_m}$ on the finite-dimensional subspaces $\mathfrak{b}^{\square_{m,n}}$ of the space $\mathfrak{b}^{\square_m}$, where

$$\mathfrak{b}^{\square_{m,n}} = \left\{x \in \mathfrak{b}^{\square_m} \,\middle|\, x = \sum_{(r,s) \in \square_{m,n}} x_{rs} E_{rs}\right\},$$

$$\square_{m,n} = \{(r, s) \in \square_m \mid 1 \le r \le m < s \le m + n\},$$

and make use of the fact that $\lim_{n \to \infty} H^{m,n} = H^m$.

We will show that for orthogonal measures $\mu_{b(1)}^{\square_m} \perp \mu_{b(2)}^{\square_m}$ the relation

$$H^m = \lim_{n \to \infty} H^{m,n} = 0$$

holds. Accordingly, by property (H3) of the Hellinger integral, $\sigma_{b(1)}^{\square_m} \perp \sigma_{b(2)}^{\square_m}$; this proves Lemma 3.1.

Calculation of the integrals $H^{m,n}$. We denote by $dl^{m,n}(x)$ Lebesgue measure on $\mathfrak{b}^{\square_{m,n}}$, and calculate the density $d\sigma_b^{\square_{m,n}}/dl^{m,n}$. Then, by definition, the Hellinger integral $H^{m,n}$ is equal to

$$H^{m,n} = \int_{\mathfrak{b}^{\square_{m,n}}} \left(\frac{d\sigma_{b(1)}^{\square_{m,n}}(x)}{dl^{m,n}(x)} \cdot \frac{d\sigma_{b(2)}^{\square_{m,n}}(x)}{dl^{m,n}(x)}\right)^{1/2} dl^{m,n}(x). \tag{3.5}$$

Suppose that f is a one-to-one measurable mapping $f\colon \mathbf{R}^n \to \mathbf{R}^n$, dx denotes Lebesgue measure on $\mathbf{R}^n$, and $d\mu$ is a measure equivalent to Lebesgue measure:

$$\mu(A) = \int_A g(x)\, d(x), \qquad \frac{d\mu(x)}{dx} = g(x).$$

Then

$$\mu^f(A) = \mu(f^{-1}(A)) = \int_{f^{-1}(A)} g(x)\, d(x) = \int_A g(f^{-1}(y)) \frac{df^{-1}(y)}{dy} \cdot dy, \tag{3.6}$$

in which we have made the substitution $x = f^{-1}(y)$.

Since, for any $t \in \mathfrak{b}^{\nabla_m}$, $L_{\rho(t^T)}$ is an automorphism of the space $\mathfrak{b}^{\square_{m,n}}$ for any $m, n \in \mathbf{N}$, then, for $A \in \mathscr{B}(\mathfrak{b}^{\square_{m,n}})$ we have, by formula (3.6), with $f(x) = L^{-1}_{\rho(t^T)}(x)$:

$$\begin{aligned}\sigma_b^{\square_{m,n}}(A) &= \int_{\mathfrak{b}^{\nabla_m}} (\mu_{b^{-1}}^{\square_{m,n}})^{L^{-1}_{\rho(t^T)}}(A)\, d\mu_1^{\nabla_m}(t)\\ &= \int_{\mathfrak{b}^{\nabla_m}} \int_A \left(\frac{d\mu_{b^{-1}}^{\square_{m,n}}}{dl^{m,n}}\right)(L_{\rho(t^T)}(x)) \frac{dl^{m,n}(L_{\rho(t^T)}(x))}{dl^{m,n}(x)}\, dl^{m,n}(x)\, d\mu_1^{\nabla_m}(t).\end{aligned} \tag{3.7}$$

It follows from (3.2) that the Jacobian of the mapping $z \to L_{\rho(t^T)}(x)$ is equal to unity for any $t \in \mathfrak{b}^{\nabla_m}$. Therefore (3.7) implies the following formula for the density $d\sigma_b^{\square_{m,n}}/dl^{m,n}$:

$$\begin{aligned}\left(\frac{d\sigma_b^{\square_{m,n}}}{dl^{m,n}}\right)(x) &= \int_{\mathfrak{b}^{\nabla_m}} \left(\frac{d\mu_{b^{-1}}^{\square_{m,n}}}{dl^{m,n}}\right)(L_{\rho(t^T)}(x))\, d\mu_1^{\nabla_m}(t)\\ &= \int_{\mathfrak{b}^{\nabla_m}} \left(\prod_{(r,s) \in \square_{m,n}} \sqrt{\frac{b_{rs}^{-1}}{\pi}}\right) \exp\left(-\sum_{(r,s) \in \square_{m,n}} \left(x_{rs} + \sum_{k=1}^{r-1} t_{kr} x_{ks}\right)^2\right) \bigotimes_{(p,q) \in \nabla_m}\\ &\quad \times \left(\frac{1}{\sqrt{\pi}} \exp(-t_{pq}^2)\, dt_{pq}\right) = \prod_{r=1}^{m} \psi^{r,n}(b^{-1}, x^{r,n}),\end{aligned} \tag{3.8}$$

where

$$x^{r,n} \in \mathbf{R}^r \times \mathbf{R}^n, \qquad x^{r,n} = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ & \cdots & \\ x_{r1} & \cdots & x_{rn} \end{pmatrix},$$

$$\psi^{r,n}(b^{-1}, x^{r,n}) = \left(\prod_{k=1}^{n} \sqrt{\frac{b_{rk}^{-1}}{\pi}}\right) \int_{\mathbf{R}^{r-1}} \exp\left(-\sum_{k=m+1}^{n+m} b_{rk}^{-1} \left(x_{rk} + \sum_{i=1}^{r-1} t_{ir} x_{ik}\right)^2\right)$$

$$\times \bigotimes_{i=1}^{r-1} \left(\frac{1}{\sqrt{\pi}} \exp(-t_{ir}^2)\, dt_{ir}\right). \tag{3.9}$$

Consider the matrix $x^{m,n} \in \mathbf{R}^m \times \mathbf{R}^n$ and vectors $t, x_1, \ldots, x_n \in \mathbf{R}^m$:

$$x^{m,n} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ & & \cdots & \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix}, \qquad t = \begin{bmatrix} t_1 \\ t_2 \\ \cdots \\ t_n \end{bmatrix},$$

$$x_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ \cdots \\ x_{m1} \end{bmatrix}, \quad x_2 = \begin{bmatrix} x_{12} \\ x_{22} \\ \cdots \\ x_{m2} \end{bmatrix}, \quad x_n = \begin{bmatrix} x_{1n} \\ x_{2n} \\ \cdots \\ x_{mn} \end{bmatrix}.$$

Then (3.9) may be written down in a more convenient form:

$$\psi^{m+1,n}(b^{-1}, x^{m+1,n}) = \prod_{k=1}^{n} \frac{\sqrt{b_{m+1k}^{-1}}}{\pi} \int_{\mathbf{R}^m} \exp\left(-\sum_{k=m+2}^{m+n+1} b_{m+1k}^{-1} (x_{m+1k} + (x_k, t))^2\right)$$

$$\times \frac{1}{\sqrt{\pi^m}} \exp(-\|t\|^2)\, dt.$$

By (3.8),

$$\left(\frac{d\sigma_b^{\square_{m+1,n}}}{dl^{m+1,n}}\right)(x) = \psi^{m+1,n}(b^{-1}, x^{m+1,n}) \left(\frac{d\sigma_b^{\square_{m,n}}}{dl^{m,n}}\right)(x^{m,n}) \tag{3.10}$$

Therefore, writing $a = (a_k)_{k=1}^n$, $b = (b_k)_{k=1}^n$, $a_k = (b_{m+1,m+1k}^{(1)})^{-1}$, $b_k = (b_{m+1,m+1+k}^{(2)})^{-1}$, $k = 1, \ldots, n$, we have:

$$H^{m+1,n} = \int_{b\square_{m+1,n}} \left(\psi^{m+1,n}(a, x^{m+1,n}) \left(\frac{d\sigma_{a^{-1}}^{\square_{m,n}}}{dl^{m,n}}\right)(x^{m,n})\right.$$

$$\left. \times \psi^{m+1,n}(b, x^{m+1,n}) \left(\frac{d\sigma_{b^{-1}}^{\square_{m,n}}}{dl^{m,n}}\right)(x^{m,n})\right)^{1/2} \left(\bigotimes_{k=1}^{n} dx_{m+1k} \otimes dl^{m,n}(x^{m,n})\right). \tag{3.11}$$

Write

$$G^{m,n}(a, b, x^{m,n}) = \int_{\mathbf{R}^n} (\psi^{m+1,n}(a, x^{m+1,n}) \psi^{m+1,n}(b, x^{m+1,n}))^{1/2} \bigotimes_{k=1}^{n} dx_{m+1k}$$

$$= \int_{\mathbf{R}^n} \left(\prod_{k=1}^{n} \sqrt{\frac{a_k}{\pi}} \int_{\mathbf{R}^m} \exp\left(-\sum_{k=1}^{n} a_k (x_{m+1k} + (x_k, t))^2\right)\right.$$

$$\times \frac{1}{\sqrt{\pi^m}} \exp(-\|t\|^2)\, dt \prod_{k=1}^{n} \sqrt{\frac{b_k}{\pi}} \int_{\mathbf{R}^m} \exp\left(-\sum_{k=1}^{n} b_k (x_{m+1k} + (x_k, t))^2\right)$$

$$\left. \times \frac{1}{\sqrt{\pi^m}} \exp(-\|t\|^2)\, dt\right)^{1/2} \bigotimes_{k=1}^{n} dx_{m+1k}. \tag{3.12}$$

Our goal in the calculations that follow is to obtain an explicit expression for the function $G^{m,n}(a, b, x^{m,n})$, $m, n \in \mathbf{N}$.

Lemma 3.2. *For any sequences $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$, $m, n \in \mathbf{N}$, of positive numbers, and any matrix*

$$x^{m,n} = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ x_{m1} & \cdots & x_{mn} \end{pmatrix},$$

the representation

$$G^{m,n}(a, b, x^{m,n}) = \left\{\prod_{k=1}^{n} \frac{4 a_k b_k}{(a_k + b_k)^2} \frac{\xi^{m,n}(a, x^{m,n}) \xi^{m,n}(b, x^{m,n})}{\left[\xi^{m,n}\left(\frac{2ab}{a+b}, x^{m,n}\right)\right]^2}\right\}^{1/4} \tag{3.13}$$

holds, where

$$\xi^{m,n}(a, x^{m,n}) = 1 + \sum_{r=1}^{m} \sum_{1 \le k_1 < k_2 < \cdots < k_r \le n} \prod_{i=1}^{r} a_{k_i} [x_{k_1}, x_{k_2}, \ldots, x_{k_r}], \tag{3.14}$$

$[x_1, x_2, \dots, x_r] = \det((x_i, x_j)_{i,j=1}^r)$ *is the Gram determinant of the vectors* $x_1, x_2, \dots, x_r \in \mathbf{R}^m$, *and* $2ab/(a+b)$ *is the sequence*

$$\left(\frac{2ab}{a+b}\right)_k = \frac{2a_k b_k}{a_k + b_k}.$$

Now we state two more lemmas needed in the proof of Lemma 3.1.

Lemma 3.4. *The estimate*

$$\frac{\xi^{m,n}(a, x^{m,n})\xi^{m,n}(b, x^{m,n})}{\left[\xi^{m,n}\left(\frac{2ab}{a+b}, x^{m,n}\right)\right]^2} \le 2^{2m+1} \prod\nolimits^{2m,n}(a, b) \tag{3.15}$$

holds, where

$$\prod\nolimits^{p,n}(a, b) = \max\left\{\sum_{i=1}^{p} \frac{(a_{k_i} + b_{k_i})^2}{4a_{k_i} b_{k_i}} \,\middle|\, 1 \leqslant k_1 < k_2 < \cdots < k_p \le n\right\}, \quad 1 \le p \leqslant n.$$

Lemma 3.5. *If the product* $\prod = \prod_{k=1}^{\infty} m_k = \infty$ *of positive numbers* $m_k \ge 1$ *diverges, then, for any* $p \in \mathbf{N}$, $\lim_{n\to\infty} \prod^{p,n}/\prod^{n,n} = 0$, *where*

$$\prod\nolimits^{p,n} = \max\left\{\prod_{i=1}^{p} m_{k_i} \,\middle|\, 1 \le k_1 < k_2 < \cdots < k_p \le n\right\}, \quad \prod\nolimits^{n,n} = \prod_{k=1}^{n} m_k.$$

Suppose that Lemmas 3.3–3.5 hold. Lemma 3.1 follows from them.

Proof of Lemma 3.1. We will show from its opposite, i.e. from the condition

$$\mu_{b(1)}^{\square_{m+1}} \perp \mu_{b(2)}^{\square_{m+2}}, \tag{3.16}$$

that $H^{m+1} = \lim_{n\to\infty} H^{m+1,n} = 0$, $m = 0, 1, \dots$. Condition (3.16) is equivalent (see (3.1)) to the condition

$$\prod_{(k,n)\in\square_{m+1}} \frac{(b_{kn}^{(1)} + b_{kn}^{(2)})^2}{4b_{kn}^{(1)} b_{kn}^{(2)}} = \infty. \tag{3.17}$$

Suppose that $m = 0$, and $\mu_{b(1)}^{\square_1} \perp \mu_{b(2)}^{\square_1}$, i.e.

$$\prod_{k=2}^{\infty} \frac{(b_{1k}^{(1)} + b_{1k}^{(2)})^2}{4b_{1k}^{(1)} b_{1k}^{(2)}} = \infty.$$

We will show that then $\sigma_{b(1)}^{\square_1} \perp \sigma_{b(2)}^{\square_1}$. Indeed,

$$\sigma_{b(1)}^{\square_1} = \sigma\{iA_{1k}^R = \partial_{1k} - b_{1k}x_{1k} \mid 1 < k\}_{\mathscr{H}^{\square_1(b)}} \sim \sigma\{x_{1k} \mid 1 < k\}_{\mathscr{H}^{\square_1(b^{-1})}},$$

since the partial Fourier transform $\mathscr{F}_1$ carries the $i(\partial_{1k} - b_{1k}x_{1k})$ into operators of multiplication by x_{1k} (see (3.2)). Accordingly,

$$\sigma_{b(1)}^{\square_1} \perp \sigma_{b(2)}^{\square_1} \Leftrightarrow \mu_{(b(1))^{-1}}^{\square_1} \perp \mu_{(b(2))^{-1}}^{\square_1} \Leftrightarrow \prod_{k=2}^{\infty} \frac{((b_{1k}^{(1)})^{-1} + (b_{1k}^{(2)})^{-1})^2}{4(b_{1k}^{(1)} b_{1k}^{(2)})^{-1}}$$

$$= \prod_{k=2}^{\infty} \frac{(b_{1k}^{(1)} + b_{1k}^{(2)})^2}{4b_{1k}^{(1)} b_{1k}^{(2)}} = \infty \Leftrightarrow \mu_{b(1)}^{\square_1} \perp \mu_{b(2)}^{\square_1}.$$

Suppose that (3.17) holds. Write

$$p = \max_{1 \le k \le m+1} \left\{k \,\middle|\, \prod_{n=m+2}^{\infty} \frac{(b_{kn}^{(1)} + b_{kn}^{(2)})^2}{4b_{kn}^{(1)} b_{kn}^{(2)}} = \infty\right\}.$$

Then, in view of Lemmas 3.3 and 3.4 and formulas (3.11) and (3.12), we get

$$H^{m+1,n} \le \left\{2^{2m-1} \frac{\prod^{2m,n}(a^{(m+1)}, b^{(m+1)})}{\prod^{n,n}(a^{(m+1)}, b^{(m+1)})}\right\}^{1/4} H^{m,n}.$$

Applying that inequality sufficiently many times, we get

$$H^{m+1,n} \le \left\{\prod_{k=p}^{m+1} 2^{2k-3} \frac{\prod^{2k-2,n}(a^{(k)}, b^{(k)})}{\prod^{n,n}(a^{(k)}, b^{(k)})}\right\}^{1/4} H^{p-1,n}, \tag{3.18}$$

where

$$a^{(k)} = (a_r^{(k)})_{r=1}^n, \quad a_r^{(k)} = (b_{k,m+1+r}^{(1)})^{-1}, \quad b^{(k)} = (b_r^{(k)})_{r=1}^n, \quad b_r^{(k)} = (b_{k,m+1+r}^{(r)})^{-1}.$$

Since $\lim_{n\to\infty} \prod^{n,n}(a^{(k)}, b^{(k)}) < \infty$ for $p < k \le m+1$ and $\lim_{n\to\infty} \prod^{n,n}(a^{(p)}, b^{(p)}) = \infty$, then, by Lemma 3.5,

$$\lim_{n\to\infty} \prod\nolimits^{2k-2,n}(a^{(k)}, b^{(k)}) \Big/ \prod\nolimits^{n,n}(a^{(k)}, b^{(k)}) < \infty, \qquad p < k \le m+1,$$

and

$$\lim_{n\to\infty} \frac{\prod^{2p-2,n}(a^{(p)}, b^{(p)})}{\prod^{n,n}(a^{(p)}, b^{(p)})} = 0.$$

By property (H1) of the Hellinger integral, $0 \le H^p = \lim_{n\to\infty} H^{p,n} \leqslant 1$. Therefore we finally find from (3.18) that

$$H^{m+1} = \lim_{n\to\infty} H^{m+1,n} = 0.$$

This completes the proof of Lemma 3.1.

§4. Proofs of Lemmas 3.2–3.5

Proof of Lemma 3.5. If all $m_k \le C$, then $\prod^{p,n} \le C^p$, so that $\prod^{p,n} / \prod^{n,n} \le C^p / \prod^{n,n} \to 0$ as $n \to \infty$. In the contrary case there exists an infinite sequence $(k(n))_{n=1}^{\infty}$, $k(1) = 1$, $k(2) = \min\{k \mid m_k > m_1\}, \ldots, k(n+1) = \min\{k \mid m_k > m_{k(n)}\}$, for which

$$\lim_{n\to\infty} m_{k(n)} = \infty, \qquad m_{k(n)} < m_{k(n+1)}, \quad n \in \mathbf{N}.$$

For $r, n \in \mathbf{N}$, $r \leqslant p$, we denote by $k(r, n) \in \mathbf{N}$ numbers such that

$$k(p,n) < k(p-1,n) < \cdots < k(1,n), \qquad \prod_{r=1}^{p} m_{k(r,n)} = \prod\nolimits^{p,n}.$$

Suppose that $n \in [k(r), k(r+1)) \cap \mathbf{N}$. Then $k(1,n) \ge k(r), k(2,n) \ge k(r-1), \ldots, k(p,n) \ge k(r-p+1)$, so that

$$\prod\nolimits^{p,n} \Big/ \prod\nolimits^{n,n} = m_{k(1,n)} m_{k(2,n)} \cdots m_{k(p,n)} \Big/ \prod_{k=1}^{n} m_k$$
$$\le \Big(\prod_{k=1}^{n} m_k\Big)^{-1} \le \Big(\prod_{k=1}^{k(r-p+1)-1} m_k\Big)^{-1} \to 0 \quad \text{if } n \to \infty,$$
$$k \ne k(1,n), \ldots, k(p,n).$$

Proof of Lemma 3.4. We rewrite inequality (3.15) in the following form:

$$\Big(1 + \sum_{r=1}^{m} \sum_{1 \le k_1 < \cdots < k_r \le n} \prod_{i=1}^{r} a_{k_i} [x_{k_1}, \ldots, x_{k_r}]\Big)$$
$$\times \Big(1 + \sum_{s=1}^{m} \sum_{1 \le q_1 < \cdots < q_s \leqslant n} \prod_{i=1}^{s} b_{q_i} [x_{q_1}, \ldots, x_{q_s}]\Big)$$
$$\le C \Big(1 + \sum_{t=1}^{m} \sum_{1 \le p_1 < \cdots < p_t \le n} \prod_{i=1}^{t} \frac{2a_{p_i} b_{p_i}}{a_{p_i} + b_{p_i}} [x_{p_1}, \ldots, x_{p_t}]\Big)^2. \quad (3.19)$$

Then inequality (3.19) follows from the following system of inequalities, obtained by comparing the coefficients on $[x_{k_1}, \ldots, x_{k_r}]$, $[x_{k_1}, \ldots, x_{k_r}] \times [x_{q_1}, \ldots, x_{q_s}]$, $1 \le r, s \le n$, in the right and left hand sides of (3.19).

$$1 \leqslant C \cdot 1;$$
$$[x_{k_1}, \ldots, x_{k_r}]: \prod_{i=1}^{r} a_{k_i} + \prod_{i=1}^{r} b_{k_i} \le C \cdot 2 \prod_{i=1}^{r} \frac{2a_{k_i} b_{k_i}}{a_{k_i} + b_{k_i}}, \qquad 1 \le r \le n,$$
$$[x_{k_1}, \ldots, x_{k_r}] \cdot [x_{q_1}, \ldots, x_{q_s}]: \prod_{i=1}^{r} a_{k_i} \prod_{t=1}^{s} b_{q_t} + \prod_{t=1}^{s} a_{q_t} \prod_{i=1}^{r} b_{k_i}$$
$$\le C \cdot 2 \prod_{i=1}^{r} \frac{2a_{k_i} b_{k_i}}{a_{k_i} + b_{k_i}} \cdot \prod_{t=1}^{s} \frac{2a_{q_t} b_{q_t}}{a_{q_t} + b_{q_t}}, \qquad 1 \le r, s \le n. \quad (3.20)$$

Now we relax the inequalities (3.20):

$$\prod_{i=1}^{r} a_{k_i} + \prod_{i=1}^{r} b_{k_i} \le \prod_{i=1}^{r} (a_{k_i} + b_{k_i}) \le C \cdot 2 \prod_{i=1}^{r} \frac{2a_{k_i} b_{k_i}}{a_{k_i} + b_{k_i}}, \qquad 1 \le r \le n,$$
$$\prod_{i=1}^{r} a_{k_i} \prod_{t=1}^{s} b_{q_t} + \prod_{t=1}^{s} a_{q_t} \cdot \prod_{i=1}^{r} b_{k_i} < \prod_{i=1}^{r} (a_{k_i} + b_{k_i}) \prod_{t=1}^{s} (a_{q_t} + b_{q_t})$$
$$\le C \cdot 2 \prod_{i=1}^{r} \frac{2a_{k_i} b_{k_i}}{a_{k_i} + b_{k_i}} \cdot \prod_{t=1}^{s} \frac{2a_{q_t} \cdot b_{q_t}}{a_{q_t} + b_{q_t}}, \qquad 1 \le r, \quad s \le n. \quad (3.21)$$

From (3.21) we get

$$C = \max\Big\{2^{r-1} \prod_{i=1}^{r} \frac{(a_{k_i} + b_{k_i})}{4a_{k_i} b_{k_i}}, 2^{r+s-1} \prod_{i=1}^{r} \frac{(a_{k_i} + b_{k_i})^2}{4a_{k_i} b_{k_i}} \prod_{t=1}^{s} \frac{(a_{q_t} + b_{q_t})^2}{4a_{q_t} \cdot b_{q_t}} \,\Big|\, 1 \le r, s \le n\Big\}$$
$$= \max\Big\{2^{2m-1} \prod_{i=1}^{2m} \frac{(a_{k_i} + b_{k_i})^2}{4a_{k_i} b_{k_i}} \,\Big|\, 1 \le k_1 < \cdots < k_{2m} \le n\Big\} = 2^{2m-1} \prod\nolimits^{2m,n};$$

this proves Lemma 3.4.

Proof of Lemma 3.2. Suppose that $T^{R,b^{(1)}}$ and $T^{R,b^{(2)}}$ are equivalent irreducible unitary representations, $U: \mathscr{H}(b^{(1)}) \to \mathscr{H}(b^{(2)})$ is their intertwining operator: $UT_t^{R,b^{(1)}} = T_t^{R,b^{(2)}} U$, $t \in B_0^{\infty}$. We will show that then

$$Ux_{kn} = x_{kn} U, \qquad k, n \in \mathbf{N}, \quad k < n. \quad (3.22)$$

Indeed, by Lemma 3.1, then for any $m \in \mathbf{N}$, $\mu_{b^{(1)}}^{\square_m} \sim \mu_{b^{(2)}}^{\square_m}$, i.e.

$$\prod_{(k,n) \in \square_m} \frac{(b_{kn}^{(1)} + b_{kn}^{(2)})^2}{4 b_{kn}^{(1)} b_{kn}^{(2)}} < \infty. \quad (3.23)$$

We know from the proof of Lemma 2.2 that $\sum_{n=N_1}^{N_2} t_n(b^{(1)})A_{1n}^{R,b^{(1)}}A_{2n}^{R,b^{(1)}} \to x_{12}$ in $\mathscr{H}(b^{(1)})$, where $\gamma_n = (b_{1n}^{(1)})^2 + b_{1n}^{(1)}b_{2n}^{(1)}$ and

$$t_n = -2b_{1n}^{(1)}\left(\gamma_n \sum_{k=N_1}^{N_2}(b_{1k}^{(1)})^2\gamma_k^{-1}\right)^{-1} = -2\left((b_{1n}^{(1)}+b_{2n}^{(1)})\cdot \sum_{k=N_1}^{N_2} b_{1k}^{(1)}(b_{1k}^{(1)}+b_{2k}^{(1)})^{-1}\right)^{-1} \tag{3.24}$$

$$\begin{aligned}\|\omega\|_{12}(b^{(1)})\|^2_{\mathscr{H}(b^{(1)})} &= \left\|\left(\sum_{n=N_1}^{N_2} t_n(b^{(1)})A_{1n}^{R,b^{(1)}}A_{2n}^{R,b^{(2)}} + \sum_{n=N_1}^{N_2} t_n(b^{(1)})\tfrac{1}{2}b_{1n}^{(1)}x_{12}\right)\mathbf{1}\right\|^2\\ &= \sum_{n=N_1}^{N_2} t_n^2(b^{(1)})\tfrac{1}{4}[(b_{1n}^{(1)})^2(b_{12}^{(1)})^{-1} + b_{1n}^{(1)}b_{2n}^{(1)}]\\ &\asymp \sum_{n=N_1}^{N_2} t_n^2(b^{(1)})[(b_{1n}^{(1)})^{(2)} + b_{1n}^{(1)}b_{2n}^{(1)}]\\ &= 4\left(\sum_{n=N_1}^{N_2} b_{1n}^{(1)}(b_{1n}^{(1)}+b_{2n}^{(1)})^{-1}\right)^{-1}.\end{aligned} \tag{3.25}$$

We will show that $U(\sum_{n=N_1}^{N_2} t_n(b^{(1)})A_{1n}^{R,b^{(1)}}A_{2n}^{R,b^{(2)}}) \to x_{12}$ in $\mathscr{H}(b^{(2)})$. As in the proof of Theorem 2.1, for this it suffices to show the convergence of the operators in question on the unit vector $\mathbf{1} \in \mathscr{H}(b^{(2)})$. We have

$$\begin{aligned}\|\hat\omega\|_{12}(b^{(1)})\|^2_{\mathscr{H}(b^{(2)})} &= \left\|\left(\sum_{n=N_1}^{N_2} t_n(b^{(1)})A_{1n}^{R,b^{(2)}}A_{2n}^{R,b^{(2)}} + \sum_{n=N_1}^{N_2} t_n(b^{(1)})\tfrac{1}{2}b_{1n}^{(2)}x_{1n}\right)\mathbf{1}\right\|^2_{\mathscr{H}(b^{(2)})}\\ &= \sum_{n=N_1}^{N_2} t_n^2(b^{(1)})\tfrac{1}{4}[(b_{1n}^{(2)})^2(b_{12}^{(2)})^{-1} + b_{1n}^{(2)}b_{2n}^{(2)}] \asymp \sum_{n=N_1}^{N_2} t_n^2(b^{(1)})[(b_{1n}^{(2)})^2 + b_{1n}^{(2)}b_{2n}^{(2)}]\\ &= 4\left(\sum_{n=N_1}^{N_2} b_{1n}^{(1)}(b_{1n}^{(1)}+b_{2n}^{(1)})^{-1}\right)^{-2}\sum_{n=N_1}^{N_2}\frac{b_{1n}^{(1)}}{b_{1n}^{(1)}+b_{2n}^{(1)}}\cdot\frac{b_{1n}^{(2)}}{b_{1n}^{(1)}}\cdot\frac{b_{1n}^{(2)}+b_{2n}^{(2)}}{b_{1n}^{(1)}+b_{2n}^{(1)}}.\end{aligned} \tag{3.26}$$

We will show that

$$\lim_{n\to\infty}\frac{b_{1n}^{(2)}}{b_{1n}^{(1)}}\frac{b_{1n}^{(2)}+b_{2n}^{(2)}}{b_{1n}^{(1)}+b_{2n}^{(1)}} = 1.$$

Indeed, condition (3.23) is equivalent to the convergence of the series

$$\sum_{(k,n)\in\square_m}\frac{(b_{kn}^{(1)}-b_{kn}^{(2)})^2}{b_{kn}^{(1)}b_{kn}^{(2)}} < \infty,$$

which is equivalent to the convergence of the series

$$\sum_{k,n\in\square_m}(b_{kn}^{(2)}/b_{kn}^{(1)}-1)^2 < \infty.$$

It follows from this that for any $m \in \mathbf{N}$ and $\varepsilon > 0$ there exists of an $n_0 \in \mathbf{N}$ such that

$$|b_{kn}^{(2)}/b_{kn}^{(1)} - 1| < \varepsilon, \qquad k = 1, 2, \ldots, m, \quad n \geq n_0.$$

Therefore

$$\left|\sum_{k=1}^{m} b_{kn}^{(2)}\left(\sum_{k=1}^{m} b_{kn}^{(1)}\right)^{-1} - 1\right| \leq \sum_{k=1}^{m}|b_{kn}^{(2)} - b_{kn}^{(1)}|\left(\sum_{k=1}^{m} b_{kn}^{(1)}\right)^{-1} \leq m\varepsilon.$$

Accordingly

$$\lim_{n\to\infty}\sum_{k=1}^{m} b_{kn}^{(2)}\left(\sum_{k=1}^{m} b_{kn}^{(1)}\right)^{-1} = 1, \qquad m \in \mathbf{N}. \tag{3.27}$$

(3.27) implies an estimate for the right side of (3.26):

$$\|\hat\omega_{12}(b^{(1)})\|^2_{\mathscr{H}(b^{(2)})} \leq C\left(\sum_{n=N_1}^{N_2} b_{1n}^{(1)}(b_{1n}^{(1)}+b_{2n}^{(1)})^{-1}\right)^{-1},$$

and this is small for appropriate N_1 and N_2. Here we take account of the fact that

$$S_{12}^{L}(b^{(1)}) = \sum_{n=3}^{\infty} b_{1n}^{(1)}(b_{2n}^{(1)})^{-1} = \infty$$

(see (1.3)).

We will write $(N_1, N_2) \to \infty$, $(N_1, N_2) = (N_1(p), N_2(p))$, if

$$\lim_{p\to\infty}\sum_{N_1(p)}^{N_2(p)} b_{1n}^{(1)}(b_{1n}^{(1)}+b_{2n}^{(1)})^{-1} = \infty.$$

In the left side of the expression (3.26) there is the expression

$$\frac{1}{2}\sum_{n=N_1}^{N_2} t_n(b^{(1)})b_{1n}^{(2)}x_{12}.$$

We will show that

$$\lim_{(N_1,N_2)\to\infty}\frac{1}{2}\sum_{n=N_1}^{N_2} t_n(b^{(1)})b_{1n}^{(2)} = -1.$$

Indeed,

$$\begin{aligned}&\lim_{(N_1,N_2)\to\infty}\frac{1}{2}\sum_{n=N_1}^{N_2} t_n(b^{(1)})b_{1n}^{(2)}\\ &\quad= -\lim_{(N_1,N_2)\to\infty}\left(\sum_{n=N_1}^{N_2}\frac{b_{1n}^{(1)}}{b_{1n}^{(1)}+b_{2n}^{(1)}}\right)^{-1}\sum_{n=N_1}^{N_2}\frac{b_{1n}^{(1)}}{b_{1n}^{(1)}+b_{2n}^{(1)}}\cdot\frac{b_{1n}^{(2)}}{b_{1n}^{(1)}} = -1,\end{aligned}$$

since

$$\left|\sum_{n=N_1}^{N_2} a_n b_n \left(\sum_{n=N_1}^{N_2} a_n\right)^{-1} - 1\right| \le \left|\sum_{n=N_1}^{N_2} a_n(b_n-1)\right|\left(\sum_{n=N_1}^{N_2} a_n\right)^{-1}$$
$$\le \max_{N_1\le n\le N_2}|b_n-1|\to 0$$

as $N_1\to\infty$. Here we have written

$$a_n = b_{1n}^{(1)}(b_{1n}^{(1)}+b_{2n}^{(1)})^{-1}, \qquad b_n = b_{1n}^{(2)}(b_{1n}^{(1)})^{-1},$$

and used the fact that

$$\lim_{n\to\infty} b_{1n}^{(2)}(b_{1n}^{(1)})^{-1}=1.$$

Thus, in the left side of (3.26),

$$\tfrac12\sum_{n=N_1}^{N_2} t_n(b^{(1)}) b_{1n}^{(2)} x_{12} \to -x_{12}.$$

Accordingly

$$U\left(\sum_{n=N_1}^{N_2} t_n(b^{(1)})A_{1n}^{R,b^{(1)}}A_{2n}^{R,b^{(1)}}\right) = \sum_{n=N_2}^{N_2} t_n(b^{(1)}) A_{1n}^{R,b^{(2)}}A_{2n}^{R,b^{(2)}} \to x_{12}$$

in $\mathscr{H}(b^{(2)})$. Hence $Ux_{12}=x_{12}U$.

Analogously we show that $Ux_{lk}=x_{lk}U$, $l<k$. Indeed, by Lemma 2.4, x_{lk} is approximated by the operators

$$\sum_{n=N_1}^{N_2} t_n(b^{(1)})(\partial_{ln} - b_{ln}x_{ln})A_{kn}^{R,b^{(1)}}$$

and

$$\|\omega_{lk}(b^{(1)})\|^2_{\mathscr{H}(b^{(1)})} = \left\|\left(\sum_{n=N_1}^{N_2} t_n(b^{(1)})(\partial_{ln}-b_{ln}^{(1)}x_{ln})A_{kn}^{R,b^{(1)}} - x_{lk}\right)\mathbf{1}\right\|^2_{\mathscr{H}(b^{(1)})}$$
$$\asymp \sum_{n=N_1}^{N_2} t_n^2(b^{(1)})\sum_{m=1}^k b_{ln}^{(1)}b_{mn}^{(1)} = 4\left(\sum_{n=N_1}^{N_2} b_{ln}^{(1)}\left(\sum_{m=1}^k b_{mn}^{(1)}\right)^{-1}\right)^{-1}\to 0,$$

where

$$t_n(b^{(1)}) = -2b_{ln}^{(1)}\left(\sum_{m=1}^k b_{ln}^{(1)}b_{mn}^{(1)}\cdot\sum_{n=N_1}^{N_2}(b_{ln}^{(1)})^2\left(\sum_{m=1}^k b_{ln}^{(1)}b_{mn}^{(1)}\right)^{-1}\right)^{-1}$$
$$= -2\left(\sum_{m=1}^k b_{mn}^{(1)}\sum_{n=N_1}^{N_2} b_{ln}^{(1)}\left(\sum_{m=1}^k b_{mn}^{(1)}\right)^{-1}\right)^{-1}.$$

We will show that

$$U\left(\sum_{n=N_1}^{N_2} t_n(b^{(1)})(\partial_{ln} - b_{ln}^{(1)}x_{ln})A_{kn}^{R,b^{(1)}}\right)\to x_{lk}$$

in $\mathscr{H}(b^{(2)})$. Indeed,

$$\|\hat\omega_{lk}(b^{(1)})\|^2_{\mathscr{H}(b^{(2)})} = \left\|\left(\sum_{n=N_1}^{N_2} t_n(b^{(1)})(\partial_{ln}-b_{ln}^{(2)}x_{ln})A_{kn}^{R,b^{(2)}} + \tfrac12\sum_{n=N_1}^{N_2} t_n(b^{(1)})b_{ln}^{(2)}x_{lk}\right)\mathbf{1}\right\|^2_{\mathscr{H}(b^{(2)})}$$
$$\asymp \sum_{n=N_1}^{N_2} t_n^2(b^{(1)})\sum_{m=1}^k b_{ln}^{(2)}b_{mn}^{(2)} = 4\left(\sum_{n=N_1}^{N_2} b_{ln}^{(1)}\left(\sum_{m=1}^k b_{mn}^{(1)}\right)^{-1}\right)^{-2}$$
$$\times \sum_{n=N_1}^{N_2} b_{ln}^{(2)}\sum_{m=1}^k b_{mn}^{(2)}\left(\sum_{m=1}^k b_{mn}^{(1)}\right)^{-2} = 4\left(\sum_{n=N_1}^{N_2} b_{ln}^{(1)}\left(\sum_{m=1}^k b_{mn}^{(1)}\right)^{-1}\right)^{-2}$$
$$\times\sum_{n=N_1}^{N_2} b_{ln}^{(1)}\left(\sum_{m=1}^k b_{mn}^{(1)}\right)^{-1} b_{ln}^{(2)}(b_{ln}^{(1)})^{-1}\sum_{m=1}^k b_{mn}^{(2)}\left(\sum_{m=1}^k b_{mn}^{(1)}\right)^{-1}.$$

Using (3.27), we get the estimate

$$\|\hat\omega_{lk}(b^{(1)})\|^2_{\mathscr{H}(b^{(2)})} \le C\left(\sum_{n=N_1}^{N_2} b_{ln}^{(1)}\left(\sum_{m=1}^k b_{mn}^{(1)}\right)^{-1}\right)^{-1},$$

which is small for appropriate N_1, N_2 (see the proof of Lemma 2.4). The equation

$$\lim_{(N_1,N_2)\to\infty}\tfrac12\sum_{n=N_1}^{N_2} t_n(b^{(1)})b_{ln}^{(2)}$$
$$= -\lim_{(N_1,N_2)\to\infty}\left(\sum_{n=N_1}^{N_2}\frac{b_{ln}^{(1)}}{\sum_{m=1}^k b_{mn}^{(1)}}\right)^{-1}\sum_{n=N_1}^{N_2}\frac{b_{ln}^{(1)}}{\sum_{m=1}^k b_{mn}^{(1)}}\cdot\frac{b_{ln}^{(2)}}{b_{ln}^{(1)}} = -1$$

completes the proof of Lemma 3.2.

Proof of Lemma 3.5. We note that the function $G^{m,n}(a,b,X^{m,n})$, $X^{m,n}\in\mathbf{R}^m\times\mathbf{R}^n$, is invariant relative to $O(m)$, the orthogonal group of space $\mathbf{R}^m$. Indeed,

$$G^{m,n}(a,b;OX^{m,n}) = G^{m,n}(a,b,Ox_1,\dots,Ox_n)$$
$$= \int_{\mathbf{R}^n}\left(\prod_{k=1}^n\sqrt{\frac{a_k}{\pi}}\int_{\mathbf{R}^m}\exp\left(-\sum_{k=1}^n a_k(x_{m+1k}+(Ox_k,t))^2\right)\frac{1}{\sqrt{\pi^m}}\exp(-\|t\|^2)\,dt\right.$$
$$\left.\times\prod_{k=1}^n\sqrt{\frac{b_k}{\pi}}\int_{\mathbf{R}^m}\exp\left(-\sum_{k=1}^n b_k(x_{m+1k}+(Ox_k,t))^2\right)\frac{1}{\sqrt{\pi^m}}\exp(-\|t\|^2)\,dt\right)^{1/2}$$
$$\times\bigotimes_{k=1}^n dx_{m+1k} = G^{m,n}(a,b;x_1,\dots,x_n) = G^{m,n}(a,b,X^{m,n}). \tag{3.28}$$

It suffices to prove equation (3.13) for $G^{n,n+1}$, $n \in \mathbf{N}$. Indeed, for $m < n$, equation (3.13) for $G^{m,n}$ is a special case of the equation for $G^{n-1,n}$, since for $x_1, \ldots, x_n \in \mathbf{R}^m \subset \mathbf{R}^n$,

$$[x_{k_1}, \ldots, x_{k_r}] = \det((x_{k_i}, x_{k_j})_{i,j=1}^r) = 0$$

for $r > m$.

We will need some formulas. It is well known that for $A \in \operatorname{End}(\mathbf{R}^k)$, $A > 0$,

$$\frac{1}{\sqrt{\pi^k}} \int_{\mathbf{R}^k} \exp(-(Ax, x))\, dx = \frac{1}{\sqrt{\det A}}. \tag{3.29}$$

Suppose that

$$m, n \in \mathbf{N}, \quad c_1, c_2, \ldots, c_n \in \mathbf{R}^m, \quad d \in \mathbf{R}^m, \quad c_k = (c_{pk})_{p=1}^m, \quad k = 1, \ldots, n,$$
$$C = (c_{pk}) \in \mathbf{R}^m \times \mathbf{R}^n, \quad \tilde{c}_1, \ldots, \tilde{c}_m \in \mathbf{R}^n, \quad \tilde{c}_p = (c_{pk})_{k=1}^n.$$

Then

$$\nu(c_1, \ldots, c_n, d) = \frac{1}{\sqrt{\pi^m}} \int_{\mathbf{R}^m} \exp\left(-\sum_{k=1}^n (c_k, t)^2 - (d, t)\right) \exp(-(t, t))\, dt$$
$$= \frac{1}{\sqrt{\det(I + A(C))}} \exp(\tfrac{1}{4}((I + A(C))^{-1} d, d)), \tag{3.30}$$

where $A(C) = (A_{ij}(C))_{i,j=1}^m$, $A_{ij}(C) = (\tilde{c}_i, \tilde{c}_j)_{\mathbf{R}^n}$.

Indeed, we make a substitution $t = T + T_0$ in the left side of (3.30) such that the terms linear in T vanish.

Since

$$\sum_{k=1}^n (c_k, t)^2 + (t, t) = ((I + A(C))t, t),$$

then

$$\sum_{k=1}^n (c_k, T + T_0)^2 + (d, T + T_0) + (T + T_0, T + T_0)$$
$$= ((I + A(C))(T + T_0), T + T_0) + (d, T + T_0)$$
$$= ((I + A(C))T, T) + 2((I + A(C))T_0, T) + (d, T)$$
$$+ ((I + A(C))T_0, T_0) + (d, T_0).$$

We choose T_0 so that $(2(I + A(C))T_0 + d, T) = 0$, i.e. $T_0 = -(I + A(C))^{-1} d/2$. Then

$$((I + A(C))T_0, T_0) + (d, T_0) = \left(\frac{d}{2}, T_0\right) = -\tfrac{1}{4}((I + A(C))^{-1} d, d).$$

Therefore

$$\nu(c_1, \ldots, c_n, d) = \frac{1}{\sqrt{\pi^m}} \int_{\mathbf{R}^m} \exp(-((I + A(C))T, T))\, dT \cdot \exp(\tfrac{1}{4}((I + A(C))^{-1} d, d)),$$

so that formula (3.30) follows from formula (3.29).

We now use formula (3.30) in order to transform $G^{m,n}$:

$$G^{m,n}(a, b, X) = \frac{\left(\prod_{k=1}^n a_k b_k\right)^{1/4}}{\sqrt{\pi^n}} \int_{\mathbf{R}^n} \left(\left(\int_{\mathbf{R}^m} \exp\left(-\sum_{k=1}^n a_k (x_{m+1k} + (x_k, t))^2\right)\right.\right.$$
$$\times \frac{1}{\sqrt{\pi^m}} \exp(-(t, t))\, dt \int_{\mathbf{R}^m} \exp\left(-\sum_{k=1}^n b_k (x_{m+1k} + (x_k, t))^2\right)$$
$$\left.\times \frac{1}{\sqrt{\pi^m}} \exp(-(t, t))\, dt\right)^{1/2} \bigotimes_{k=1}^n dx_{m+1k}$$
$$= \frac{\left(\prod_{k=1}^n a_k b_k\right)^{1/4}}{\sqrt{\pi^n}} \int_{\mathbf{R}^n} \left(\left(\int_{\mathbf{R}^m} \exp\left(-\sum_{k=1}^n a_k (x_k, t)^2 - 2\sum_{k=1}^n a_k x_{m+1k} (x_k, t)\right)\right.\right.$$
$$\times \frac{1}{\sqrt{\pi^m}} \exp(-(t, t))\, dt \int_{\mathbf{R}^m} \exp\left(-\sum_{k=1}^n b_k (x_k, t)^2 - 2\sum_{k=1}^n b_k x_{m+1k} (x_k, t)\right)$$
$$\left.\times \frac{1}{\sqrt{\pi^m}} \exp(-(t, t))\, dt\right)^{1/2} \bigotimes_{k=1}^n \exp\left(-\frac{a_k + b_k}{2} x_{m+1k}^2\right) dx_{m+1k}.$$

Now we make the substitutions

$$\frac{a_k + b_k}{2} x_{m+1k}^2 = z_k^2, \quad x_{m+1k} = \frac{\sqrt{2} z_k}{\sqrt{a_k + b_k}}, \quad x_k(a) = x_k \sqrt{a_k}, \quad x_k(b) = x_k \sqrt{b_k},$$

$$d(a) = (d_p(a))_{p=1}^m, \quad d_p(a) = \sum_{k=1}^n \frac{a_k x_{pk} z_k}{\sqrt{a_k + b_k}} = (\hat{x}_p(a), z), \quad \hat{x}_p(a) = (\hat{x}_{pk}(a))_{k=1}^n,$$

$$\hat{x}_{pk}(a) = \frac{a_k x_{pk}}{\sqrt{a_k + b_k}}, \quad d(b) = (d_p(b))_{p=1}^n, \quad d_p(b) = \sum_{k=1}^n \frac{b_k x_{pk} z_k}{\sqrt{a_k + b_k}} = (\hat{x}_p(b), z),$$

$$\hat{x}_p(a) = (\hat{x}_{pk}(b))_{k=1}^n, \quad \hat{x}_{pk}(b) = \frac{b_k x_{pk}}{\sqrt{a_k + b_k}}.$$

We get

$$G^{m,n}(a,b,X)=\left\{\prod_{k=1}^{n}\frac{4a_kb_k}{(a_k+b_k)^2}\right\}^{1/4}\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^m}\exp\left(-\sum_{p=1}^{n}(x_p(a),t)^2-(2\sqrt{2}d(a),t)\right)\right.$$

$$\times\frac{1}{\sqrt{\pi^m}}\exp(-(t,t)\,dt\cdot\int_{\mathbf{R}^m}\exp\left(-\sum_{p=1}^{n}(x_p(b),t)^2-(2\sqrt{2}\,d_p(b),t)\right)$$

$$\left.\times\frac{1}{\sqrt{\pi^m}}\exp(-(t,t))\,dt\right)^{1/2}\frac{1}{\sqrt{\pi^n}}\exp(-(z,z))\,dz$$

$$=\left\{\prod_{k=1}^{n}\frac{4a_kb_k}{(a_k+b_k)^2}\frac{1}{\det(I+A(X(a)))\det(I+A(X(b)))}\right\}^{1/4}$$

$$\times\int_{\mathbf{R}^n}\exp\{((I+A(X(a)))^{-1}d(a),d(a))+((I+A(X(b)))^{-1}d(b),d(b)\}$$

$$\times\frac{1}{\sqrt{\pi^n}}\exp(-(z,z))\,dz. \tag{3.31}$$

Denote by $R_{pq}(a)$, $A_{pq}(a)$, $1\le p,q\le n$, respectively, the elements of the matrix $(1+A(X(a)))^{-1}$ and the cofactors of the matrix $(1+A(X(a)))^{-1}$. Then

$$R_{pq}(a)=\frac{A_{pq}}{\det(I+A(X(a)))}.$$

Therefore

$$((I+A(X(a)))^{-1}d(a),d(a))+((I+A(X(b)))^{-1}d(b),d(b))$$

$$=\sum_{p,q=1}^{m}(R_{pq}(a)(\hat{x}_p(a),z)(\hat{x}_q(a),z)+R_{pq}(\hat{x}_p(b),z)(\hat{x}_q(b),z))$$

$$=\sum_{p,q=1}^{n}\left(R_{pq}(a)\sum_{i,j=1}^{n}\hat{x}_{pi}(a)z_i\hat{x}_{qj}(a)z_j+R_{pq}(b)\sum_{i,j=1}^{n}\hat{x}_{pi}(b)z_i\hat{x}_{qj}(b)z_j\right)$$

$$=\sum_{i,j=1}^{n}\left(\sum_{p,q=1}^{n}R_{pq}(a)\hat{x}_{pi}(a)\hat{x}_{qj}(a)+\sum_{p,q=1}^{n}R_{pq}(b)\hat{x}_{pi}(b)\hat{x}_{qj}(b)\right)z_iz_j$$

$$=\sum_{i,j=1}^{n}R_{ij}(a,b,X)z_iz_j,$$

where

$$R_{ij}(a,b,X)=\sum_{p,q=1}^{n}(R_{pq}(a)\hat{x}_{pi}(a)\hat{x}_{qj}(a)+R_{pq}(b)\hat{x}_{pi}(b)\hat{x}_{qj}(b))$$

$$=\sum_{p,q=1}^{n}\left(\frac{A_{pq}(a)}{\det(I+A(X(a)))}\hat{x}_{pi}(a)\hat{x}_{qj}(a)+\frac{A_{pq}(b)}{\det(I+A(X(b)))}\hat{x}_{pi}(b)\hat{x}_{qj}(b)\right)$$

$$=\frac{1}{\det(I+A(X(a)))\det(I+A(X(b)))}\sum_{p,q=1}^{n}(\det(I+A(X(b)))A_{pq}(a)\hat{x}_{pi}(a)\hat{x}_{qj}(a)$$

$$+\det(I+A(X(a)))A_{pq}(b)\hat{x}_{pi}(b)\hat{x}_{qj}(b))=\frac{r_{ij}(a,b,X)}{\det(I+A(X(a)))\det(I+A(X(b)))}.$$

Accordingly,

$$((I+A(X(a)))^{-1}d(a),d(a))+((I+A(X(b)))^{-1}d(b),d(b))$$

$$=((\det(I+A(X(a)))\det(I+A(X(b))))^{-1}\sum_{i,j=1}^{n}r_{ij}(a,b,X)z_iz_j. \tag{3.32}$$

Substituting (3.32) into (3.31), we get

$$G^{m,n}(a,b,X)=\left\{\prod_{k=1}^{n}\frac{4a_kb_k}{(a_k+b_k)^2}\frac{1}{\det(I+A(X(a)))\det(I+A(X(b)))}\right\}^{1/4}$$

$$\times\int_{\mathbf{R}^n}\exp\left\{\frac{(r(a,b,X)z,z)}{\det(I+A(X(a)))\det(I+A(X(b)))}\right\}\frac{1}{\sqrt{\pi^n}}\exp(-(z,z))\,dz$$

$$=\left\{\begin{array}{l}\displaystyle\prod_{k=1}^{n}\frac{4a_kb_k}{(a_k+b_k)^2}\frac{1}{\det(I+A(X(a)))\det(I+A(X(b)))}\\ \displaystyle\qquad\qquad\times\det{}^2\left(I-\frac{r(a,b,X)}{\det(I+A(X(a)))\det(I+A(X(b)))}\right)\end{array}\right\}^{1/4}$$

$$=\left\{\prod_{k=1}^{n}\frac{4a_kb_k}{(a_k+b_k)^2}\frac{\det(I+A(X(a)))\det(I+A(X(b)))}{\det^2(\det(I+A(X(a)))\det(I+A(X(b)))I-r(a,b,X))}\right\}^{1/4}. \tag{3.33}$$

Since $A_{pq}(a)$, $A_{pq}(b)$, $\det(I+A(X(a)))$, $\det(I+A(X(b)))$ are polynomials in the x_{ij}, $i=1,\ldots,m$, $j=1,\ldots,n$, then the $r_{ij}(a,b,X)$, and accordingly $\det(\det(I+A(X(a)))\det(I+A(X(b)))I-r(a,b,X))$, are also polynomials in x_{ij}. Suppose that

$$X=(x_{pk}),\quad X(a)=(x_{pk}\sqrt{a_k})\in\mathbf{R}^m\times\mathbf{R}^n;\quad x_1(a),\ldots,x_n(a)\in\mathbf{R}^m,$$

$$x_k(a)=(x_{pk}\sqrt{a_k})_{p=1}^{m},\quad \tilde{x}_1(a),\ldots,\tilde{x}_m(a)\in\mathbf{R}^n,\quad \tilde{x}_p(a)=(x_{pk}\sqrt{a_k})_{k=1}^{n}.$$

Write

$$A(X(a)) = (A_{ij}(X(a)))_{i,j=1}^{n},$$

$$A_{ij}(X(a)) = (\tilde{x}_i(a), \tilde{x}_j(a)), \quad \tilde{A}(X(a)) = (\tilde{A}_{pq}(X(a)))_{p,q=1}^{m},$$

$$\tilde{A}_{pq}(X(a)) = (x_p(a), x_q(a)).$$

Then the following equation holds:

$$\det(I + A(X(a))) = \det(I + \tilde{A}(X(a))) = \xi^{m,n}(a, X). \tag{3.34}$$

Indeed, since the equation

$$[x_{k_1}, \ldots, x_{k_2}] = \det((x_{k_i}, x_{k_j})_{i,j=1}^{r}) = \sum_{1 \le p_1 < \cdots < p_r \le m} |M_{k_1,\ldots,k_r}^{p_1,\ldots,p_r}(X)|^2, \tag{3.35}$$

in which the $M_{k_1,\ldots,k_r}^{p_1,\ldots,p_r}(x)$ are minors of the matrix X, holds for the vectors $x_{k_1}, \ldots, x_{k_r} \in \mathbf{R}^m$, then

$\det(I + A(X(a)))$

$$= \det \begin{Vmatrix} 1 + (\tilde{x}_1(a), \tilde{x}_1(a)) & (\tilde{x}_1(a), \tilde{x}_2(a)) & \cdots & (\tilde{x}_1(a), \tilde{x}_m(a)) \\ (\tilde{x}_2(a), \tilde{x}_1(a)) & 1 + \tilde{x}_2(a), \tilde{x}_2(a)) & \cdots & (\tilde{x}_2(a), \tilde{x}_m(a)) \\ \cdots & & \cdots & \\ (\tilde{x}_m(a), \tilde{x}_1(a)) & (\tilde{x}_m(a), \tilde{x}_2(a)) & \cdots & 1 + (\tilde{x}_m(a), \tilde{x}_m(a)) \end{Vmatrix}$$

$$= 1 + \sum_{s=1}^{m} \sum_{1 \le p_1 < \cdots < p_s \le m} [\tilde{x}_{p_1}(a), \ldots, \tilde{x}_{p_s}(a)]$$

$$= 1 + \sum_{s=1}^{m} \sum_{\substack{1 \le p_1 < \cdots < p_s \le m \\ 1 \le k_1 < \cdots < k_s \le n}} |M_{k_1,\ldots,k_s}^{p_1,\ldots,p_s}(X(a))|^2$$

$$= 1 + \sum_{s=1}^{m} \sum_{1 \le k_1 < \cdots < k_s \le n} \prod_{i=1}^{s} a_{k_i}[x_{k_1}, \ldots, x_{k_s}] = (\xi^{m,n}(a, X) = \det(I + A(X(b))).$$

We will prove equation (3.13) for $G^{n,n+1}$ by induction on $n \in \mathbf{N}$. Suppose that $n = 1$. Write

$$x_1(a) = x_{11}\sqrt{a_1}, \quad x_2(a) = x_{12}\sqrt{a_2}, \quad \tilde{x}_1(a) = (x_{11}\sqrt{a_1}, x_{12}\sqrt{a_2}),$$

$$\det(I + A(X(a))) = 1 + (\tilde{x}_1(a), \tilde{x}_1(a)) = 1 + \sum_{k=1}^{2} a_k x_{1k}^2,$$

$$\det(I + A(X(b))) = 1 + \sum_{k=1}^{2} b_k x_{1k}^2, \quad d(a) = \sum_{k=1}^{2} \frac{a_k x_{1k} z_k}{\sqrt{a_k + b_k}}, \quad d(b) = \sum_{k=1}^{2} \frac{b_k x_{1k} z_k}{\sqrt{a_k + b_k}}.$$

$$((I + A(X(a)))^{-1} d(a), d(a)) + ((I + A(X(b)))^{-1} d(b), d(b)) = (c_1, z)^2 + (c_2, z)^2,$$

where

$$c_1 = \frac{1}{\sqrt{1 + \sum_{k=1}^{2} a_k x_{1k}^2}} \left(\frac{a_1 x_{11}}{\sqrt{a_1 + b_1}}, \frac{a_2 x_{12}}{\sqrt{a_2 + b_2}} \right),$$

$$c_2 = \frac{1}{\sqrt{1 + \sum_{k=1}^{2} b_k x_{1k}^2}} \left(\frac{b_1 x_{11}}{\sqrt{a_1 + b_1}}, \frac{b_2 x_{12}}{\sqrt{a_2 + b_2}} \right).$$

By (3.31) and (3.30) we have

$G^{1,2}(a, b; x_{11}, x_{12})$

$$= \left\{ \prod_{k=1}^{2} \frac{4 a_k b_k}{(a_k + b_k)^2} \frac{1}{\det(I + A(X(a))) \det(I + A(X(b))) \det^2(I - A(C))} \right\}^{1/4}.$$

But, by (3.34),

$$\det(I + A(C)) = \det(I + \tilde{A}(C)) = \det \begin{Vmatrix} 1 - (c_1, c_1) & -(c_1, c_2) \\ -(c_2, c_1) & 1 - (c_2, c_2) \end{Vmatrix}$$

$$= \left(1 - \frac{\sum_{k=1}^{2} \frac{a_k^2 x_{1k}^2}{a_k + b_k}}{1 + \sum_{k=1}^{2} a_k x_{1k}^2} \right) \left(1 - \frac{\sum_{k=1}^{2} \frac{b_k^2 x_{1k}^2}{a_k + b_k}}{1 + \sum_{k=1}^{2} b_k x_{1k}^2} \right)$$

$$- \frac{\left(\sum_{k=1}^{2} \frac{a_k b_k x_{1k}^2}{a_k + b_k} \right)^2}{\left(1 + \sum_{k=1}^{2} a_k x_{1k}^2\right)\left(1 + \sum_{k=1}^{2} b_k x_{1k}^2\right)} = \frac{\left(1 + \sum_{k=1}^{2} \frac{2 a_k b_k}{a_k + b_k} x_{1k}^2\right)^2}{\left(1 + \sum_{k=1}^{2} a_k x_{1k}^2\right)\left(1 + \sum_{k=1}^{2} b_k x_{1k}^2\right)},$$

which implies (3.13) for $G^{1,2}(a, b; x_{11}, x_{12})$.

We will prove that if equation (3.13) holds for $G^{n-1,n}(a, b, X^{n-1,n})$, then it holds for $G^{n,n}(a, b, X^{n,n})$. In view of the invariance of $G^{n,n}(a, b, OX) = G^{n,n}(a, b, X)$, $O \in O(n)$, we may suppose that the matrix $X = X^{n,n}$ is triangular:

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n-1} & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n-1} & 0 \\ & & \cdots & & \\ x_{n-11} & x_{n-12} & \cdots & 0 & 0 \\ x_{n1} & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Then we may express the function $\xi^{n,n}(a, X)$ as follows:

$$\xi^{n,n}(a, X) = 1 + \sum_{r=1}^{n} \sum_{1 \le k_1 < \cdots < k_r \le n} \prod_{l=1}^{r} a_{k_l}[x_{k_1}, \ldots, x_{k_r}]$$

$$= 1 + \sum_{r=1}^{n} \sum_{1 \le k_1 < \cdots < k_r \le n-1} \prod_{l=1}^{r} a_{k_l}[x_{k_1}, \ldots, x_{k_r}]$$

$$+ \sum_{r=1}^{n} \sum_{1 \le k_1 < \cdots < k_r = n} \prod_{l=1}^{r} a_{k_l}[x_{k_1}, \ldots, x_{k_{r-1}}, x_n].$$

For the matrix X we find from property (3.35) that

$$[x_{k_1}, \ldots, x_{k_{r-1}}, x_n] = \sum_{1 \le p_1 < \cdots < p_r = n} |M^{p_1, \ldots, p_{r-1}, p_r}_{k_1, \ldots, k_{r-1}, n}(X)|^2$$

$$= \sum_{1 = p_1 < p_2 < \cdots < p_r \le n} |M^{1, p_2, \ldots, p_{r-1}, p_r}_{k_1, k_2, \ldots, k_{r-1}, n}(X)|^2$$

$$= x_{1n}^2 \sum_{2 \le p_2 < \cdots < p_r \le n} |M^{p_2, \ldots, p_r}_{k_1, \ldots, k_{r-1}}({}^1X)|^2 = x_{1n}^2[{}^1x_{k_1}, \ldots, {}^1x_{k_{r-1}}],$$

where the $M^{p_2, \ldots, p_r}_{k_1, \ldots, k_{r-1}}({}^1X)$ are minors of the matrix

$${}^1X = \begin{pmatrix} x_{21} & \cdots & x_{2n-1} \\ x_{31} & \cdots & 0 \\ & \cdots & \\ x_{n1} & \cdots & 0 \end{pmatrix}, \qquad {}_1X = \begin{pmatrix} x_{12} & \cdots & x_{1n} \\ x_{22} & \cdots & 0 \\ & \cdots & \\ x_{n-12} & \cdots & 0 \end{pmatrix},$$

${}^1x_{k_1}, \ldots, {}^1x_{k_{r-1}}$ the columns of the matrix ${}^1X$. Therefore

$$\xi^{n,n}(a, X^{n,n}) = \xi^{n,n-1}(a, X^{n,n-1}) + a_n x_{1n}^2 \xi^{n-1,n-1}(a, {}^1X). \tag{3.36}$$

Analogously one proves that

$$\xi^{n,n}(a, X^{n,n}) = \xi^{n-1,n}(a, X^{n-1,n}) + a_1 x_{n1}^2 \xi^{n-1,n-1}(a, {}_1X). \tag{3.37}$$

Since

$$\int_{\mathbf{R}^1} \exp(-a(y + tx)^2) \frac{1}{\sqrt{\pi}} \exp(-t^2)\, dt = \frac{1}{\sqrt{1 + ax^2}} \exp\left(-\frac{ay^2}{1 + ax^2}\right), \tag{3.38}$$

then

$$\int_{\mathbf{R}^1} \exp\left(-a_1\left(x_{n+11} + \sum_{r=1}^{n-1} t_r x_{r1} + t_n x_{n1}\right)^2\right) \frac{1}{\sqrt{\pi}} \exp(-t_n^2)\, dt_n$$

$$= \frac{1}{\sqrt{1 + a_1 x_{n1}^2}} \exp\left(-\frac{a_1\left(x_{n+11} + \sum_{r=1}^{n-1} t_r x_{r1}\right)^2}{1 + a_1 x_{n1}^2}\right),$$

$$\int_{\mathbf{R}^1} \exp\left(-b_1\left(x_{n+11} + \sum_{r=1}^{n-1} t_r x_{r1} + t_n x_{n1}\right)^2\right) \frac{1}{\sqrt{\pi}} \exp(-t_n^2)\, dt_n$$

$$= \frac{1}{\sqrt{1 + b_1 x_{n1}^2}} \exp\left(-\frac{b_1\left(x_{n+11} + \sum_{r=1}^{n-1} t_r x_{r1}\right)^2}{1 + b_1 x_{n1}^2}\right).$$

Therefore

$$G^{n,n}(a, b, X^{n,n}) = ((1 + a_1 x_{n1}^2)(1 + b_1 x_{n1}^2))^{-1/4} G^{n-1,n}(\tilde{a}, \tilde{b}, X^{n-1,n}),$$

where

$$\tilde{a} = (\tilde{a}_k)_{k=1}^n, \quad \tilde{b} = (\tilde{b}_k)_{k=1}^n, \quad \tilde{a}_1 = \frac{a_1}{1 + a_1 x_{n1}^2}, \quad \tilde{b}_1 = \frac{b_1}{1 + b_1 x_{n1}^2},$$

$$\tilde{a}_k = a_k, \tilde{b}_k = b_k, 2 \le k \le n.$$

Since

$$\frac{2\tilde{a}_1 \tilde{b}_1}{\tilde{a}_1 + \tilde{b}_1} = \frac{2a_1 b_1}{(1 + a_1 x_{n1}^2)(1 + b_1 x_{n1}^2)} \cdot \frac{1}{a_1(1 + a_1 x_{n1}^2)^{-1} + b_1(1 + b_1 x_{n1}^2)^{-1}}$$

$$= \frac{2a_1 b_1 (a_1 + b_1)^{-1}}{1 + 2a_1 b_1 x_{n1}^2 (a_1 + b_1)^{-1}},$$

$$\frac{4\tilde{a}_1 \tilde{b}_1}{(\tilde{a}_1 + \tilde{b}_1)^2} = \frac{4a_1 b_1 (1 + a_1 x_{n1}^2)^2 (1 + b_1 x_{n1}^2)^2}{(1 + a_1 x_{n1}^2)(1 + b_1 x_{n1}^2)(a_1 + b_1)^2 (1 + 2a_1 b_1 x_{n1}^2 (a_1 + b_1)^{-1})^2}$$

$$= \frac{4a_1 b_1 (1 + a_1 x_{n1}^2)(1 + b_1 x_{n1}^2)}{(a_1 + b_1)^2 (1 + 2a_1 b_1 x_{n1}^2 (a_1 + b_1)^{-1})^2},$$

then

$$\xi^{n-1,n}(\tilde{a}, X^{n-1,n}) = 1 + \sum_{r=1}^{n} \sum_{1 \le k_1 < \cdots < k_r \le n} \prod_{i=1}^{r} \tilde{a}_{k_i}[x_{k_1}, \ldots, x_{k_r}]$$

$$= 1 + \sum_{r=1}^{n} \left(\frac{a_1}{1 + a_1 x_{n1}^2} \sum_{2 \le k_2 < \cdots < k_r \le n} \prod_{i=2}^{r} a_{k_i}[x_1, x_{k_2}, \ldots, x_{k_r}] + \sum_{2 \le k_1 < \cdots < k_r \le n} \prod_{i=1}^{r} a_{k_i}[x_{k_1}, \ldots, x_{k_r}] \right)$$

$$= \frac{1}{1 + a_1 x_{n1}^2} \left(1 + a_1 x_{n1}^2 + \sum_{r=1}^{n} \left(\sum_{2 \le k_2 < \cdots < k_r \le n} a_1 \prod_{i=2}^{r} a_{k_i}[x_1, x_{k_2}, \ldots, x_{k_r}] + \sum_{2 \le k_1 < \cdots < k_r \le n} \prod_{i=1}^{r} a_{k_i}[x_{k_1}, \ldots, x_{k_r}] + a_1 x_{n1}^2 \times \left(\sum_{2 \le k_1 < \cdots < k_r \le n} \prod_{i=1}^{r} a_{k_i}[x_{k_1}, \ldots, x_{k_r}] \right) \right) \right)$$

$$= \frac{1}{1 + a_1 x_{n1}^2} \left(1 + \sum_{r=1}^{n} \sum_{1 \le k_1 < \cdots < k_r \le n} \prod_{i=1}^{r} a_{k_i}[x_{k_1}, \ldots, x_{k_r}] + a_1 x_{n1}^2 \times \left(1 + \sum_{r=1}^{n} \sum_{2 \le k_1 < \cdots < k_r \le n} \prod_{i=1}^{r} a_{k_i}[x_{k_1}, \ldots, x_{k_r}] \right) \right)$$

$$= \frac{1}{1 + a_1 x_{n1}^2} \left(\xi^{n-1,n}(a, X^{n-1,n}) + a_1 x_{n1}^2 \xi^{n-1,n-1}(a, {}_1X) \right).$$

Therefore

$$G^{n,n}(a, b, X) = ((1 + a_1 x_{n1}^2)(1 + b_1 x_{n1}^2))^{-1/4} G^{n-1,n}(\tilde{a}, \tilde{b}, X^{n-1,n})$$

$$= \left\{ \prod_{k=1}^{n} \frac{4a_k b_k}{(a_k + b_k)^2} \frac{(1 + a_1 x_{n1}^2)(1 + b_1 x_{n1}^2)(1 + 2a_1 b_1 x_{n1}^2 (a_1 + b_1)^{-1})}{(1 + 2a_1 b_1 x_{n1}^2 (a_1 + b_1)^{-1})(1 + a_1 x_{n1}^2)(1 + b_1 x_{n1}^2)} \times \frac{(\xi^{n-1,n}(a, X^{n-1,n}) + a_1 x_{n1}^2 \xi^{n-1,n-1}(a, {}_1X)) \times (\xi^{n-1,n}(b, X^{n-1,n}) + b_1 x_{n1}^2 \xi^{n-1,n-1}(b, {}_1X))}{\left(\left(\xi^{n-1,n}\left(\frac{2ab}{a+b}, X^{n-1,n}\right) + \frac{2a_1 b_1}{a_1 + b_1} x_{n1}^2 \xi^{n-1,n-1}\left(\frac{2ab}{a+b}, {}_1X\right) \right) \right)^2} \right\}^{1/4};$$

when (3.37) is taken into account, this completes the proof of (3.13) for $G^{n,n}(a, b, X^{n,n})$.

Now we shall prove that formula (3.13) is valid for $G^{n,n+1}(a, b, X^{n,n+1})$. If we write

$$X = X^{n,n-1} = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n+1} \\ x_{21} & x_{22} & x_{23} & \cdots & 0 \\ & & & \cdots & \\ x_{n1} & x_{n2} & 0 & \cdots & 0 \end{pmatrix}, \qquad X^{n,n} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ & & \cdots & \\ x_{n1} & x_{n2} & \cdots & 0 \end{pmatrix},$$

$${}^1X = \begin{pmatrix} x_{21} & x_{22} & \cdots & x_{2n} \\ x_{31} & z_{32} & \cdots & 0 \\ & & \cdots & \\ x_{n1} & x_{n2} & \cdots & 0 \end{pmatrix}$$

(we may bring the matrix X into such a form, since $G^{n,n+1}(a, b, X)$ is $O(n)$-invariant), and use (3.36) for $\xi^{n,n+1}(a, X)$, then equation (3.13) may be rewritten in the form

$$\theta(a, b, X) := \left\{ \frac{(\xi^{n,n}(a, X^{n,n}) + a_{n+1} x_{1n+1}^2 \xi^{n-1,n}(a, {}^1X)) \times (\xi^{n,n}(b, X^{n,n}) + b_{n+1} x_{1n+1}^2 \xi^{n-1,n}(b, {}^1X))}{(G^{n,n+1}(a, b, X))^4} \right\}^{1/2}$$

$$= \left\{ \prod_{k=1}^{n+1} \frac{(a_k + b_k)^2}{4a_k b_k} \right\}^{1/2} \left(\xi^{n,n}\left(\frac{2ab}{a+b}, X^{n,n}\right) + \frac{2a_{n+1} b_{n+1}}{a_{n+1} + b_{n+1}} x_{1n+1}^2 \xi^{n-1,n}\left(\frac{2ab}{a+b}, {}^1X\right) \right). \tag{3.39}$$

For $x_{1n+1} = 0$, (3.39) is (3.13) for $G^{n,n}(a, b, X)$. Since $G^{n,n+1}(a, b, X)$ is an even function in x_{1n+1}, then $\theta(a, b, X)$ is also even. Taking account of (3.33), we conclude that $\theta(a, b, X)$ is a polynomial:

$$\theta(a, b, X) = \sum_{k=0}^{p} \theta_k(a, b, X) x_{1n+1}^{2k}, \; p < \infty,$$

$$\theta_0(a, b, X) = \left\{ \prod_{k=1}^{n+1} \frac{(a_k + b_k)^2}{4a_k b_k} \right\}^{1/2} \xi^{n,n}\left(\frac{2ab}{a+b}, X^{n,n}\right). \tag{3.40}$$

A direct calculation yields

$$2\theta_1(a, b, X) = \left.\frac{\partial^2\theta(a, b, X)}{\partial x_{1n+1}^2}\right|_{x_{1n+1}=0}$$

$$= \left\{\prod_{k=1}^{n+1} \frac{(a_k + b_k)^2}{4a_k b_k}\right\}^{1/2} \frac{4a_{n+1}b_{n+1}}{a_{n+1} + b_{n+1}} \xi^{n,n}\left(\frac{2ab}{a+b}, {}^1X^{n,n}\right). \quad (3.41)$$

Indeed,

$$\left.\frac{\partial^2\theta}{\partial x_{1n+1}^2}\right|_{x_{1n+1}} = \theta(a, b, X)\left[\frac{a_{n+1}\xi^{n-1,n}(a, {}^1X)}{\xi^{n,n}(a, X^{n,n})} + \frac{b_{n+1}\xi^{n-1,n}(b, {}^1X)}{\xi^{n,n}(b, X^{n,n})}\right.$$

$$\left.\left. - 2\,\frac{\dfrac{\partial^2 G^{n,n+1}}{\partial x_{1n+1}^2}}{G^{n,n+1}}\right]\right|_{x_{1n+1}=0}.$$

But for the function

$$\hat{G}(a, b, X) = \left(\prod_{k=1}^{n+1} a_k b_k\right)^{-1/4} G^{n,n+1}(a, b, X)$$

we have

$$(G^{n,n+1})^{-1}\left.\frac{\partial^2 G^{n,n+1}}{\partial^2 x_{1n+1}}\right|_{x_{1n+1}=0} = (\hat{G})^{-1}\left.\frac{\partial^2\hat{G}}{\partial x_{1n+1}^2}\right|_{x_{1n+1}=0}$$

$$= -2(\hat{G})^{-1}\left.\left(a_{n+1}^2\frac{\partial\hat{G}}{\partial a_{n+1}} + b_{n+1}^2\frac{\partial\hat{G}}{\partial b_{n+1}}\right)\right|_{x_{1n+1}=0} - \frac{a_{n+1}+b_{n+1}}{2},$$

which makes it possible to use the following formula, which was already proved:

$$\hat{G}(a, b, X)|_{x_{1n+1}=0} = \left(\prod_{k=1}^{n+1} a_k b_k\right)^{-1/4} G^{n,n+1}(a, b, X)\Big|_{x_{1n+1}=0}$$

$$= \left\{\frac{4^{n+1}}{\prod_{k=1}^{n+1}(a_k+b_k)^2}\,\frac{\xi^{n,n}(a, X^{n,n})\xi^{n,n}(b, X^{n,n})}{\left(\xi^{n,n}\left(\dfrac{2ab}{a+b}, X^{n,n}\right)\right)^2}\right\}^{1/4}.$$

On calculating

$$(G^{n,n+1})^{-1}\left.\frac{\partial^2 G^{n,n+1}}{\partial x_{1n+1}^2}\right|_{x_{1n+1}=0}$$

and substituting into $\partial^2\theta/\partial x_{1n+1}^2$, we get (3.41).

We will prove that

$$\left.\frac{\partial^{2k}\theta}{\partial x_{1n+1}^{2k}}\right|_{x_{1n+1}=0} = 0, \qquad 2 \le k \le p.$$

To do this we will show that the following limit exists:

$$\lim_{x_{1n+1}\to\infty} G^{n,n+1}(a, b, X) > 0. \quad (3.42)$$

Indeed, integrating over $t_1 \in \mathbf{R}^1$ and using (3.38), we get

$$G^{n,n+1}(a, b, X) = \frac{\left(\prod_{k=1}^{n+1} a_k b_k\right)^{1/4}}{\sqrt{\pi^{n+1}}}\int_{\mathbf{R}^{n+1}}$$

$$\times\left(\int_{\mathbf{R}^{m-1}} \exp\left(-\sum_{k=1}^{n} a_k\left(x_{m+1k} + \sum_{r=2}^{m} t_r x_{rk}\right)^2 - a_{n+1}x_{m+1n+1}^2\right.\right.$$

$$\left. + \left(\sum_{k=1}^{n+1} a_k x_{1k}\left(x_{m+1k} + \sum_{r=2}^{m} t_r x_{rk}\right)\right)^2\left(1 + \sum_{k=1}^{n+1} a_k x_{1k}^2\right)^{-1}\right)\bigotimes_{k=2}^{m}\frac{1}{\sqrt{\pi}}\exp(-t_k^2)\,dt_k$$

$$\times\int_{\mathbf{R}^{m-1}} \exp\left(-\sum_{k=1}^{n} b_k\left(x_{m+1k} + \sum_{r=2}^{m} t_r x_{rk}\right)^2 - b_{k+1}x_{m+1n+1}^2\right.$$

$$\left. + \left(\sum_{k=1}^{n+1} b_k x_{1k}\left(x_{m+1k} + \sum_{r=2}^{m} t_r x_{rk}\right)\right)^2\left(1 + \sum_{k=1}^{n+1} b_k x_{1k}^2\right)^{-1}\right)$$

$$\left.\times\bigotimes_{k=2}^{m}\frac{1}{\sqrt{\pi}}\exp(-t_k^2)\,dt_k\right)^{1/2}\bigotimes_{k=1}^{n+1} dx_{m+1k}\left(\left(1 + \sum_{k=1}^{n+1} a_k x_{1k}^2\right)\left(1 + \sum_{k=1}^{n+1} b_k x_{1k}^2\right)\right)^{-1/4}.$$

Since

$$\left(\sum_{k=1}^{n+1} a_k x_{1k}\left(x_{m+1k} + \sum_{r=2}^{m} t_r x_{rk}\right)\right)^2\left(1 + \sum_{k=1}^{n+1} a_k x_{1k}^2\right)^{-1}$$

$$= \left(\sum_{r=2}^{m} t_r \sum_{k=1}^{n} a_k x_{1k} x_{rk} + \sum_{k=1}^{n} a_k x_{1k} x_{m+1k}\right)^2\left(1 + \sum_{k=1}^{n+1} a_k x_{1k}^2\right)^{-1}$$

$$\ge \frac{(a_{n+1}x_{1n+1}x_{m+1n+1})^2}{1 + \sum_{k=1}^{n+1} a_k x_{1k}^2}$$

for

$$t = (t_2, t_3, \ldots, t_m) \in L_{m-1}(a) = \left\{t \in \mathbf{R}^{m-1}\ \Bigg|\ \sum_{r=2}^{m} t_r \sum_{k=1}^{n} a_k x_{1k} x_{rk} \geqslant 0\right\}$$

and

$$\left(\sum_{k=1}^{n+1} b_k x_{1k}\left(x_{m+1k}+\sum_{r=2}^{m} x_{rk}s_r\right)\right)^2\left(1+\sum_{k=1}^{n+1}a_kx_{1k}^2\right)^{-1} \geq \frac{(b_{n+1}x_{1n+1}x_{m+1n+1})^2}{1+\sum_{k=1}^{n+1}b_kx_{1k}^2},$$

for

$$s=(s_2,\dots,s_m)\in L_{m-1}(b)=\left\{s\in\mathbf{R}^{m-1}\;\middle|\;\sum_{r=2}^m s_r\sum_{k=1}^n b_kx_{1k}x_{rk}\geq 0\right\}$$

and

$$x^{(m+1)}=(x_{m+11},\dots,x_{m+1n})\in D_n(a)\cap D_n(b),$$

$$D_n(a)=\left\{x^{m+1}\in\mathbf{R}^n\;\middle|\;\sum_{k=1}^n a_kx_{1k}x_{m+1k}\geq 0\right\},$$

$$D_n(b)=\left\{x^{(m+1)}\in\mathbf{R}^n\;\middle|\;\sum_{k=1}^n b_kx_{1k}x_{m+1k}\geq 0\right\}$$

and $D_n(a,b)=D_n(a)\cap D_n(b)\neq\varnothing$, because of $a_k, b_k>0$, $k=1,\dots,n$, the following estimate holds for $G^{n,n+1}(a,b,X)$:

$$\begin{aligned}
G^{n,n+1}(a,b,X)\geq{}& \frac{\left(\prod_{k=1}^{n+1}a_kb_k\right)^{1/4}}{\sqrt{\pi}^{n+1}}\int_{\mathbf{R}^1_+}\int_{D_n(a,b)}\\
&\times\left(\int_{L_{m-1}(a)}\exp\left(-\sum_{k=1}^n a_k\left(x_{m+1k}+\sum_{r=2}^m t_rx_{rk}\right)^2\right)\bigotimes_{r=2}^m\frac{1}{\sqrt{\pi}}\exp(-t_r^2)\,dt_k\right.\\
&\left.\times\int_{L_{m-1}(b)}\exp\left(-\sum_{k=1}^n b_k\left(x_{m+1k}+\sum_{r=2}^m s_rx_{rk}\right)^2\right)\bigotimes_{k=2}^m\frac{1}{\sqrt{\pi}}\exp(-s_k^2)\,ds_k\right)^{1/2}\\
&\times\exp\frac{1}{2}\left(\frac{(a_{n+1}x_{1n+1}x_{m+1n+1})^2}{1+\sum_{k=1}^{n+1}a_kx_{1k}^2}-a_{n+1}x_{m+1n+1}^2\right.\\
&\left.+\frac{(b_{n+1}x_{1n+1}x_{m+1n+1})^2}{1+\sum_{k=1}^{n+1}b_kx_{1k}^2}-b_{n+1}x_{m+1n+1}^2\right)\\
&\times\left(\bigotimes_{k=1}^n dx_{m+1k}\right)\otimes dx_{m+1n+1}:\left(\left(1+\sum_{k=1}^{n+1}a_kx_{1k}^2\right)\left(1+\sum_{k=1}^{n+1}b_kx_{1k}^2\right)\right)^{-1/4}.
\end{aligned}$$

Since

$$\begin{aligned}
&\left(\left(1+\sum_{k=1}^{n+1}a_kx_{1k}^2\right)\left(1+\sum_{k=1}^{n+1}b_kx_{1k}^2\right)\right)^{-1/4}\frac{1}{\sqrt{\pi}}\int_{\mathbf{R}^1_+}\\
&\quad\times\exp\frac{1}{2}\left(\frac{(a_{n+1}x_{1n+1}x_{m+1n+1})^2}{1+\sum_{k=1}^{n+1}a_kx_{1k}^2}+\frac{(b_{n+1}x_{1n+1}x_{m+1n+1})^2}{1+\sum_{k=1}^{n+1}b_kx_{1k}^2}\right.\\
&\quad\left.-a_{n+1}x_{m+1n+1}^2-b_{n+1}x_{m+1n+1}^2\right)dx_{m+1n+1}\\
&=\left(\left(1+\sum_{k=1}^{n+1}a_kx_{1k}^2\right)\left(1+\sum_{k=1}^{n+1}b_kx_{1k}^2\right)\right)^{-1/4}\frac{1}{\sqrt{\pi}}\int_{\mathbf{R}^1_+}\\
&\quad\times\exp\frac{1}{2}\left\{\left[-\frac{a_{n+1}\left(1+\sum_{k=1}^{n}a_kx_{1k}^2\right)}{1+\sum_{k=1}^{n+1}a_kx_{1k}^2}-\frac{b_{n+1}\left(1+\sum_{k=1}^{n}b_kx_{1k}^2\right)}{1+\sum_{k=1}^{n+1}b_kx_{1k}^2}\right]\right.\\
&\quad\left.\times x_{m+1n+1}^2\right\}dx_{m+1n+1}\\
&=\frac{1}{\sqrt{2}}\left\{\frac{\left(1+\sum_{k=1}^{n+1}a_kx_{1k}^2\right)\left(1+\sum_{k=1}^{n+1}b_kx_{1k}^2\right)}{\left(a_{n+1}\left(1+\sum_{k=1}^{n}a_kx_{1k}^2\right)\left(1+\sum_{k=1}^{n+1}b_kx_{1k}^2\right)+b_{n+1}\left(1+\sum_{k=1}^{n}b_kx_{1k}^2\right)\left(1+\sum_{k=1}^{n+1}a_kx_{1k}^2\right)\right)^2}\right\}^{1/4}\\
&\to\frac{1}{\sqrt{2}}\left(\frac{a_{n+1}b_{n+1}}{\left(a_{n+1}\left(1+\sum_{k=1}^{n}a_kx_{1k}^2\right)b_{n+1}+b_{n+1}\left(1+\sum_{k=1}^{n}b_kx_{1k}^2\right)a_{n+1}\right)^2}\right)^{1/4}\\
&=\frac{1}{\sqrt{2}}\left(\frac{1}{a_{n+1}b_{n+1}\left(2+\sum_{k=1}^{n}(a_k+b_k)x_{1k}^2\right)^2}\right)>0
\end{aligned}$$

for any collection $(x_{11}, \dots, x_{1n}) \in \mathbf{R}^n$, then

$$\lim_{x_{1n+1}\to\infty} G^{n,n+1}(a, b, X) \geqslant \frac{\left(\prod_{k=1}^{n+1} a_k b_k\right)^{1/4}}{\sqrt{\pi^n}} \int_{D_n(a,b)}$$

$$\times \left(\int_{L_{m-1}(a)} \exp\left(-\sum_{k=1}^{n} a_k \left(x_{m+1k} + \sum_{r=2}^{m} t_r x_{rk}\right)^2\right) \bigotimes_{k=2}^{m} \frac{1}{\sqrt{\pi}} \exp(-t_k^2)\, dt_k\right.$$

$$\left.\times \int_{L_{m-1}(b)} \exp\left(-\sum_{k=1}^{n} b_k \left(x_{m+1k} + \sum_{r=2}^{m} s_r x_{rk}\right)^2\right) \bigotimes_{k=2}^{m} \frac{1}{\sqrt{\pi}} \exp(-s_k^2)\, ds_k\right)^{1/2}$$

$$\times \bigotimes_{k=1}^{n} dx_{m+1k} \cdot 2^{-1/2}(a_{n+1}b_{n+1})^{-1/4}\left(2 + \sum_{k=1}^{n} (a_k + b_k)x_{1k}^2\right)^{-1/2} > 0.$$

Thus (3.41) is proved.

Accordingly, $\theta_k(a, b, X) = 0$ for $2 \leq k \leq p$, since, in the contrary case, taking account of (3.39) and (3.40), we would arrive at a contradiction with (3.42). This completes the proof of Lemma 3.3.

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