Criteria for Irreducibility and Equivalence of Regular Gaussian Representations of Groups of Finite Upper-Triangular Matrices of Infinite Order*

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Introduction

Regular representations play an important role in the theory of representations of locally compact groups. The decomposition of a regular representation into irreducible representations contains all the irreducible representations for finite and compact groups and many irreducible representations of locally compact Lie groups. In the case of locally compact groups a regular representation itself is always reducible, since along with a right regular representation there exists a left regular one commuting with it. It is known (see Dixmier [5], 1969) that the following theorem holds for unimodular groups.

Theorem A. The commutant of a right regular representation is generated by operators of a left representation, and the commutant of a left representation by operators of a right representation.

Therefore it is natural to wish to construct an analogue of the regular representation in the case of infinite-dimensional groups and to investigate its properties. By an analogue of a regular representation (right or left) of an infinite-dimensional group \( G \) we mean homomorphisms

\[
\begin{align*}
U^R, U^L; & G \mapsto \mathcal{U}(G') = L_2(G, d\mu): \\
\mathcal{H} \ni f(x) & \mapsto (U^R(t)f)(x) = (d\mu(xt)/d\mu(x))^{1/2}f(xt) \in \mathcal{H}, \\
\mathcal{H} \ni f(x) & \mapsto (U^L(t)f)(x) = (d\mu(t^{-1}x)/d\mu(x))^{1/2}f(t^{-1}x) \in \mathcal{H},
\end{align*}
\]

where \( G \) is a topological group, or a topological \( G \)-space containing \( G \) as a dense subgroup: \( G \subset G \), and \( \mu \) is a quasi-invariant measure on \( G \).

It seems that the first, analogue to a regular representation \( \phi \to U(L_2(\phi, \phi, d\mu)) \) of an infinite-dimensional commutative group of a kernel space \( \phi \), where \( \phi \) is the space conjugate to \( \phi \), appeared in the 1961 monograph [7], by Gel'fand and Vilenkin.

Regular representations \( R^\circ \to U(L_2(\mathbb{R}^\infty, R^\circ, dw)) \) of the commutative group \( R^\circ \) of finite sequences of real numbers, connected with various \( R^\circ \)-quasi-invariant measures on the group \( \mathbb{R}^\infty = R_1 \times R_1 \times \cdots \to R^\circ \), were studied by Samoilenko in the monograph [18].

The so-called energy representation \( E \) of the group \( C^\circ(X, G) \) of smooth mappings with compact support of a Riemannian manifold \( X \) into a compact semisimple Lie group \( G \) was studied in the papers [8, 1, 21, 9, 2, 3]. Ismagilov introduced that representation in [8] for \( G = SU_2 \) and \( X \) a domain in \( \mathbb{R}^n \). In the general case it was introduced in [1] and [21]. The irreducibility and mutual nonequivalence of such representations for various metrics were first proved in [8] in the case \( d = \dim X \geq 5 \) and \( G = SU_2 \). In [21] Vershik, Gel'fand and Graev proved the irreducibility and nonequivalence in the case \( d \geq 4 \) and \( G \) a compact semisimple Lie group. In [2] Albeverio, Høegh-Krohn, and Testard proved irreducibility for \( d \geq 3 \), and, under additional conditions, for \( d = 2 \). Reducibility for \( d = 1 \) was proved in [2] and [9].

The connection with the regular representation in the case \( d = 1 \) was noted in the papers [1], [2], [3], [9]. In [1] Albeverio and Høegh-Krohn proved that, in the case \( \alpha = (0, 1) \), the energy representation \( E \) is unitarily equivalent to the right regular representation

\[
U^R: C_0((0, 1), G) \to U(L_2(C((0, 1), G), C_0((0, 1), G), dW)),
\]

where \( C((0, 1), G) \) is the space of smooth paths and \( dW \) is the Wiener measure on \( C((0, 1), G) \), defined by the left Brownian motion on \( G \). In [9] Ismagilov proved, for the group \( C(0, 1], G \), that along with a right representation \( U^R \), equivalent to the energy representation \( E \), there exists a left representation \( U^L \).

This proved the reducibility of the energy representation in the case \( d = 1 \). We have already noted that the authors of [2] had proved this fact as well. There they studied the right and left regular representations \( U^R, U^L \) of the groups \( C^\circ(\mathbb{R}^1, G) \) and \( C^\infty(\mathbb{R}^1, G) \). Together with Vershik, they proved in [3] that the representations of \( U^R \) and \( U^L \), constructed in [2], are factor-representations, and that Theorem A holds for them. They also presented expansions of the representations \( U^R \) and \( U^L \) into direct integrals of irreducible representations.
paper is the proof of these conjectures for the group $B_0^\infty$ and Gaussian product-measures (see also [11]). It is likely that these conjectures are valid for other infinite-dimensional groups, and for measures which are not necessarily Gaussian. The question as to the decomposition of a reducible regular representation of the group $B_0^\infty$ remains open.

In §1, we construct on the group $B^\infty$ a family of Gaussian measures $\mu_x$ that have property (B) and a family of right regular representations $T^R, B$ of the group $B_0^\infty$. In §2 we show that property (B) is equivalent to the irreducibility of $T^R, B$. The proof of that irreducibility is based on the $B_0^\infty$-ergodicity of the measure $\mu_x$ and on the fact that the operators $T^R, B$ are all connected by a common variable may be approximated by the generators of one-parameter groups. In §3 we prove that to nonequivalent measures there correspond nonequivalent representations. The proof is based on the calculation, using partial Fourier transforms on the group $B^\infty$, of the spectral measures of a family of commutative subgroups $B_0^\infty = B_0^\infty, m \in \mathbb{N}$, and on a comparison of those spectral measures using Hellinger integrals. In §4 we carry out the proofs of some technical lemmas.

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§1. Regular representations

Suppose that $B^\infty$ is the group of finite upper-triangular real matrices of infinite order with units on the principal diagonal, $B_0^\infty$ the group of all upper-triangular matrices with units on the principal diagonal, and $b^\infty$ its Lie algebra, i.e. the set of all strictly upper-triangular matrices. If one denotes by $E_{kn}$, $k, n \in \mathbb{N}$, the matrix units of infinite order, then the elements of the group $B_0^\infty$ (resp. $B^\infty$) are matrices $I + x, x = \sum_{k \leq n} x_{kn}E_{kn}$, where only a finite number of elements $x_{kn}$ are nonzero (the $x_{kn}$ are arbitrary).

$b^\infty = \left\{ x = \sum_{k < n} x_{kn}E_{kn} \right\}$.

Suppose that $B(m, R)$ is the subgroup of $B_0^\infty$ of matrices of the form

$B(m, R) = \left\{ t = I + \sum_{k < n, k \leq m} x_{kn}E_{kn} \right\}$.

Obviously $B_0^\infty = \lim_{m \to \infty} B(m, R)$. We will equip $B_0^\infty$ with the inductive limit topology.

Since the group $G = B_0^\infty$ is not locally compact, there is no $G$-invariant measure on it (A. Weil [22]), nor any $G$-quasi-invariant measure either (Da-Xing Xia [23]). Accordingly, some kind of analogue has to be constructed or some completion $\mathcal{G}$ of the group $G$. If one chooses the group $B^\infty$ to serve in the role of such a completion $\mathcal{G}$, then on the group $B^\infty$ there already exist many different $B_0^\infty$-quasi-invariant measures, for instance Gaussian measures. There is no basis whatever for giving preference to any of those measures. Therefore it makes sense to consider all measures or all measures in a certain class.

It is convenient to construct the measure first on the corresponding Lie algebra $b^\infty$, and then to transfer it to the group $B^\infty$ using the exponential mapping.

We will be dealing with matrices $b = (b_{kn})_{k < n}$ of positive numbers. We will denote the set of such matrices by $B$. We define a Gaussian measure $\mu_b$ on the space $b^\infty$ as follows:

$dp_b(x) = \bigotimes_{k < n} dp_{k,n}(x_{kn}) = \bigotimes_{k < n} \frac{b_{kn}}{\pi} \exp(-b_{kn}x_{kn}) dx_{kn}$.

Let $\mu_b$ be the measure on $b^\infty$ which is the image of the measure $\mu_b$ under the mapping $\rho$:

$b^\infty \ni x \mapsto \rho(x) = I + x \in B^\infty$, $\mu_b(A) = \mu_b(\rho^{-1}(A))$.

In fact $x = \sum_{k < n} x_{kn}E_{kn}$ are the canonical coordinates of the second kind for $\rho(x) = I + x$. Indeed, write $x_m = \sum_{k=1}^{n-1} x_{km}E_{km}$. Then obviously

$\rho(x) = I + x = \cdots (I + x_m) \cdots (I + x_3)(I + x_2) \cdots \exp(x_n) \cdots \exp(x_3) \exp(x_2)$.

Consider the right and left actions $R_t, L_t$ of the group $B_0^\infty$ on $B^\infty$:

$R_t = st, \quad L_t = ts, \quad t \in B_0^\infty, \quad s \in B^\infty$.

Denote by $(\mu_b)^R_t, (\mu_b)^L_t$ the images of the measure $\mu_b$ under the mapping: $R_t, L_t: B^\infty \to B^\infty$. It turns out that the measure $\mu_b$ is always $B_0^\infty$-right-quasi-invariant (Lemma 1.1), but it is not always $B_0^\infty$-left-quasi-invariant (Lemma 1.2). Therefore we can construct a family of analogues of the right $T^R, B$ and left $T^L, B$ representations of $B^\infty$. The construction of them is based on a decomposition of $B^\infty$ into $B_0^\infty$-quasi-invariant subgroups.
(if they exist) regular representations of the group \(B^\infty_6\) in the space
\[ \mathcal{H}(b) = L_2(B^\infty_6; db^g), \quad b \in \mathcal{A}. \]
They are
\[ \mathcal{H}(b) \ni f(x) \mapsto (T_b^b f)(x) = \left( \frac{d\mu(x)}{d\mu_2(x)} \right)^{1/2} f(x) \in \mathcal{H}(b) \quad (1.1) \]
and
\[ \mathcal{H}(b) \ni f(x) \mapsto (T_b^t f)(x) = \left( \frac{d\mu(t^{-1} x)}{d\mu_2(x)} \right)^{1/2} f(t^{-1} x) \in \mathcal{H}(b) \quad (1.2) \]

**Theorem 1.1.** The right regular representation \(T_b^b\) of the group \(B^\infty_6\) is irreducible if and only if no left shifts \(L_t\), \(t \in B^\infty_6\), are admissible for the measure \(\mu^g_b\), \(b \in \mathcal{A}\).

The proof of this theorem will be given at the beginning of \$2\) below.

We constructed the representations (1.1) and (1.2) in \([10]\), but we did not consider the question of their irreducibility. The analogue to the representation \(T_b^b\), for the standard Gaussian measure \(\mu^g\), \(b \in B^\infty_6\), was constructed independently by N. I. Nessonov \([13]\), who proved its irreducibility. However, Nessonov's method, based on the Fourier transform and the law of large numbers, did not include the case of an arbitrary \(b \in \mathcal{A}\).

**Lemma 1.1.** For \(t \in B^\infty_6\) the measures \((\mu_b^g)^{t_b}\) and \(\mu_b^g\) are always equivalent.

**Proof.** Under the transformation \(R_t: B^\infty_6 \to B^\infty_6\) only a finite number of coordinates change:
\[ B^\infty_6 \ni x = I + \sum_{k < n} x_k E_k \mapsto R_t(x) = I + \sum_{k < n} \bar{x}_k E_k, \]
where
\[ \bar{x}_k = x_k + \sum_{r = k + 1}^{n-1} x_r t_r + t_{k_n} \quad \text{if } k < n \leq N = N(t), \]
\[ \bar{x}_k = x_k \quad \text{if } n > N. \]

It follows that the question reduces to the equivalence of two nondegenerate Gaussian measures in finite-dimensional space. But then they are obviously equivalent because each of them is equivalent to Lebesgue measure.
product \( \prod \) 

\[
\prod = \int \frac{d\mu_{kk+1}}{d\mu_{kk+1}}^{1/2} (x_{kk+1} + 1) \, d\mu_{kk+1} \, (x_{kk+1}) \\
\times \int_{k+2}^{\infty} \frac{d\mu_{km} \otimes \mu_{km+1}}{d(\mu_{km} \otimes \mu_{km+1})} \, (x_{km}, x_{km+1}) \\
\times d(\mu_{km} \otimes \mu_{km+1})(x_{km}, x_{km+1}) \\
= \left( \frac{b_{kk+1}}{\pi} \right)^{1/2} \int \exp \left( -\frac{1}{2} (x_{kk+1} + \pi)^2 - b_{kk+1} x_{kk+1}^2 \right) \\
\times \exp \left( -b_{kk+1} x_{kk+1}^2 \right) \, dx_{kk+1} \\
\times \int_{k+2}^{\infty} \frac{b_{km} b_{km+1}}{\pi^2} \, dx_{km} \, dx_{km+1} \\
= \exp \left( -\frac{b_{kk+1}^2}{4} \right) \int \exp \left( -\frac{1}{2} (x_{kk+1} + \pi)^2 - b_{kk+1} x_{kk+1}^2 \right) \\
\times \int_{k+2}^{\infty} \frac{b_{km} b_{km+1}}{\pi^2} \, dx_{km} \, dx_{km+1} \\
\times \exp \left( -\frac{b_{km} b_{km+1}}{4} \right) \, dx_{km+1} 
\]

Thus the convergence of the product \( \prod \) is equivalent to the convergence of the series

\[
S_{kk+1}(b) = \sum_{m-k+1}^{\infty} b_{km} b_{km+1}^{-1} 
\]

Since the one-parameter groups

\[
G_{kk+1} = \{ t_{kk+1} \in B_{0}^{\infty} \mid t_{kk+1} = I + t E_{kk+1}, t \in R^1 \}, \quad k \in N, 
\]

generate the group \( B_{0}^{\infty} \), then the condition that \( \mu_{kk+1} \sim \mu_{kk} \), \( t \in B_{0}^{\infty} \), is equivalent to the condition that \( \mu_{kk+1} \sim \mu_{kk} \), \( k \in N \), which proves the lemma.

\section*{Irreducibility and Equivalence of Finite Upper-Triangular Matrices}

Remark 1.1. From what we have proved it follows in particular that the conditions \( S_{kk+1}(b) < \infty, k \in N, \) and

\[
S_{kk}(b) = \sum_{m-n+1}^{\infty} b_{kn} b_{kn+1}^{-1} < \infty, \quad k, n \in N, \quad k < n, 
\]

are equivalent.

Lemma 1.3. The measure \( \mu_{p} \) on \( B_{0}^{\infty} \), is \( B_{0}^{\infty} \)-ergodic relative to the right action.

Proof. It is well known that any measurable function on \( R^\infty \) with the standard Gaussian measure, invariant under any change of the first coordinates, coincides almost everywhere with a constant function (§3, Corollary 1). Therefore the proof follows from the fact that the measure \( \mu_{t} \) is a tensor product of measures, and the fact that the subgroup \( B(m, R) \) of the group \( B_{0}^{\infty} \) acts transitively on the subgroup \( B(m, R) \subset B_{0}^{\infty} \).

\section*{§2. Irreducibility of representations}

The proof of the irreducibility of a right regular representation is based on the ergodicity of the measure \( \mu_{p} \) relative to right shifts by elements of the group \( B_{0}^{\infty} \), and on the fact that the operators of multiplication by the independent variables may be approximated by the generators of one-parameter groups.

Proof of Theorem 1.1. The necessity is obvious. We will prove the sufficiency. Suppose that \( (\mu_{p})^{t} \perp \mu_{t}, \quad t \in B_{0}^{\infty} \). Then, by Lemma 1.2,

\[
S_{kn}(b) = \infty, \quad k, n \in N, \quad k < n. 
\]

We denote by \( \mathcal{W}(b) \) the set of selfadjoint or skew-selfadjoint operators in \( \mathcal{A}(b) \) adjoined to the algebra \( \mathcal{W}(b) = (B_{0}^{\infty} \mid t \in B_{0}^{\infty}) \), and show that

\[
\mathcal{W}(b) = \{ x_{kn}, \partial_{pq} - b_{pq} x_{pq} \mid k < n, p < q, k, n, p, q \in N \}. 
\]

We give the notation for the generators of the right shift \( A_{kn}^{R} = A_{kn}^{R,b} \):

\[
A_{kn}^{R,b} = \frac{d}{dt} T^{R,b}(I + t E_{kn}) \bigg|_{t=0}, \quad k, n \in N, \quad k < n. 
\]

We calculate directly that

\[
A_{kn}^{R,b} = \sum_{m=1}^{k} x_{kn}(\partial_{mn} - b_{mn} x_{mn}), \quad x_{kk} = 1, \quad k < n, \quad \partial_{kn} = \partial/\partial k_{kn}. 
\]
Lemma 2.1. \( \{x_{mn}, \partial_{pq} - b_{mn}x_{pq} | m < n, p < q; m, n, p, q \in \mathbb{N} \} \subset \mathcal{W}(b) \).

We will carry out the proof by induction.

**Basis of the induction.** We will prove that
\[
\{x_{12}, \partial_{1k} - b_{1k}x_{1k}, \partial_{2k+1} + b_{2k+1}x_{2k+1}, k = 2, 3, \ldots \} \subset \mathcal{W}(b).
\]

Indeed, the operator \( x_{12} \) may be approximated by a linear combination of operators \( A_{n}^{k}, A_{n}^{k}, n > 2 \). For the proof of this we use a method of calculation due to R. S. Ismagilov (Lemmas 2.2-2.4). The original proof of Lemmas 2.2 and 2.4 was more complicated (see [11]).

**Lemma 2.2.** The operator \( x_{12} \) may be approximated by a linear combination of operators \( A_{n}^{k}, A_{n}^{k}, k > 2 \) if and only if
\[
\sigma_{12}(b) = \sum_{n=3}^{\infty} \frac{b_{2n}}{b_{n}} = \infty.
\]

**Proof.** We calculate the deviation of \( x_{12} \) from the linear span of the vectors
\[
A_{n}^{k} = \partial_{1n} - b_{1n}x_{1n}, \quad A_{n}^{k} = x_{12}(\partial_{1n} - b_{1n}x_{1n}) + (\partial_{2n} - b_{2n}x_{2n})
\]
(see (2.1)), then
\[
A_{n}^{k}A_{n}^{k} = x_{12}(b_{1n}^{2}x_{1n}^{2} - b_{1n}^{2}) + b_{1n}b_{2n}x_{1n}x_{2n} = -b_{1n}x_{12} + b_{1n}x_{1n}y_{1n} + b_{1n}b_{2n}x_{1n}x_{2n}.
\]

We have made a change of variables:
\[
x_{1n}^{2} = \left( x_{1n}^{2} - \frac{1}{2b_{1n}} \right) + \frac{1}{2b_{1n}} = y_{1n} + \frac{1}{2b_{1n}};
\]
then
\[
\int y_{1n} d\mu_{b} = 0 \quad \text{and} \quad \int y_{1n}^{2} d\mu_{b} = \frac{1}{2b_{1n}}.
\]
We multiply both sides of (2.2) by \( t_{n}, N_{1} \leq n \leq N_{2} \), such that
\[
\frac{1}{2} \sum_{n=N_{2}}^{N_{2}} b_{1n}t_{n} = 1, \quad \text{and sum on } n:
\]
\[
\sum_{n=N_{1}}^{N_{2}} t_{n}A_{n}^{k}A_{n}^{k} = x_{12} + \sum_{n=N_{1}}^{N_{2}} t_{n}b_{1n}x_{1n}x_{2n} + \frac{1}{2} \sum_{n=N_{2}}^{N_{2}} t_{n}b_{1n}b_{2n}x_{1n}x_{2n}.
\]

We write \( \omega_{12}(b) = \sum_{n=N_{1}}^{N_{2}} t_{n}A_{n}^{k}A_{n}^{k}x_{12} = 1 = \sum_{n=N_{1}}^{N_{2}} t_{n}b_{1n}x_{1n}x_{2n} + \frac{1}{2} \sum_{n=N_{2}}^{N_{2}} t_{n}b_{1n}b_{2n}x_{1n}x_{2n} \).

Since all the terms are uncorrelated, then
\[
||\omega_{12}(b)||^{2} = \sum_{n=N_{1}}^{N_{2}} t_{n}^{2} \left[ \frac{b_{1n}^{4}}{4b_{1n}^{2}b_{2n}^{2} + 2b_{1n}^{2}b_{2n}^{2}} \right]^{2} \geq \sum_{n=N_{1}}^{N_{2}} t_{n}^{2}b_{1n}^{2}b_{2n}^{2} = \sum_{n=N_{1}}^{N_{2}} t_{n}^{2}y_{n}.
\]

Now we choose the \( t_{n} \) so as to minimize \( \omega_{12}(b) \). It is easy to see that
\[
\min \left\{ \sum_{n=N_{1}}^{N_{2}} t_{n}^{2}y_{n}, \sum_{n=N_{1}}^{N_{2}} t_{n}b_{2n} = -2 \right\} = 4 \left( \sum_{n=N_{1}}^{N_{2}} \frac{b_{2n}}{b_{n}} \right)^{-1},
\]
the minimum being taken on at
\[
t_{n} = -\frac{2b_{1n}}{b_{n}^{2}} \left( \sum_{n=N_{1}}^{N_{2}} \frac{b_{2n}}{b_{n}} \right)^{-1}.
\]

Hence, with the optimal choice for \( t_{n} \), we get
\[
||\omega_{12}(b)||^{2} = 4 \left( \sum_{n=N_{1}}^{N_{2}} \frac{b_{2n}}{b_{n}} \right)^{-1}.
\]

We will require that \( \sum_{n=3}^{\infty} b_{n}^{2}y_{n}^{-1} = \infty, \) i.e.
\[
\sum_{n=3}^{\infty} \frac{b_{n}^{2}}{b_{n}^{2} + b_{n}^{2}} = \sum_{n=3}^{\infty} \frac{1}{1 + b_{n}^{2}}, \quad \text{and the fact that the } b_{n} \text{ are real}
\]
\[
\sum_{n=3}^{\infty} b_{n} = \infty \quad \Rightarrow \quad \sum_{n=3}^{\infty} b_{n} = \infty. \quad \square
\]

Thus
\[
x_{12} \in \mathcal{W}(b), \quad \partial_{1k} - b_{1k}x_{1k} = A_{n}^{k} \in \mathcal{W}(b), \quad k > 1,
\]

\[
\partial_{2k} - b_{2k}x_{2k} = A_{n}^{k} = x_{12} \left( \partial_{1k} - b_{1k}x_{1k} \right) \in \mathcal{W}(b), \quad k > 2.
\]

Now we will show that the convergence \( \sum_{n=N_{1}}^{N_{2}} t_{n}A_{n}^{k}A_{n}^{k} \rightarrow x_{12} \) of the self-adjoint operators \( A_{N_{1}}, A_{N_{2}} \) follows from the commutation relations \( [A_{n}^{k}, A_{n}^{k}] = 0, n, q \geq 3, \) the skew-selfadjointness \( A_{n}^{k} = k > 2, \) and the fact that the \( t_{n} \) are real to the selfadjoint operator \( A = x_{12} \) holds in the strong resolvent sense. By Theorem VIII.25 of [17], it suffices to show the convergence \( A_{n}^{k}f \rightarrow Af \) for any \( f \in D, \) where \( D \) is a common essential domain for all the operators \( A_{n}, n \geq 2. \)
and \( A \). For the role of \( D \) we choose a dense set consisting of finite linear combinations of arbitrary monomials:

\[
x_k = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} x_{k+1}^{\alpha_{k+1}} \cdots, \quad \alpha_j = 0, 1, \ldots,
\]

\( i < j \). Obviously \( D \) is a common essential domain for the operators \( A_{N_1}, A_{N_2} \) and \( A \), since \( D \) consists of analytic vectors for the operators \( A_{N_1}, A_{N_2} \) and \( A \). Suppose that \( f \) is cylindrical, then there exists an \( n_0 \in \mathbb{N} \) such that \( f \) does not depend on the variables \( x_{n+1}, x_{n+2} \) for \( n > n_0 \).

Suppose that \( N_1 > n_0 \). Then

\[
\left\| (x_{12} - \sum_{n=N_1}^{N_2} t_n A_{n_N} A_{m_n}) f \right\|^2 = \left\| (x_{12} - \sum_{n=N_1}^{N_2} t_n (x_{12} (b_{12} x_{12}^2 - b_{12}) + b_{12} x_{12}^2 x_{12}^2)) \right\|^2 \\
= \left\| x_{12} f \right\|^2 \cdot \left( (1 - \sum_{n=N_1}^{N_2} t_n (b_{12} x_{12}^2 - b_{12})) \right) \right\|^2 \\
+ \left\| f \right\|^2 \cdot \left( \sum_{n=N_1}^{N_2} t_n b_{12} x_{12}^2 x_{12}^2 \right) \rightarrow 0,
\]

since, by what has been proved,

\[
\left\| \omega_{12}(b) \right\|^2 = \left\| (x_{12} - \sum_{n=N_1}^{N_2} t_n A_{n_N} A_{m_n}) \right\|^2 \\
= \left\| x_{12} \right\|^2 \cdot \left( (1 - \sum_{n=N_1}^{N_2} t_n (b_{12} x_{12}^2 - b_{12})) \right) \right\|^2 \\
+ \left\| f \right\|^2 \cdot \left( \sum_{n=N_1}^{N_2} t_n b_{12} x_{12}^2 x_{12}^2 \right) \rightarrow 0,
\]

will be small for appropriate \( N_1, N_2 \).

**The induction step.** Suppose that the inclusion

\[
\{x_{nm}, n < m \leq p, \partial_{nm} - b_{nm} x_{nm}, 1 \leq n \leq p, m > n \} \subset W(b)
\]

holds. We will show that then

\[
\{x_{np+1}, \partial_{p+1,m} - b_{p+1,m} x_{p+1,m} | 1 < p + 1 < m \} \subset W(b).
\]

We will present the proof of this assertion in the form of some lemmas.

It may happen that the operators \( x_{ik}, l < k \), can be approximated, in analogy with \( x_{12} \), by operators \( A_{n_N} A_{m_n}, k < n \). However, the following considerations show that this is not always possible for

\[
S_{k_N}(b) = \infty, \quad k < n.
\]
Obviously, $SL(bt_1') = \frac{k}{n}$, but it follows from Lemma 2.3 that for the weight $b^{(i)}$ it is not possible to approximate any operator $X^k_{ik}$, $k \leq 3$, by operators $A_i A_j$. The operator $x_{i2}$ may however be so approximated. It is better to approximate with operators of the type $X^k_{il}$.

Lemma 2.4. For the approximation of the variables $x_{ik}$, $l < k$, by the operators $(\partial_{in} - b_{in}x_{in})A^k_{in}$, $k < n$, it is necessary and sufficient that

$$\sigma_k(b) = \sum_{n=k+1}^{\infty} b_{in} \left( \sum_{m=1}^{k} b_{mn} \right)^{-1} = \infty, \quad l < k.$$
which is equivalent to
\[ \sigma_k(b) = \sum_{n=k+1}^{\infty} b_n \left( \sum_{m=1, m \neq i}^{k} b_{mn} \right)^{-1} = \infty, \quad l < k. \]
\[ \square \]

However, Example 2.1 shows that, for the weight \( b^{(1)} \), it is, as before, not possible to approximate any operator \( f \to x_{ik} f, \ k \geq 3 \), by operators \( (\delta_{in} - b_{in} x_{in}) A^k_{in} \). As before, \( x_{i2} \) can be.

It turns out that for any \( q = 2, 3, \ldots \) there exists a \( p < q \) such that the variables \( x_{pq} \) can be approximated by operators \( (\delta_{mp} - b_{mp} x_{mp}) A^k_{mn}, \ n > q. \)

**Lemma 2.5.** Suppose that \( S^q_{k}(b) = \infty, \ k = 1, 2, \ldots, q - 1 \). Then there exists a \( p < q \) such that \( \sigma_{pq}(b) = \infty \).

**Proof.** We will carry out an induction. Suppose that \( q = 3 \) and
\[ S^3_{13}(b) = \sum_{k=4}^{\infty} b_{3k} b^2_{3k} = \infty, \quad S^6_{12}(b) = \sum_{k=4}^{\infty} b_{3k} = \infty. \]
Suppose the contrary, i.e. that
\[ \sigma_{13}(b) = \sum_{k=4}^{\infty} \frac{b_{1k}}{b_{3k}} < \infty \quad \text{and} \quad \sigma_{23}(b) = \sum_{k=4}^{\infty} \frac{b_{2k}}{b_{3k}} < \infty. \]
From the fact that \( \sigma_{13}(b) < \infty \) it follows that \( b_{1k} < b_{2k} + b_{3k}, \ k \geq 3, \) so that
\[ \sigma_{23}(b) > \sum_{k=4}^{\infty} b_{2k} > \sum_{k=4}^{\infty} \frac{b_{2k}}{b_{3k}} = \infty, \]
which contradicts the induction hypothesis, after the notations \( b_{kn} = b_{k+1,n+1} \), \( k < n \), are adjusted.

**Example 2.2.** Suppose that \( b^{(1)}_{13} = n^2, 1 = k < n, b^{(1)}_{13} = 1, 1 < k < n. \) Then
\[ S^q_{k}(b^{(1)}) = \infty, \ k < q, \quad \sigma_{1q}(b^{(1)}) = \infty, \quad \sigma_{pq}(b^{(1)}) < \infty, \ 1 < p < q. \]

There remains some “adjustment” of the operators \( (\delta_{in} - b_{in} x_{in}) A^k_{in}, \ n < q, \) in order that the variables \( x_{ik}, l < k, \) might be approximated.

**Lemma 2.6.** Suppose that \( S^q_{k}(b) = \infty, \ k < n. \) Then the variables \( x_{ik}, 1 < k, \) are approximated by the operators
\[ \left( \delta_{in} - b_{in} x_{in} \right) A^k_{in}, \]
where
\[ A^{[n_1, \ldots, n_k]}_{kn} = \sum_{m=1, m \neq n_1, \ldots, n_k}^{k} x_{mk} (\delta_{mn} - b_{mn} x_{mn}), \]
in view of \( \sigma_{pq}(b) = \infty \) for some \( p < q \). We will prove this for \( q + 1. \) Suppose the contrary, i.e. that \( \sigma_{q+1}(b) = \infty, r = 1, 2, \ldots, q. \) From
\[ \sigma_{1q+1}(b) = \sum_{n=q+2}^{\infty} \frac{b_{1n}}{b_{mn}} < \infty \]
it follows that \( b_{1n} < \sum_{m=1}^{q+1} b_{mn}, \ n \geq n_0. \) We substitute this into \( \sigma_{q+1}(b) < \infty, \)
\[ \sigma_{q+1}(b) > \sum_{n=n_0}^{\infty} b_{mn} \left( \sum_{m=2, m \neq r}^{q+1} b_{mn} \right)^{-1} = \infty, \quad r = 2, 3, \ldots, q. \]
This last is equivalent to
\[ \sigma_{p+1}(b) = \sum_{n=p+2}^{\infty} b_{mn} \left( \sum_{m=1, m \neq r}^{p+1} b_{mn} \right)^{-1} = \infty, \quad 2 \leq r \leq q, \]
while \( S^q_{k+1}(b) = \infty, 2 \leq k \leq q, \) which contradicts the induction hypothesis, after the notations \( b_{kn} = b_{k+1,n+1} \), \( k < n \), are adjusted.

The following example shows that, for any \( q \), there might be only one \( p < q \) with \( \sigma_{pq}(b) = \infty \).

**Example 2.2.** Suppose that \( b^{(1)}_{13} = n^2, 1 = k < n, b^{(1)}_{13} = 1, 1 < k < n. \) Then
\[ S^q_{k}(b^{(1)}) = \infty, \ k < q, \quad \sigma_{1q}(b^{(1)}) = \infty, \quad \sigma_{pq}(b^{(1)}) < \infty, \ 1 < p < q. \]

It turns out that for any \( q = 2, 3, \ldots \) there exists a \( p < q \) such that the variables \( x_{ik}, l < k, \) might be approximated.

**Lemma 2.5.** Suppose that \( S^q_{k}(b) = \infty, k = 1, 2, \ldots, q - 1 \). Then there exists a \( p < q \) such that \( \sigma_{pq}(b) = \infty \).

**Proof.** We will carry out an induction. Suppose that \( q = 3 \) and
\[ S^3_{13}(b) = \sum_{k=4}^{\infty} b_{3k} b^2_{3k} = \infty, \quad S^6_{12}(b) = \sum_{k=4}^{\infty} b_{3k} = \infty. \]
Suppose the contrary, i.e. that
\[ \sigma_{13}(b) = \sum_{k=4}^{\infty} \frac{b_{1k}}{b_{3k}} < \infty \quad \text{and} \quad \sigma_{23}(b) = \sum_{k=4}^{\infty} \frac{b_{2k}}{b_{3k}} < \infty. \]
From the fact that \( \sigma_{13}(b) < \infty \) it follows that \( b_{1k} < b_{2k} + b_{3k}, \ k \geq 3, \) so that
\[ \sigma_{23}(b) > \sum_{k=4}^{\infty} b_{2k} > \sum_{k=4}^{\infty} \frac{b_{2k}}{b_{3k}} = \infty, \]
which contradicts the induction hypothesis, after the notations \( b_{kn} = b_{k+1,n+1} \), \( k < n \), are adjusted.

**Example 2.2.** Suppose that \( b^{(1)}_{13} = n^2, 1 = k < n, b^{(1)}_{13} = 1, 1 < k < n. \) Then
\[ S^q_{k}(b^{(1)}) = \infty, \ k < q, \quad \sigma_{1q}(b^{(1)}) = \infty, \quad \sigma_{pq}(b^{(1)}) < \infty, \ 1 < p < q. \]

There remains some “adjustment” of the operators \( (\delta_{in} - b_{in} x_{in}) A^k_{in}, \ n < q, \) in order that the variables \( x_{ik}, l < k, \) might be approximated.

**Lemma 2.6.** Suppose that \( S^q_{k}(b) = \infty, k < n. \) Then the variables \( x_{ik}, 1 < k, \) are approximated by the operators
\[ \left( \delta_{in} - b_{in} x_{in} \right) A^k_{in}, \]
where
\[ A^{[n_1, \ldots, n_k]}_{kn} = \sum_{m=1, m \neq n_1, \ldots, n_k}^{k} x_{mk} (\delta_{mn} - b_{mn} x_{mn}), \]
in view of \( \sigma_{pq}(b) = \infty \) for some \( p < q \). We will prove this for \( q + 1. \) Suppose the contrary, i.e. that \( \sigma_{q+1}(b) = \infty, r = 1, 2, \ldots, q. \) From
\[ \sigma_{1q+1}(b) = \sum_{n=q+2}^{\infty} \frac{b_{1n}}{b_{mn}} < \infty \]
it follows that \( b_{1n} < \sum_{m=1}^{q+1} b_{mn}, \ n \geq n_0. \) We substitute this into \( \sigma_{q+1}(b) < \infty, \)
\[ \sigma_{q+1}(b) > \sum_{n=n_0}^{\infty} b_{mn} \left( \sum_{m=2, m \neq r}^{q+1} b_{mn} \right)^{-1} = \infty, \quad r = 2, 3, \ldots, q. \]
Hence, for some $n_1 < p + 1$, $\sigma_{n_1 p}^{(1)}(b) = \infty$, i.e., we may approximate $x_{n_1 p + 1}$ by the operators $(\delta_{n_1 p} - b_{n_1}x_{n_1})A^{p+1}_{n_1}$. In view of Lemma 2.5, $x_{n_1 p + 1}$ may be approximated by the operators $\delta_{n_1}A^{p+1}_{n_1}$ if and only if

$$\sigma_{n_1 p + 1}^{(1)}(b) = \sum_{m = n_1 + 1}^{\infty} b_m \left( \sum_{r = m-n_1+1}^{p+1} b_{2r} \right)^{-1} = \infty, \quad 1 \leq k \leq p, \quad r \neq n_1.$$

By Lemma 2.5, one of the series $\sigma_{n_1 p + 1}(b)$ diverges. Suppose that $\sigma_{n_1 p + 1}(b) = \infty$. Then we may approximate $x_{n_2 p + 1}$, where $n_2 \neq n_1$, by operators $\delta_{n_2}A^{p+1}_{n_2}$, and so forth. As a result we obtain a sequence $n_1, \ldots, n_p$ such that the variables $x_{n_k p + 1}, k = 1, \ldots, p$, may be approximated by operators $\delta_{n_k}A^{p+1}_{n_k}$, and $\{n_1, \ldots, n_p\}$ is a permutation of $\{1, \ldots, p\}$. From

$$\partial_{p+1}x_m - b_{p+1}x_{p+1} = A^{p+1}_m - \sum_{r=1}^{p} x_r (\partial_{m} - b_m x_{m})$$

it follows that the inclusion

$$\partial_{p+1}x_m - b_{p+1}x_{p+1} \in \mathbb{W}(b), \quad m > p + 1,$$

holds; this completes the proof of Lemma 2.1.

Thus we have adjoined to the von Neumann algebra $W(b) = (T^{k+1}_{\varepsilon} \mid \varepsilon \in B^{0})$ the operators of multiplication by the independent variables $x_{kn}, k < n, k, n \in \mathbb{N}$. Therefore the von Neumann algebra contains the operators

$$\{U_{kn}(t) = \exp(itx_{kn}) \mid t \in \mathbb{R}, k, n \in \mathbb{N}, k < n\}.$$ 

Now suppose that the bounded operator $A \in L(L_{2}(B^{0}, \mu_{b})), \mu_{b} \in B^{0}$, commutes with all the operators $(T^{k+1}_{\varepsilon} \mid \varepsilon \in B^{0})$. We will show that then it is a multiple of the identity: $A = \lambda I, \lambda \in \mathbb{C}$. Indeed, in this case $A$ commutes with the operators $U_{kn}(t)$. Hence $A$ is the operator of multiplication by an essentially bounded function: $A = f_{A}(x)$. In view of the commutation relations $[f_{A}(x), T^{k+1}_{\varepsilon}] = 0$, we conclude that the function $f_{A}(x)$ is invariant relative to the action of the group $B_{0}^{0}$. We will show that $f_{A}(x) = \mu_{b}(x)\psi_{b}(x)$ almost everywhere, where $\mu_{b}(x) = L_{2}(B^{0}, \mu_{b})$, will intertwine the representations $T^{k+1}_{\varepsilon}$ of the group $B^{0}$. Thus we have constructed a family $T^{k+1}_{\varepsilon}, b \in \mathcal{A}$, of analogues to regular representations of the group $B^{0}$. Among these the irreducible representations are distinguished by the condition $b \in \mathcal{A}$:

$$\mathcal{A}^{k} = \{b \in \mathcal{A} \mid S_{kn}(b) = \sum_{m = n+1}^{\infty} b_{kn}b_{km}^{-1} = \infty, \quad k, n \in \mathbb{N}, k < n\}.$$ 

§3. Equivalence of representations

The question naturally arises as to which of the irreducible representations $T^{k+1}_{\varepsilon}, b \in \mathcal{A}$, are equivalent.

Theorem 3.1. The irreducible representations $T^{k+1}_{\varepsilon}$ and $T^{k+1}_{\psi}$ are equivalent if and only if the measures $\mu_{b}(x)$ and $\mu_{b}(x)$ are equivalent.

It is well known (see [17], Chap. II) that two product measures $\mu_{b}(x)$ and $\mu_{b}(x)$ are equivalent if and only if

$$H(b^{(1)}, b^{(2)}) = H(\mu_{b}(x), \mu_{b}(x)) = \left( \prod_{k < \infty} \frac{(b^{(1)}_{k} + b^{(2)}_{k})^2}{4(b^{(1)}_{k}b^{(2)}_{k})} \right)^{-1} > 0.$$ (3.1)

Theorems 2.1 and 3.1 give a final description of the regular representations $T^{k+1}_{\varepsilon}, b \in \mathcal{A}$, of the group $B^{0}$.

Theorem 3.2. Among the right regular representations $T^{k+1}_{\varepsilon}, b \in \mathcal{A}$, the irreducible ones are distinguished by the condition $b \in \mathcal{A}$. Among the irreducible representations, equivalent ones, are distinguished by the condition $H(\mu_{b}(x), \mu_{b}(x)) > 0$.

The proof of Theorem 3.1 is based on an explicit computation of the spectral measures $\sigma_{b}^{(m)}$ of the restrictions of the representations $T^{k+1}_{\varepsilon}$ to the infinite-dimensional commutative subgroups

$$B_{0}^{0} = B_{0}^{0} \cdots B_{0}^{0} \cdots = B_{0}^{0} \cdots B_{0}^{0},$$

and on the comparison of these spectral measures using Hellinger integrals. The calculation of the spectral measure $\sigma_{b}^{(m)}$ makes use of a partial Fourier transform, due to D. N. Nessonov [13], carrying the generators of one-parameter groups of $B_{0}^{m}$ into operators of multiplication by a function.

The sufficiency is obvious. Indeed, suppose that $\mu_{b}(x) \sim \mu_{b}(x)$. Then $\mu_{b}(x) \sim \mu_{b}(x)$, and the unitary operator $U : \mathcal{N}(b^{(1)}) \to \mathcal{N}(b^{(2)})$ of multiplication by the function $(d\mu_{b}(x)/d\mu_{b}(x))(x)$, where $\mathcal{N}(b) = L_{2}(B^{0}, \mu_{b})$, will intertwine the representations $T^{k+1}_{\varepsilon}$ and $T^{k+1}_{\psi}$, i.e.

$$UT^{k+1}_{\varepsilon} = T^{k+1}_{\psi}U, \quad t \in B^{0}.$$ 

Necessity. We will prove that $T^{k+1}_{\varepsilon} \sim T^{k+1}_{\psi}$ implies that $\mu_{b}(x) \sim \mu_{b}(x)$. Denote by $W(b) = (T^{k+1}_{\varepsilon} \mid \varepsilon \in B^{0})$ the von Neumann algebra generated by the operators $(T^{k+1}_{\varepsilon} \mid \varepsilon \in B^{0})$, by $\mathcal{W}(b)$ the set of selfadjoint or skew-selfadjoint operators $A = \int \lambda dE(\lambda)$ adjoined to the algebra $W(b)$, i.e. such that their spectral projec-
tors $E_A(\Delta)$ lie in $W(b)$, and by $A \in \mathcal{B}(\mathbb{R})$ the $\sigma$-algebra of Borel sets on the axis. Suppose that $W_k(b) = \{x_{kn} \mid k < n\}$ is the set of operators of multiplication by the independent variables $f \mapsto x_{kn}f$ in the space $\mathcal{H}(b)$. Then, as we proved in Theorem 2.1, $W_k(b) \subseteq W(b)$. Suppose that

$$\nabla_{\mu} = \{(k, n) \mid k < n \leq m\},$$

where

$$\mathcal{A}(b) = \mathcal{B} \mu_{\mathcal{A}}(b) \cap \mathcal{B} \mu_{\mathcal{A}}(b).$$

We note that $B^\infty$ is a commutative subgroup of the group $B$. Then

$$\hat{W}(b)^{\infty} = \{iA_{kn} \mid (k, n) \in \nabla_{\mu}\}$$

is a commuting family of operators of the set $\hat{W}(b)$. We recall that the spectral measure $\sigma(\Delta)$ of a family $A = (A_k)_{k \in \mathbb{N}}$ of selfadjoint operators $A_k$, $k \in \mathbb{N}$, commutative in the sense of resolution of identity, is any scalar measure $\sigma(\Delta, \lambda)$ on the $\sigma$-algebra $\mathcal{B}(\mathbb{R}^\infty) \ni \Delta$, generated by cylindrical sets with Borel bases

$$C(\mathbb{R}^\infty) = \{C(k_1, \ldots, k_n, \Delta_1, \ldots, \Delta_n) = \{x \in \mathbb{R}^\infty \mid x_k \in \Delta_k, k = 1, \ldots, n, \Delta_k \in \mathcal{B}(\mathbb{R})\mid \Delta_k \in \mathcal{B}(\mathbb{R}), k = 1, \ldots, n\},$$

equivalent to the joint resolution of identity $E_A$ of the family $A$ of operators, defined on the cylindrical sets by the formula

$$E_A(C(k_1, \ldots, k_n, \Delta_1, \ldots, \Delta_n)) = E_{\Delta_1} \cdots E_{\Delta_n}.$$

Suppose that $\sigma^{\infty} = \sigma(\hat{W}(b)^{\infty})$ is the spectral measure of the family of operators $\hat{W}(b)^{\infty}$, $m \in \mathbb{N}$.

Assume that the representations $T_{R_k^{(1)}}$ and $T_{R_k^{(2d)}}$ are equivalent, i.e. that there exists a unitary operator $U : \mathcal{H}(b)^{(1)} \to \mathcal{H}(b)^{(2d)}$ such that $UT_{R_k^{(1)}} = T_{R_k^{(2d)}}U$, $t \in B^\infty$. We will write for short $T_{R_k^{(2d)}} \simeq T_{R_k^{(2d)}}$.

The proof of necessity rests on two lemmas.

Lemma 3.1. Suppose that $\sigma^{\infty} \sim \sigma^{\infty}$. Then

$\mu_{\mathcal{A}}^{\infty} \sim \mu_{\mathcal{A}}^{\infty}$, $m \in \mathbb{N}$.

Lemma 3.2. Assume that $T_{R_k^{(1)}}$ and $T_{R_k^{(2d)}}$ are equivalent irreducible unitary representations: $T_{R_k^{(1)}} \simeq T_{R_k^{(2d)}}$. Then $W_k(b)^{(1)} \sim W_k(b)^{(2d)}$ with the same intertwining operator $U$, and $Ux_{kn} = x_{kn}U$, $k < n$.

The necessity of Theorem 3.1 follows from Lemma 3.2. Indeed, $T_{R_k^{(1)}} \simeq T_{R_k^{(2d)}} \Rightarrow W_k(b)^{(1)} \sim W_k(b)^{(2d)} \Rightarrow \sigma(W_k(b)^{(1)}) \sim \sigma(W_k(b)^{(2d)})$.

But the spectral measure $\sigma(W_k(b))$ of the family of operators of multiplication by independent variables in the space $\mathcal{H}(b)$ is obviously equivalent to $\mu_{\mathcal{A}}^{\infty}$. Hence we have

$\sigma(W_k(b)^{(1)}) \sim \sigma(W_k(b)^{(2d)}) \Rightarrow \mu_{\mathcal{A}}^{\infty} \sim \mu_{\mathcal{A}}^{\infty} \Rightarrow \mu_{\mathcal{A}}^{\infty} \sim \mu_{\mathcal{A}}^{\infty}$.

The proof of Lemma 3.1 comes down to the explicit calculation of the spectral measures $\sigma^{\infty}$, $m \in \mathbb{N}$, and the calculation of the Hellinger integrals $H(\sigma^{\infty}, \sigma^{\infty})$.

We recall the definition and properties of the Hellinger integral ([12], Chap. 2, §2).

Suppose that $\mu$ and $v$ are two probability measures on the measure space $(\mathcal{X}, \mathcal{B})$. Assume that $\lambda$ is a probability measure such that $\mu < \lambda, v < \lambda$, for example $\lambda = (\mu + v)/2$. The Hellinger integral for $\mu$ and $v$ is defined as follows:

$$H(\mu, v) = \int_\mathcal{X} \sqrt{\frac{d\mu}{d\lambda} \frac{d\nu}{d\lambda}} d\lambda.$$

It does not depend on $\lambda$, and has the following properties:

(H1) $0 \leq H(\mu, v) \leq 1$ (the Schwarz inequality);

(H2) $H(\mu, v) = 1$ if $\mu = v$;

(H3) $H(\mu, v) = 0$ if $\mu \perp v$;

(H4) $\mu \sim v \Rightarrow H(\mu, v) > 0$.

The converse to (H4) does not hold in general.

We fix on a number $m \in \mathbb{N}$ and do a Fourier transform $\mathcal{F}_m$ of the space $\mathcal{H}_{\nu}(b) \otimes \mathcal{H}_{\sigma}(b)$, in which the operators $A_{kn}$ of the family $\hat{W}(b)^{\infty}$ act.

Write $b^\infty = p^{-1}(b^\infty)$, $b^\infty = p^{-1}(b^\infty)$. Suppose that

$$t = \sum_{k < n \leq m} x_{kn}E_{kn} \in b^\infty, \quad y = \sum_{k < m < n} x_{kn}E_{kn}, \quad x = \sum_{k < m < n} x_{kn}E_{kn} \in b^\infty.$$
Obviously $B^{\circ m}$ is a commutative normal subgroup in $B^m$. Consider the semi-direct product $B^{\circ m} \rtimes B^m$. Obviously

$$1 + t + y = (1 + y)(1 + t) \in B^{\circ m} \rtimes B^m.$$

For the function

$$f_\rho(t) = \exp \left( \frac{1}{2} \sum_{(k,n) \in \square_m} (x_{kn} b_{kn}^t) \right) \exp \left( \int_{(k,n) \in \square_m} i(x,y) f(Y) (\sigma^\rho_Y) D \rho(t) \right),$$

we define the partial Fourier–Weiner transform (see [4], or [5], Chap. II, §6)

$$(\mathcal{F}_m f)(\rho(x) \rho(t)) = \exp \left( \frac{1}{2} \sum_{(k,n) \in \square_m} (x_{kn} b_{kn}^t) \right) \exp \left( \int_{(k,n) \in \square_m} i(x,y) f(Y) (\sigma^\rho_Y) D \rho(t) \right),$$

where $(b/2)_{km} = b_{kn}/2$, $k < n$, and $B$ is the diagonal operator $(Bx)_{km} = b_{kn} x_{kn}$.

Suppose that

$$z = \sum_{k = m < n} z_{km} b_{km} \in b^{\circ m}.$$

We will calculate $\rho(y) \rho(t) \rho(z)$. We have

$$\rho(y) \rho(t) \rho(z) = \rho(y) \rho(t) \rho(z) \rho(t)^{-1} \rho(t).$$

Since $B^{\circ m}$ is a normal subgroup in $B^{\circ m} \rtimes B^m$, then

$$\rho(t) \rho(z) \rho(t)^{-1} = \text{Ad}_{\rho(t)} \rho(z) \in B^{\circ m}.$$

Therefore the mapping $\varphi_{\rho(t)}: B^{\circ m} \rightarrow B^{\circ m}$, $\varphi_{\rho(t)}(\rho(z)) = \rho(t) \rho(z) \rho(t)^{-1} \rho(t)$.

In the Fourier images the operators $T^{\circ m}_{\rho(t)}$, $\rho(z) \in B^{\circ m}$, take the form

$$\mathcal{F}_m (\rho(x) \rho(t)) \mapsto \exp \left( \langle (B^{-1} x, x) \rangle \int_{\square_m} \exp \left( i(x,y) f(y + z') \rho(t) \right) \right) \int_{\square_m} \exp \left( i(x,y) f(y + z') \rho(t) \right) \exp \left( -i(x,y)^2 \right),$$

where $w = y + z'$, $y = w - z'$, $\exp i(x,y) = \exp i(k, w)$, then $\exp i(x, y) = \exp i(k, w) \exp ( -i(x, z') ).$

Accordingly,

$$(\mathcal{F}_m T^{\circ m}_{\rho(t)} \mathcal{F}_m f)(\rho(x) \rho(t)) = \exp \left( -i(x, \rho(t) z) \right) \exp (\rho(x, \rho(t) z)).$$

The generators of $\mathcal{A}^{\circ m}$ now go over into $i\mathcal{A}^{\circ m}$, the operators of multiplication by the following functions:

$$i\mathcal{A}^{\circ m} \mapsto \mathcal{F}_m \left( \frac{d}{dz} \right) \exp \left( -i(x, \rho(t) z) \right) \exp \left( i(x, \rho(t) z) \right).$$

Since $\mathcal{F}_m: X^{\circ m} \rightarrow X^{\circ m}$, then

$$\mathcal{F}_m \left( \frac{d}{dz} \right) \exp \left( -i(x, \rho(t) z) \right) \exp \left( i(x, \rho(t) z) \right) = \mathcal{F}_m \left( \frac{d}{dz} \right) \exp \left( -i(x, \rho(t) z) \right) \exp \left( i(x, \rho(t) z) \right).$$

The measure $\sigma^\rho_{\mathcal{A}^{\circ m}}$ on the group $B^{\circ m}$ and its image $\sigma^\rho_{\mathcal{A}^{\circ m}}$ on the algebra $\mathcal{A}^{\circ m}$ will be denoted by the same symbol: $\sigma^\rho_{\mathcal{A}^{\circ m}}$. 

$\text{IRREDUCIBILITY AND EQUIVALENCE OF FINITE UPPER-TRIANGULAR MATRICES}$
We will show that the measure $\sigma_{b^m}$ on the algebra $b^m$, where $\mathcal{B}(b^m)$ is the $\sigma$-algebra of Borel sets on $b^m$:

$$\sigma_{b^m}(A) = \int_{b^m} \left( \mu_{b^m} \right)^{\ast m}(t) \mu_{b^m}(t) \, dt.$$  

(3.3)

Indeed, by the definition of $\sigma_{b^m}$ we have

$$\sigma_{b^m}(A) = \int_{b^m} \left( \mu_{b^m} \right)^{\ast m}(t) \mu_{b^m}(t) \, dt.$$

Then, by definition, the Hellinger integral $H_{b^m}$ is equal to

$$H_{b^m} = \int_{b^m} \left( \frac{\mu_{b^m}(x)}{\mu_{b^m}(x)} \right)^{1/2} \, dx.$$  

(3.5)

Suppose that $f$ is a one-to-one measurable mapping $f: \mathbb{R}^n \to \mathbb{R}^n$, $dx$ denotes Lebesgue measure on $\mathbb{R}^n$, and $d\mu$ is a measure equivalent to Lebesgue measure:

$$\mu(A) = \int_A g(x) \, dx,$$

and make use of the fact that

$$\lim_{t \to 0} H_{b^m} = 0.$$  

Then

$$\mu(f(A)) = \int_{f^{-1}(A)} g(x) \, dx = \int_A g(f^{-1}(y)) \frac{df^{-1}(y)}{dy} \, dy.$$  

(3.6)

in which we have made the substitution $x = f^{-1}(y)$.

Since, for any $t \in b^m$, $L_{\rho(t)}$ is an automorphism of the space $b^m$ for any $m,n \in \mathbb{N}$, then, for $A \in \mathcal{B}(b^m)$ we have, by formula (3.6), with $f(x) = L_{\rho(t)}(x)$:

$$\sigma_{b^m}(A) = \int_{b^m} \left( \mu_{b^m} \right)^{\ast m}(t) \mu_{b^m}(t) \, dt.$$  

(3.4)

In order to calculate the Hellinger integrals $H_{m,n} = H(\sigma_{b^m}, \sigma_{b^n})$ of the measures $\sigma_{b^m}$ and $\sigma_{b^n}$ on the space $b^m$, we calculate the Hellinger integrals $H_{m,n} = H(\sigma_{b^m}, \sigma_{b^n})$ of the projections $\sigma_{b^m}$ of the measure $\sigma_{b^n}$ on the finite-dimensional subspaces $b^m$, where

$$b^m = \left\{ x \in b^m \mid x = \sum_{i=1}^m x_i E_i \right\},$$

and make use of the fact that $\lim_{m \to \infty} H_{m,n} = H_{m,n}$.

We will show that for orthogonal measures $\mu_{b^m} \perp \mu_{b^n}$, the relation

$$H_{m,n} = \lim_{n \to \infty} H_{m,n} = 0$$

holds. Accordingly, by property (H3) of the Hellinger integral, $\sigma_{b^m} \perp \sigma_{b^n}$; this proves Lemma 3.1.
where
\[ x^{\tau,n} \in \mathbb{R} \times \mathbb{R}^n, \quad x^{\tau,n} = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix}, \]
\[ \psi^{\tau}(b^{-1}, x^{\tau,n}) = \left( \prod_{k=1}^{n} \sqrt{\frac{b_{m+1}}{\pi}} \right) \int_{\mathbb{R}^{m+1}} \exp \left( - \sum_{k=m+1}^{m+n} b_{m+1}^{-1} (x_{rk} + \sum_{i=1}^{r-1} t_{r,i} x_{ik})^2 \right) \times \prod_{i=1}^{r-1} \left( \frac{1}{\sqrt{\pi}} \exp(-t_{i,k}^2) dt_{r,i} \right). \]

Consider the matrix \( x^{m,n} \in \mathbb{R}^m \times \mathbb{R}^n \) and vectors \( t, x_1, \ldots, x_n \in \mathbb{R}^n \):
\[ x^{m,n} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix}, \quad t = \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}, \]
\[ x_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{m1} \end{pmatrix}, \quad x_2 = \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{m2} \end{pmatrix}, \quad \ldots, \quad x_n = \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{mn} \end{pmatrix}. \]

Then (3.9) may be written down in a more convenient form:
\[ \psi^{m+1,n}(b^{-1}, x^{m+1,n}) = \prod_{k=1}^{n} \sqrt{\frac{b_{m+1}}{\pi}} \int_{\mathbb{R}^{m+1}} \exp \left( - \sum_{k=m+1}^{m+n+1} b_{m+1}^{-1} (x_{rk} + (x_k, t))^2 \right) \times \frac{1}{\sqrt{\pi^m}} \exp(-\|t\|^2) dt. \]

By (3.8),
\[ \left( \frac{d\sigma^{m+1,n}}{dt^{m+1,n}} \right)(x) = \psi^{m+1,n}(b^{-1}, x^{m+1,n}) \left( \frac{d\sigma^{m,n}}{dt^{m,n}} \right)(x^{m,n}) \] (3.10)
holds, where

$$\prod_{k=1}^{n} (a_k b_k) = \max \left\{ \prod_{k=1}^{n} (a_k b_k) : 1 \leq k_1 < k_2 < \cdots < k_p \leq n \right\},$$

$$1 \leq p \leq n.$$

Lemma 3.5. If the product \( \prod_{k=1}^{n} m_k = \infty \) of positive numbers \( m_k \geq 1 \) diverges, then, for any \( p \in \mathbb{N} \), \( \lim_{n \to \infty} \frac{\prod_{k=1}^{p} m_k}{\prod_{k=1}^{n} m_k} = 0 \), where

$$\prod_{k=1}^{p} m_k = \max \left\{ \prod_{k=1}^{p} m_k : 1 \leq k_1 < k_2 < \cdots < k_p \leq n \right\}, \quad \prod_{k=1}^{n} m_k.$$

Suppose that Lemmas 3.3–3.5 hold. Lemma 3.1 follows from them.

Proof of Lemma 3.1. We will show from its opposite, i.e. from the condition

$$\mu_{\sigma_{\pi}} = \prod_{k=1}^{m+1} \mu_{\sigma_{\pi}}^{m+1},$$

that \( H^{m+1} = \lim_{n \to \infty} H^{m+1} = 0 \), \( m = 0, 1, \ldots \). Condition (3.16) is equivalent (see (3.1)) to the condition

$$\prod_{k=1}^{m+1} \frac{(b_{k(n)}^{(m)} + b_{k(n)}^{(n)})^2}{4b_{k(n)}^{(m)} b_{k(n)}^{(n)}} = \infty.$$  

(3.17)

Suppose that \( m = 0 \), and \( \mu_{\sigma_{\pi}} \perp \mu_{\sigma_{\pi}} \), i.e.

$$\prod_{k=2}^{\infty} \frac{(b_{k(n)}^{(m)} + b_{k(n)}^{(n)})^2}{4b_{k(n)}^{(m)} b_{k(n)}^{(n)}} = \infty.$$
By property (H1) of the Hellinger integral, 0 ≤ H^n = \lim_{n \to \infty} H^{p,n} ≤ 1. Therefore we finally find from (3.18) that
\[ H^{n+1} = \lim_{n \to \infty} H^{n+1,n} = 0. \]
This completes the proof of Lemma 3.1.

§4. Proofs of Lemmas 3.2–3.5

Proof of Lemma 3.5. If all m_k ≤ C, then \( \prod_m^{p,n} \leq C^n \), so that \( \prod_m^{p,n} / \prod_m^{p,n,n} \to 0 \) as \( n \to \infty \). In the contrary case there exists an infinite sequence \( (k(n))_{n=1} \), \( k(1) = 1 \), \( k(2) = \min \{ k | m_k > m_1 \} \), \( \ldots \), \( k(n) = \min \{ k | m_k > m_{k(n-1)} \} \), for which
\[ \lim_{n \to \infty} m_{k(n)} = \infty, \quad m_{k(n)} < m_{k(n+1)}, \quad n \in \mathbb{N}. \]
For \( r, n \in \mathbb{N} \), \( r < p \), we denote by \( k(r, n) \in \mathbb{N} \) numbers such that
\[ k(p, n) < k(p-1, n) < \cdots < k(1, n), \quad \prod_{r=1}^p m_{k(r,n)} = \prod_m^{p,n}. \]
Suppose that \( n \in \{ k(r), k(r+1) \} \cap \mathbb{N} \). Then \( k(1, n) \geq k(r), k(2, n) \geq k(r-1), \ldots, k(p, n) \geq k(r-p+1) \), so that
\[ \prod_{r=1}^m m_k \leq \left( \prod_{k=1}^n m_k \right)^{-1} \left( \prod_{r=1}^{k(r-1)} m_k \right)^{-1} \to 0 \text{ if } n \to \infty, \]
\[ k \neq k(1, n), \ldots, k(p, n). \]

Proof of Lemma 3.4. We rewrite inequality (3.15) in the following form:
\[ \left( 1 + \sum_{r=1}^m \sum_{k=1} \prod a_k[x_k, \ldots, x_n] \right) \times \left( 1 + \sum_{r=1}^m \sum_{q=1} \prod b_q[x_q, \ldots, x_n] \right) \leq C \left( 1 + \sum_{r=1}^m \sum_{p=1} \prod a_p b_q [x_p, \ldots, x_n] \right)^2. \]

IRREDUCIBILITY AND EQUIVALENCE OF FINITE UPPER-TRIANGULAR MATRICES

Then inequality (3.19) follows from the following system of inequalities, obtained by comparing the coefficients on \( [x_1, \ldots, x_n] \), \( [x_1, \ldots, x_n] \times [x_1, \ldots, x_n] \), \( 1 \leq r, s \leq n \), in the right and left hand sides of (3.19).

\[ \left( 1 + \sum_{r=1}^m \sum_{p=1} \prod a_p b_q [x_p, \ldots, x_n] \right)^2 \leq C \cdot \prod_{r=1}^m (a_r + b_r)^2. \]

Now we relax the inequalities (3.20):
\[ \left( 1 + \sum_{r=1}^m \sum_{p=1} \prod a_p b_q [x_p, \ldots, x_n] \right)^2 \leq C \cdot \prod_{r=1}^m (a_r + b_r)^2, \quad 1 \leq r, s \leq n. \]

From (3.21) we get
\[ \tau = \max \left\{ 2^{r-1} \prod_{i=1}^r \left( a_i + b_i \right)^2, 2^{-r+1} \prod_{i=1}^r \left( a_i + b_i \right)^2 \right\} \leq 2^{3m-1} \prod_{i=1}^m \left( a_i + b_i \right)^2, \quad 1 \leq r, s \leq n \}
\[ = 2^{3m-1} \prod_{i=1}^m \left( a_i + b_i \right)^2 \leq k(1) \leq \cdots \leq k(2m-1). \]

this proves Lemma 3.4.

Proof of Lemma 3.2. Suppose that \( T^{p,n} \) and \( T^{p,n} \) are equivalent irreducible unitary representations, \( U : U^{(1)}(b^{(1)}) \to U^{(2)}(b^{(2)}) \) is their intertwining operator:
\[ \mu^{(1)}(U^{(1)}) = T^{p,n} \mu^{(2)}(U) \in B_{p,n}^{(2)}. \]
We know from the proof of Lemma 2.2 that \( \sum_{n=N_1}^{N_2} t_n(b^{(1)}) A^{R^{(m)}}_{2m} A^{R^{(m)}}_{2m} \to x_{12} \) in \( \mathcal{H}(b^{(1)}) \), where \( \gamma_n = (b^{(1)} b^{(1)})^2 + b^{(1)} b^{(1)} \) and

\[
t_n = -2b^{(1)} \left( b^{(1)} \sum_{k=N_1}^{N_2} b^{(1)} k^{-1} \right) + 2 \left( b^{(1)} b^{(1)} \right) \cdot \sum_{k=N_1}^{N_2} b^{(1)} (b^{(1)} b^{(1)})^{-1} \right]^{-1} = -2 b^{(1)} \left( \sum_{k=N_1}^{N_2} b^{(1)} k^{-1} \right) + 2 \left( b^{(1)} b^{(1)} \right) \cdot \sum_{k=N_1}^{N_2} b^{(1)} (b^{(1)} b^{(1)})^{-1} \right]^{-1}
\]

(3.24)

\[
\left\| \omega \right\|_{12}(b^{(1)}) \left\| \bar{\omega}(b^{(1)}) \right\| = \left( \left( \sum_{n=N_1}^{N_2} t_n(b^{(1)}) A^{R^{(m)}}_{2m} A^{R^{(m)}}_{2m} \right) + \sum_{n=N_1}^{N_2} t_n(b^{(1)}) b^{(1)} x_{12} \right) \right\|_{12}(b^{(1)}) \]

(3.25)

We will show that \( U(\sum_{n=N_1}^{N_2} t_n(b^{(1)}) A^{R^{(m)}}_{2m} A^{R^{(m)}}_{2m} ) \to x_{12} \) in \( \mathcal{H}(b^{(1)}) \). As in the proof of Theorem 2.1, for this it suffices to show the convergence of the operators in question on the unit vector \( 1 \in \mathcal{H}(b^{(1)}) \). We have

\[
\left\| \omega \right\|_{12}(b^{(1)}) \left\| \bar{\omega}(b^{(1)}) \right\| = \left( \left( \sum_{n=N_1}^{N_2} t_n(b^{(1)}) A^{R^{(m)}}_{2m} A^{R^{(m)}}_{2m} \right) + \sum_{n=N_1}^{N_2} t_n(b^{(1)}) b^{(1)} x_{12} \right) \right\|_{12}(b^{(1)}) \]

\[
= \sum_{n=N_1}^{N_2} t_n(b^{(1)}) \left( (b^{(1)} k^{-1} + b^{(1)} b^{(1)}) \right) \]

\[
= 4 \left( \sum_{n=N_1}^{N_2} b^{(1)} (b^{(1)} b^{(1)})^{-1} \right)^{-1} \]

(3.26)

We will show that

\[
\lim_{n \to \infty} \frac{b^{(1)} b^{(1)} + b^{(1)} b^{(1)}}{b^{(1)} b^{(1)} + b^{(1)} b^{(1)}} = 1.
\]

Indeed, condition (3.23) is equivalent to the convergence of the series

\[
\sum_{(k,n) \in \triangle} \left( \frac{b^{(1)} b^{(1)} - b^{(1)} b^{(1)}}{b^{(1)} b^{(1)} + b^{(1)} b^{(1)}} \right)^2 < \infty,
\]

which is equivalent to the convergence of the series

\[
\sum_{(k,n) \in \triangle} \left( \frac{b^{(1)} b^{(1)} - b^{(1)} b^{(1)}}{b^{(1)} b^{(1)} + b^{(1)} b^{(1)}} \right)^2 < \infty.
\]

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\[
\sum_{(k,n) \in \triangle} \left( \frac{b^{(1)} b^{(1)} - b^{(1)} b^{(1)}}{b^{(1)} b^{(1)} + b^{(1)} b^{(1)}} \right)^2 < \infty.
\]

It follows from this that for any \( m \in \mathbb{N} \) and \( \varepsilon > 0 \) there exists of an \( n_0 \in \mathbb{N} \) such that

\[
\left| \frac{b^{(1)} b^{(1)} - b^{(1)} b^{(1)}}{b^{(1)} b^{(1)} + b^{(1)} b^{(1)}} - 1 \right| < \varepsilon, \quad k = 1, 2, \ldots, m, \quad n \geq n_0.
\]

Therefore

\[
\left| \sum_{n=N_1}^{N_2} b^{(1)} \left( \sum_{k=1}^{m} b^{(1)} \right)^{-1} \right| \leq \sum_{k=1}^{m} \left| \frac{b^{(1)} b^{(1)} - b^{(1)} b^{(1)}}{b^{(1)} b^{(1)} + b^{(1)} b^{(1)}} \right| \left( \sum_{k=1}^{m} b^{(1)} \right) \leq m \varepsilon.
\]

Accordingly

\[
\lim_{n \to \infty} \sum_{n=N_1}^{N_2} b^{(1)} \left( \sum_{k=1}^{m} b^{(1)} \right)^{-1} = 1, \quad m \in \mathbb{N}.
\]

(3.27) implies an estimate for the right side of (3.26):

\[
\left\| \omega \right\|_{12}(b^{(1)}) \left\| \bar{\omega}(b^{(1)}) \right\| \leq C \left( \sum_{n=N_1}^{N_2} b^{(1)} b^{(1)} + b^{(1)} b^{(1)} \right)^{-1},
\]

and this is small for appropriate \( N_1 \) and \( N_2 \). Here we take account of the fact that

\[
S^{(1)}(b^{(1)}) = \sum_{n=3}^{\infty} b^{(1)} b^{(1)} = \infty
\]

(see (1.3)).

We will write \( (N_1, N_2) \to \infty, (N_1, N_2) = (N_1(p), N_2(p)), \) if

\[
\lim_{p \to \infty} \sum_{n=N_1(p)}^{N_2(p)} b^{(1)} b^{(1)} + b^{(1)} b^{(1)} = \infty.
\]

In the left side of the expression (3.26) there is the expression

\[
\frac{1}{2} \sum_{n=N_1}^{N_2} t_n(b^{(1)}) b^{(1)} x_{12}.
\]

We will show that

\[
\lim_{(N_1, N_2) \to \infty} \frac{1}{2} \sum_{n=N_1}^{N_2} t_n(b^{(1)}) b^{(1)} = -1.
\]

Indeed,

\[
\lim_{(N_1, N_2) \to \infty} \frac{1}{2} \sum_{n=N_1}^{N_2} t_n(b^{(1)}) b^{(1)} = -1,
\]
since
\[
\left| \sum_{n=N_1}^{N_2} a_n b_n \left( \sum_{n=N_1}^{N_2} a_n \right)^{-1} - 1 \right| \leq \sum_{n=N_1}^{N_2} a_n(b_n - 1) \left( \sum_{n=N_1}^{N_2} a_n \right)^{-1}
\]
\[
\leq \max_{N_1 \leq n \leq N_2} |b_n - 1| \to 0
\]
as \(N_1 \to \infty\). Here we have written
\[a_n = b_n^{(1)}(b_n^{(1)} + b_n^{(2)})^{-1}, \quad b_n = b_n^{(2)}(b_n^{(1)})^{-1},\]
and used the fact that
\[\lim_{n \to \infty} b_n^{(1)}(b_n^{(1)})^{-1} = 1.\]
Thus, in the left side of (3.26),
\[
\frac{1}{N_2} \sum_{n=N_1}^{N_2} t_n(b^{(1)}) b_n^{(2)} x_{12} \to -x_{12}.
\]
Accordingly
\[
U \left( \sum_{n=N_1}^{N_2} t_n(b^{(1)}) A_n b^{(1)} A_n^{*} b^{(1)} \right) = \sum_{n=N_2}^{N_2} t_n(b^{(1)}) A_n b^{(1)} A_n^{*} b^{(1)} \to x_{12}
\]
in \(\mathcal{H}(b^{(2)})\). Hence \(U x_{12} = x_{12} U\).

Analogously we show that \(U x_{ik} = x_{ik} U, i < k\). Indeed, by Lemma 2.4, \(x_{ik}\) is approximated by the operators
\[
\sum_{n=N_1}^{N_2} t_n(b^{(1)}) (\partial_{in} - b_{in} x_n) A_n b^{(1)}
\]
and
\[
\left\| \omega_{ik}(b^{(1)}) \right\|_{\mathcal{H}(b^{(2)})} = \left\| \sum_{n=N_1}^{N_2} t_n(b^{(1)}) (\partial_{in} - b_{in} x_n) A_n b^{(1)} x_{ik} - x_{ik} \right\|_{\mathcal{H}(b^{(2)})}\]
\[
= \max_{N_1 \leq n \leq N_2} |b_n - 1| \to 0.
\]
where
\[
t_n(b^{(1)}) = -2b_n^{(1)} \left( \sum_{m=1}^{k} b_m^{(1)} b_m^{(1)} \right) \left( \sum_{m=1}^{n} b_m^{(1)} b_m^{(1)} \right)^{-1} - 1
\]
\[
= -2 \left( \sum_{m=1}^{k} b_m^{(1)} b_m^{(1)} \right) \left( \sum_{m=1}^{n} b_m^{(1)} b_m^{(1)} \right)^{-1} - 1.
\]

We will show that
\[
U \left( \sum_{n=N_1}^{N_2} t_n(b^{(1)}) (\partial_{in} - b_{in} x_n) A_n b^{(1)} \right) \to x_{ik}
\]
in \(\mathcal{H}(b^{(2)})\). Indeed,
\[
\left\| \omega_{ik}(b^{(1)}) \right\|_{\mathcal{H}(b^{(2)})} = \left\| \sum_{n=N_1}^{N_2} t_n(b^{(1)}) (\partial_{in} - b_{in} x_n) A_n b^{(1)} + \sum_{n=N_1}^{N_2} t_n(b^{(1)}) b_{in} x_{ik} \right\|_{\mathcal{H}(b^{(2)})} - 1
\]
\[
= \sum_{n=N_1}^{N_2} t_n(b^{(1)}) b_n^{(2)} x_{12} = \sum_{n=N_1}^{N_2} b_n^{(2)} b_n^{(2)} - 4 \left( \sum_{n=N_1}^{N_2} b_n^{(1)} \left( \sum_{m=1}^{k} b_m^{(1)} \right)^{-1} \right)^{-2}
\]
\[
= \sum_{n=N_1}^{N_2} b_n^{(1)} b_n^{(1)} b_n^{(1)} b_n^{(1)} - \sum_{n=N_1}^{N_2} b_n^{(1)} b_n^{(1)} - 1.
\]

Using (3.27), we get the estimate
\[
\left\| \omega_{ik}(b^{(1)}) \right\|_{\mathcal{H}(b^{(2)})} \leq C \left( \sum_{n=N_1}^{N_2} b_n^{(1)} \left( \sum_{m=1}^{k} b_m^{(1)} \right)^{-1} \right)^{-1},
\]
which is small for appropriate \(N_1, N_2\) (see the proof of Lemma 2.4). The equation
\[
\lim_{\langle N_1, N_2 \rangle \to \infty} \sum_{n=N_1}^{N_2} t_n(b^{(1)}) b_{in}^{(2)} = -1
\]
completes the proof of Lemma 3.2.

Proof of Lemma 3.5. We note that the function \(G^{m,n}(a, b, X^{m,n}), X^{m,n} \in \mathbb{R}^m \times \mathbb{R}^n\), is invariant relative to \(O(m)\), the orthogonal group of space \(\mathbb{R}^m\). Indeed,
\[
G^{m,n}(a, b; O X^{m,n}) = G^{m,n}(a, b, O x_1, \ldots, O x_n)
\]
\[
= \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^n} \sqrt{\sum_{k=1}^{n} a_k(x_{m+k} + (O x_k, t))^2} \right)^{-\frac{1}{2}} \exp(-\|t\|^2) \, dt
\]
\[
\times \left( \int_{\mathbb{R}^n} \sqrt{\sum_{k=1}^{n} b_k(x_{m+k} + (O x_k, t))^2} \right)^{-\frac{1}{2}} \exp(-\|t\|^2) \, dt\]
\[
\times \prod_{k=1}^{n} dx_{m+k} = G^{m,n}(a, b, x_1, \ldots, x_n) = G^{m,n}(a, b, X^{m,n}).
\]
(3.28)
It suffices to prove equation (3.13) for $G^{m+1}$, $n \in \mathbb{N}$. Indeed, for $m < n$, equation (3.13) for $G^{m,n}$ is a special case of the equation for $G^{n-1,n}$, since for $x_1, \ldots, x_n \in \mathbb{R}^n \subset \mathbb{R}^n$,

$$[x_{k_1}, \ldots, x_{k_r}] = \det((x_{k_i}, x_{k_j})_{i,j=1}^r) = 0$$

for $r > m$.

We will need some formulas. It is well known that for $A \in \text{End}(\mathbb{R}^n)$, $A > 0$,

$$\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}^n} \exp(-\langle Ax, x \rangle) \, dx = \frac{1}{\sqrt{\det A}}.$$ \hspace{1cm} (3.29)

Suppose that $m, n \in \mathbb{N}$, $c_1, c_2, \ldots, c_n \in \mathbb{R}^m$, $d \in \mathbb{R}^m$, $c_k = (c_{pk})_{P=1}^n$, $k = 1, \ldots, n$, $C = (c_{pk}) \in \mathbb{R}^m \times \mathbb{R}^n$, $\bar{c}_1, \ldots, \bar{c}_m \in \mathbb{R}^n$, $\bar{c}_p = (c_{pk})_{P=1}^n$.

Then

$$\nu(c_1, \ldots, c_n, d) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}^n} \exp\left(-\sum_{k=1}^n (c_k, t)^2 - (d, t)\right) \exp(-\langle t, t \rangle) \, dt$$

$$= \frac{1}{\sqrt{\det(I + A(C))}} \exp(\langle(I + A(C))^{-1} d, d\rangle),$$ \hspace{1cm} (3.30)

where $A(C) = (A_0(C))_{0=1}^m$, $A_0(C) = (\bar{c}_i, \bar{c}_i)_{i=1}^n$.

Indeed, we make a substitution $t = T + T_0$ in the left side of (3.30) such that the terms linear in $T$ vanish.

Since

$$\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}^n} \exp\left(-\sum_{k=1}^n (c_k, t)^2 - (d, t)\right) \exp(-\langle t, t \rangle) \, dt$$

$$= \frac{1}{\sqrt{\det(I + A(C))}} \exp(\langle(I + A(C))^{-1} d, d\rangle),$$

Now we make the substitutions

$$\frac{a_k + b_k}{2} x_{m+1,k}^2 = z_k^2, \quad x_{m+1,1} = \frac{\sqrt{2}a_k}{a_k + b_k}, \quad x_k(a) = x_k, \quad x_k(b) = x_k \sqrt{\frac{b_k}{a_k}},$$

$$d(a) = (d_p(a))_{p=1}^m, \quad d_p(a) = \sum_{k=1}^m \frac{a_k x_{m+1,k} z_k}{\sqrt{a_k + b_k}} = (\bar{x}_p(a), z), \quad \bar{x}_p(a) = (\bar{x}_{pk}(a))_{p=1}^m,$$

$$\bar{x}_{pk}(a) = \frac{a_k x_{m+1,k}}{\sqrt{a_k + b_k}}, \quad d(b) = (d_p(b))_{p=1}^m, \quad d_p(b) = \sum_{k=1}^m \frac{b_k x_{m+1,k}}{\sqrt{a_k + b_k}} = (\bar{x}_p(b), z),$$

$$\bar{x}_p(b) = (\bar{x}_{pk}(b))_{p=1}^m, \quad \bar{x}_{pk}(b) = \frac{b_k x_{m+1,k}}{\sqrt{a_k + b_k}}.$$
We get
\[ G_{m,n}(a, b, X) = \left\{ \prod_{k=1}^{n} \frac{4a_k b_k}{(a_k + b_k)^2} \right\}^{1/4} \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \exp\left(- \sum_{p=1}^{n} (x_p(a), t)^2 - (2\sqrt{2d(a), t}) \right) \right) \times \frac{1}{\sqrt{\pi^n}} \exp(-t, t) dt \int_{\mathbb{R}^n} \exp\left(- \sum_{p=1}^{n} (x_p(b), t)^2 - (2\sqrt{2d(b), t}) \right) \times \frac{1}{\sqrt{\pi^n}} \exp(-t, t) dt \int_{\mathbb{R}^n} \exp\left(- (x, z) \right) dz \]
\[ = \left\{ \prod_{k=1}^{n} \frac{4a_k b_k}{(a_k + b_k)^2 \det(I + A(X(a))) \det(I + A(X(b)))} \right\}^{1/4} \int_{\mathbb{R}^n} \exp\left(((I + A(X(a)))^{-1} d(a), d(a)) + ((I + A(X(b)))^{-1} d(b), d(b) \right) \times \frac{1}{\sqrt{\pi^n}} \exp(-(z, z)) dz. \quad (3.31) \]

Denote by \( R_{pq}(a), A_{pq}(a), 1 \leq p, q \leq n, \) respectively, the elements of the matrix \((1 + A(X(a)))^{-1}\) and the cofactors of the matrix \((1 + A(X(a)))^{-1}\). Then
\[ R_{pq}(a) = \frac{A_{pq}}{\det(I + A(X(a)))}. \]

Therefore
\[ ((I + A(X(a)))^{-1} d(a), d(a)) + ((I + A(X(b)))^{-1} d(b), d(b) \]
\[ = \sum_{p,q=1}^{n} (R_{pq}(a)\hat{x}_p(a), z)(\hat{x}_q(a), z) + R_{pq}(b)\hat{x}_p(b), \hat{x}_q(b), z) \]
\[ = \sum_{p,q=1}^{n} \left( \sum_{i,j=1}^{n} R_{pq}(a)\hat{x}_p(a)\hat{x}_q(a)z_i z_j + R_{pq}(b)\sum_{i,j=1}^{n} \hat{x}_p(b)\hat{x}_q(b)z_i z_j \right) \]
\[ = \sum_{i,j=1}^{n} \left( \sum_{p,q=1}^{n} R_{pq}(a)\hat{x}_p(a)\hat{x}_q(a) + R_{pq}(b)\hat{x}_p(b)\hat{x}_q(b) \right) z_i z_j \]
\[ = \sum_{i,j=1}^{n} R_{ij}(a, b, X)z_i z_j, \]

where
\[ R_{ij}(a, b, X) = \sum_{p,q=1}^{n} (R_{pq}(a)\hat{x}_p(a)\hat{x}_q(a) + R_{pq}(b)\hat{x}_p(b)\hat{x}_q(b)) \]
\[ = \sum_{p,q=1}^{n} \left( \frac{A_{pq}(a)}{\det(I + A(X(a)))} \hat{x}_p(a)\hat{x}_q(a) + \frac{A_{pq}(b)}{\det(I + A(X(b)))} \hat{x}_p(b)\hat{x}_q(b) \right) \]
\[ = \frac{1}{\det(I + A(X(a))) \det(I + A(X(b)))} \sum_{p,q=1}^{n} (\det(I + A(X(a)))A_{pq}(a)\hat{x}_p(a)\hat{x}_q(a) + \det(I + A(X(b)))A_{pq}(b)\hat{x}_p(b)\hat{x}_q(b)) \]
\[ = \frac{r_{ij}(a, b, X)}{\det(I + A(X(a))) \det(I + A(X(b)))}. \]

Accordingly,
\[ ((I + A(X(a)))^{-1} d(a), d(a)) + ((I + A(X(b)))^{-1} d(b), d(b) \]
\[ = ((\det(I + A(X(a))) \det(I + A(X(b))))^{-1} \sum_{i,j=1}^{n} r_{ij}(a, b, X)z_i z_j. \quad (3.32) \]

Substituting (3.32) into (3.31), we get
\[ G_{m,n}(a, b, X) = \left\{ \prod_{k=1}^{n} \frac{4a_k b_k}{(a_k + b_k)^2 \det(I + A(X(a))) \det(I + A(X(b)))} \right\}^{1/4} \int_{\mathbb{R}^n} \exp\left(\frac{r(a, b, X)z_i z_j}{\det(I + A(X(a))) \det(I + A(X(b)))} \right) \times \frac{1}{\sqrt{\pi^n}} \exp(-(z, z)) dz \]
\[ = \left\{ \prod_{k=1}^{n} \frac{4a_k b_k}{(a_k + b_k)^2 \det(I + A(X(a))) \det(I + A(X(b)))} \right\}^{1/4} \int_{\mathbb{R}^n} \exp\left(\frac{r(a, b, X)z_i z_j}{\det(I + A(X(a))) \det(I + A(X(b)))} \right) \times \frac{1}{\sqrt{\pi^n}} \exp(-(z, z)) dz \]
\[ = \left\{ \prod_{k=1}^{n} \frac{4a_k b_k}{(a_k + b_k)^2 \det(I + A(X(a))) \det(I + A(X(b)))} \right\}^{1/4} \int_{\mathbb{R}^n} \exp\left(\frac{r(a, b, X)z_i z_j}{\det(I + A(X(a))) \det(I + A(X(b)))} \right) \times \frac{1}{\sqrt{\pi^n}} \exp(-(z, z)) dz \]
\[ = \left\{ \prod_{k=1}^{n} \frac{4a_k b_k}{(a_k + b_k)^2 \det(I + A(X(a))) \det(I + A(X(b)))} \right\}^{1/4} \int_{\mathbb{R}^n} \exp\left(\frac{r(a, b, X)z_i z_j}{\det(I + A(X(a))) \det(I + A(X(b)))} \right) \times \frac{1}{\sqrt{\pi^n}} \exp(-(z, z)) dz. \quad (3.33) \]

Since \( A_{pq}(a), A_{pq}(b), \det(I + A(X(a))), \det(I + A(X(b))) \) are polynomials in the \( x_{ij}, \; i = 1, \ldots, m, \; j = 1, \ldots, n, \) then the \( r_{ij}(a, b, X), \) and accordingly \( \det(I + A(X(a))) \det(I + A(X(b))) I - r(a, b, X) \), are also polynomials in \( x_{ij} \).

Suppose that
\[ X = (x_{pk}), \; X(a) = (x_{pk} \sqrt{a_k}) \in \mathbb{R}^n \times \mathbb{R}^m, \; x_1(a), \ldots, x_{n}(a) \in \mathbb{R}^m, \]
\[ x_i(a) = (x_{pk} \sqrt{a_k})_{p=1}^{n-1}, \; \hat{x}_1(a), \ldots, \hat{x}_{m}(a) \in \mathbb{R}^n, \; \hat{x}_p(a) = (x_{pk} \sqrt{a_k})_{k=1}^{n-1}. \]
Write

\[ A(X(a)) = (A_{ij}(X(a)))_{i,j=1}^{n-1}, \]
\[ A_{ij}(X(a)) = (\tilde{x}_i(a), \tilde{x}_j(a)), \quad \tilde{A}(X(a)) = (\tilde{A}_{pq}(X(a)))_{p,q=1}^{n-1}, \]
\[ \tilde{A}_{pq}(X(a)) = (x_p(a), x_q(a)). \]

Then the following equation holds:

\[ \det(I + A(X(a))) = \det(I + \tilde{A}(X(a))) = \xi^{mn}(a, X). \quad (3.34) \]

Indeed, since the equation

\[ [x_{k_1}, \ldots, x_{k_2}] = \det((x_{k_1}, x_{k_2})_{i,j=1}^{n-1}) = \sum_{1 \leq p_1 < \cdots < p_2 \leq m} [M_{p_1 \cdots p_2}(X)]^2, \quad (3.35) \]

in which the \( M_{p_1 \cdots p_2}(X) \) are minors of the matrix \( X \), holds for the vectors \( x_{k_1}, \ldots, x_{k_2} \in \mathbb{R}^n \), then

\[
\det(I + A(X(a))) = 1 + \sum_{s=1}^n \sum_{1 \leq p_1 < \cdots < p_s \leq m} [x_{p_1}(a), \ldots, x_{p_s}(a)]
\]

\[
= 1 + \sum_{s=1}^n \sum_{1 \leq p_1 < \cdots < p_s \leq m} [M_{p_1 \cdots p_s}(X(a))]^2
\]

\[
= 1 + \sum_{s=1}^n \sum_{1 \leq p_1 < \cdots < p_s \leq m} \prod_{k=1}^s a_k [x_{k_1}, \ldots, x_{k_s}] = (\xi^{mn}(a, X) = \det(I + A(X(b))).
\]

We will prove equation (3.13) for \( G^{n,n+1} \) by induction on \( n \in \mathbb{N} \). Suppose that \( n = 1 \). Write

\[ x_1(a) = x_{11}, x_2(a) = x_{12}, x_3(a) = x_{11} - x_{12}, x_4(a) = (x_{11}, x_{12}, \sqrt{x_12}), \]

\[ \det(I + A(X(a))) = 1 + (\tilde{x}_1(a), \tilde{x}_1(a)) = 1 + \sum_{k=1}^2 a_k x_k^2, \]

\[ \det(I + A(X(b))) = 1 + \sum_{k=1}^2 b_k x_k^2, \quad d(a) = \sum_{k=1}^2 a_k x_k^2, \quad d(b) = \sum_{k=1}^2 b_k x_k^2, \]

\[ ((I + A(X(a)))^{-1} d(a), d(a)) + ((I + A(X(b)))^{-1} d(b), d(b)) = (c_1, z)^2 + (c_2, z)^2. \]

We will prove that if equation (3.13) holds for \( G^{n-1,n}(a, b, X^{n-1}) \), then it holds for \( G^{n,n}(a, b, X^n) \). In view of the invariance of \( G^{n,n}(a, b, OX) = G^{n,n}(a, b, X), O \in O(n) \), we may suppose that the matrix \( X = X^{n,n} \) is triangular:

\[
\begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1,n} \\
  a_{21} & a_{22} & \cdots & a_{2,n-1} & 0 \\
  \vdots & \vdots & & \vdots & \vdots \\
  a_{n-1,1} & a_{n-1,2} & \cdots & 0 & 0 \\
  a_{n,1} & 0 & \cdots & 0 & 0
\end{bmatrix}
\]

where

\[
c_1 = \frac{1}{\sqrt{1 + \sum_{k=1}^2 a_k x_k^2}} \left( \frac{a_{1,1} x_{1,1}}{a_1 + b_1}, \frac{a_{1,2} x_{1,2}}{a_2 + b_2} \right),
\]

\[
c_2 = \frac{1}{\sqrt{1 + \sum_{k=1}^2 b_k x_k^2}} \left( \frac{b_{1,1} x_{1,1}}{b_1 + c_1}, \frac{b_{1,2} x_{1,2}}{b_2 + c_2} \right).
\]

By (3.31) and (3.30) we have

\[
G^{1,2}(a, b; x_{11}, x_{12}) = \frac{1}{(a + b)^2} \det(I + A(X(a))) \det(I + A(X(b))) \det(I + A(X(c))) \quad (3.34).
\]

But, by (3.34),

\[
\det(I + A(C)) = \det(I + \tilde{A}(C)) = \det \begin{bmatrix}
  1 - (c_1, c_1) & - (c_1, c_2) \\
  - (c_2, c_1) & 1 - (c_2, c_2)
\end{bmatrix}
\]

\[
= \left( 1 - \sum_{k=1}^2 \frac{a_k x_k^2}{a_k + b_k} \right) \left( 1 - \sum_{k=1}^2 b_k x_k^2 \right)
\]

\[
= \left( \frac{\sum_{k=1}^2 a_k b_k x_k^2}{a_k + b_k} \right)^2 \left( \frac{\sum_{k=1}^2 b_k x_k^2}{a_k + b_k} \right)^2
\]

\[
= \left( \frac{\sum_{k=1}^2 a_k b_k x_k^2}{a_k + b_k} \right)^2 \left( \frac{\sum_{k=1}^2 b_k x_k^2}{a_k + b_k} \right)^2
\]

\[
= \left( 1 - \frac{\sum_{k=1}^2 a_k x_k^2}{a_k + b_k} \right) \left( 1 + \frac{\sum_{k=1}^2 b_k x_k^2}{a_k + b_k} \right)
\]

which implies (3.13) for \( G^{1,2}(a, b; x_{11}, x_{12}) \).
Then we may express the function $\xi^n(a, X)$ as follows:

$$\xi^n(a, X) = 1 + \sum_{r=1}^{n} \sum_{1 \leq i < j \leq n} \prod_{k=i+1}^{j} a_k[x_k, \ldots, x_{k_r}]$$

$$= 1 + \sum_{r=1}^{n} \sum_{1 \leq i < j \leq n} \prod_{k=i+1}^{j} a_k[x_k, \ldots, x_{k_r}]$$

$$+ \sum_{r=1}^{n} \sum_{1 \leq i < j \leq n} \prod_{k=i+1}^{j} a_k[x_k, \ldots, x_{k_r}].$$

For the matrix $X$ we find from property (3.35) that

$$[x_1, \ldots, x_{n-1}, x_n] = \sum_{1 \leq p_1 < \cdots < p_r \leq n} |M_{k_1, \ldots, k_r-1}(X)|^2$$

$$= \sum_{1 < p_2 < \cdots < p_r \leq n} |M_{k_1, k_2, \ldots, k_r-1}(X)|^2$$

$$= x_n^2 \sum_{2 < p_2 < \cdots < p_r \leq n} |M_{k_1, k_2, \ldots, k_r-1}(1X)|^2 = x_n^2 \sum_{i=1}^{n} [x_{1}, \ldots, x_{k_i-1}].$$

where the $M_{k_1, \ldots, k_r-1}(1X)$ are minors of the matrix

$$1X = \begin{pmatrix} x_{11} & \cdots & x_{1n-1} \\ x_{21} & \cdots & 0 \\ \vdots & \vdots & \vdots \\ x_{n1} & \cdots & 0 \end{pmatrix}, \quad X = \begin{pmatrix} x_{12} & \cdots & x_{1n} \\ x_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots \\ x_{n2} & \cdots & 0 \end{pmatrix},$$

$1x_k, \ldots, 1x_{k_r-1}$ the columns of the matrix $1X$. Therefore

$$\xi^n(a, X^n) = \xi^n-1(a, X^n-1) + a_n x_n^2 \xi^{n-1,n-1}(a, 1X). \quad (3.36)$$

Analogously one proves that

$$\xi^n(a, X^n) = \xi^n-1(a, X^n-1) + a_n x_n^2 \xi^{n-1,n-1}(a, 1X). \quad (3.37)$$

Since

$$\int_{\mathbb{R}^n} \exp(-a(y + tx)^2) \frac{1}{\sqrt{\pi}} \exp(-t^2) \, dt = \frac{1}{\sqrt{1 + a x^2}} \exp\left(-\frac{a y^2}{1 + a x^2}\right), \quad (3.38)$$

then

$$\int_{\mathbb{R}^n} \exp\left(-a(x_{n+1} + \sum_{r=1}^{n-1} t_r x_r + t_n x_n)^2\right) \frac{1}{\sqrt{\pi}} \exp(-t^2) \, dt = \frac{1}{\sqrt{1 + a x_n^2}} \exp\left(-\frac{a x_n^2}{1 + a x_n^2}\right),$$

$$\int_{\mathbb{R}^n} \exp\left(-b(x_{n+1} + \sum_{r=1}^{n-1} t_r x_r + t_n x_n)^2\right) \frac{1}{\sqrt{\pi}} \exp(-t^2) \, dt = \frac{1}{\sqrt{1 + b x_n^2}} \exp\left(-\frac{b x_n^2}{1 + b x_n^2}\right).$$

Therefore

$$G^{a,n}(a, b, X^n) = ((1 + a_1 x_n^2)(1 + b_1 x_n^2))^{-1/2} \Theta^a, b, X^n, \quad (3.39)$$

where

$$\bar{a} = (\bar{a}_k)^2 = b = \delta_k^2, \quad b_1 = \frac{b_1}{b_1}, \quad \bar{a}_k = a_k, \delta_k = b_k, \quad 2 \leq k \leq n.$$
\[
\xi^{n-1,n}(\alpha, X^{n-1,n}) = 1 + \sum_{r=1}^{n} \left( \frac{a_{r}}{1 + a_{r}X_{n_1}} \right) \sum_{2 \leq k_1 < \cdots < k_r \leq n} \prod_{i=1}^{k_r} a_{k_i} [x_{k_1}, \ldots, x_{k_r}]
\]
\[
= 1 + \sum_{r=1}^{n} \left( \frac{a_{r}}{1 + a_{r}X_{n_1}} \right) \sum_{2 \leq k_1 < \cdots < k_r \leq n} \prod_{i=1}^{k_r} a_{k_i} [x_{k_1}, \ldots, x_{k_r}]
+ \sum_{2 \leq k_1 < \cdots < k_r \leq n} \prod_{i=1}^{k_r} a_{k_i} [x_{k_1}, \ldots, x_{k_r}]
\]
\[
= \frac{1}{1 + a_{1}X_{n_1}} \left( 1 + \sum_{r=1}^{n} \left( \frac{a_{r}}{1 + a_{r}X_{n_1}} \right) \sum_{2 \leq k_1 < \cdots < k_r \leq n} \prod_{i=1}^{k_r} a_{k_i} [x_{k_1}, \ldots, x_{k_r}] + a_{1}X_{n_1}^{2} \right)
\times \left( 1 + \sum_{r=1}^{n} \left( \frac{a_{r}}{1 + a_{r}X_{n_1}} \right) \prod_{i=1}^{k_r} a_{k_i} [x_{k_1}, \ldots, x_{k_r}] \right)
\]
\[
= \frac{1}{1 + a_{1}X_{n_1}} \left( \xi^{n-1,n}(\alpha, X^{n-1,n}) + a_{1}X_{n_1}^{2} \xi^{n-1,n-1}(\alpha, 1, X) \right).
\]

Therefore
\[
G^{n,n}(a, b, X) = (1 + a_{1}X_{n_1})(1 + b_{1}X_{n_1}^{2})^{-1/4} \xi^{n-1,n}(\alpha, X, X^{n-1,n})
\]
\[
= \prod_{k=1}^{n} \frac{4a_{k}b_{k}}{(a_{k} + b_{k})^{2}} \left( 1 + a_{k}X_{n_1}^{2} \right) \left( 1 + b_{k}X_{n_1}^{2} \right)
\times \left( \xi^{n-1,n}(\alpha, X^{n-1,n}) + a_{1}X_{n_1}^{2} \xi^{n-1,n-1}(\alpha, 1, X) \right)
\times \left( \xi^{n-1,n}(\alpha, X^{n-1,n}) + b_{1}X_{n_1}^{2} \xi^{n-1,n-1}(\alpha, 1, X) \right)
\times \left( \xi^{n-1,n}(\alpha, X^{n-1,n}) + 2a_{1}b_{1}X_{n_1}^{2} \xi^{n-1,n}(\alpha, 1, X) \right)
\times \left( \xi^{n-1,n}(\alpha, X^{n-1,n}) + a_{1}X_{n_1}^{2} \xi^{n-1,n}(\alpha, 1, X) \right)^{1/4}
\]
when (3.37) is taken into account, this completes the proof of (3.13) for \(G^{n,n}(a, b, X^{n,n})\).

Now we shall prove that formula (3.13) is valid for \(G^{n,n+1}(a, b, X^{n,n+1})\). If we write
\[
X = X^{n,n+1} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n+1} \\ x_{21} & x_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix},
\]
\[
X^{n,n} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix},
\]
\[
\theta(a, b, X) := \left( \frac{\xi^{n,n}(\alpha, X^{n,n}) + a_{n+1}X_{n+1}^{2} \xi^{n-1,n}(\alpha, 1, X)}{\xi^{n,n}(\alpha, X^{n,n}) + b_{n+1}X_{n+1}^{2} \xi^{n-1,n}(\alpha, 1, X)} \right)^{1/2}
\]
\[
= \left( \prod_{k=1}^{n+1} \frac{(a_{k} + b_{k})^{2}}{4a_{k}b_{k}} \right)^{1/2} \left( \frac{\xi^{n,n}(\alpha, X^{n,n}) + a_{n+1}X_{n+1}^{2} \xi^{n-1,n}(\alpha, 1, X)}{\xi^{n,n}(\alpha, X^{n,n}) + b_{n+1}X_{n+1}^{2} \xi^{n-1,n}(\alpha, 1, X)} \right)^{1/2}
\]
\[
\times \left( \frac{2a_{n+1}b_{n+1}X_{n+1}^{2} \xi^{n-1,n}(\alpha, 1, X)}{a_{n+1} + b_{n+1}} \right)^{1/2}
\]
\[
\theta_{0}(a, b, X) = \left( \prod_{k=1}^{n+1} \frac{(a_{k} + b_{k})^{2}}{4a_{k}b_{k}} \right)^{1/2} \xi^{n,n}(\alpha, X^{n,n}) \frac{2ab}{a + b},
\]
\[
\theta(a, b, X) = \sum_{k=0}^{p} \theta_{k}(a, b, X)X^{k}_{n+1},
\]
where \(p < \infty\),
\[
\theta_{0}(a, b, X) = \left( \prod_{k=1}^{n+1} \frac{(a_{k} + b_{k})^{2}}{4a_{k}b_{k}} \right)^{1/2} \xi^{n,n}(\alpha, X^{n,n}) \frac{2ab}{a + b},
\]
\[
(3.40)\]
A direct calculation yields

\[ 2\theta_1(a, b, X) = \frac{\partial^2 \theta(a, b, X)}{\partial x_{1n+1}^2} \bigg|_{x_{1n+1} = 0} = \left( \prod_{k=1}^{n+1} \left( \frac{a_k + b_k}{4a_k b_k} \right)^2 \right) \frac{1}{4} \frac{4a_{n+1} b_{n+1} + 1}{a_{n+1} + b_{n+1}} \xi_{n,n} \left( \frac{2ab}{a + b}, X^{n,n} \right). \]

(3.41)

Indeed,

\[ \frac{\partial^2 \theta}{\partial x_{1n+1}^2} \bigg|_{x_{1n+1} = 0} = \theta(a, b, X) \left[ \frac{a_{n+1} \xi_{n,n}^{-1}(a, X) + b_{n+1} \xi_{n,n}^{-1}(b, X)}{\xi_{n,n}(a, X^{n,n})} + \frac{\partial^2 \xi_{n,n}(a, X^{n,n})}{\partial x_{1n+1}^2} \right] \]

\[ -2 \frac{\partial^2 \xi_{n,n}^{-1}(a, X)}}{\partial x_{1n+1}^2} \bigg|_{x_{1n+1} = 0}. \]

But for the function

\[ G(a, b, X) = \left( \prod_{k=1}^{n+1} a_k b_k \right)^{-1/4} G^{n+1}(a, b, X) \]

we have

\[ (G^{n+1})^{-1} \frac{\partial^2 G^{n+1}}{\partial x_{1n+1}^2} \bigg|_{x_{1n+1} = 0} = (G)^{-1} \frac{\partial^2 G}{\partial x_{1n+1}^2} \bigg|_{x_{1n+1} = 0} = -2(G)^{-1} \left( \frac{\partial G}{\partial a_{n+1}} + \frac{\partial G}{\partial b_{n+1}} \right) \frac{a_{n+1} + b_{n+1}}{2}, \]

which makes it possible to use the following formula, which was already proved:

\[ G(a, b, X) \bigg|_{x_{1n+1} = 0} = \left( \prod_{k=1}^{n+1} a_k b_k \right)^{-1/4} G^{n+1}(a, b, X) \bigg|_{x_{1n+1} = 0} = \left[ \frac{\xi_{n,n}(a, X^{n,n}) \xi_{n,n}(b, X^{n,n})}{\prod_{k=1}^{n+1} (a_k + b_k)^2} \right]^{1/4} \]

On calculating

\[ (G^{n+1})^{-1} \frac{\partial^2 G^{n+1}}{\partial x_{1n+1}^2} \bigg|_{x_{1n+1} = 0} \]

and substituting into \( \partial^2 \theta / \partial x_{1n+1}^2 \), we get (3.41).
and
\[
\left( \sum_{k=1}^{n+1} b_k x_{ik} \left( x_{m+1k} + \sum_{r=2}^{m} s_r x_{rk} \right) \right)^2 \left( 1 + \sum_{k=1}^{n+1} a_k x_{ik} \right)^{-1} \geq \left( \sum_{k=1}^{n+1} b_k x_{ik} \right)^2 \frac{1}{1 + \sum_{k=1}^{n+1} b_k x_{ik}^2},
\]
for
\[
s = (s_2, \ldots, s_m) \in L_{m-1}(b) = \left\{ s \in \mathbb{R}^{m-1} \left| \sum_{r=2}^{m} s_r b_{ik} x_{rk} \geq 0 \right. \right\}
\]
and
\[
x^{(m+1)} = (x_{m+11}, \ldots, x_{m+1n}) \in D_n(a) \cap D_n(b),
\]
\[
D_n(a) = \left\{ x^{(m+1)} \in \mathbb{R}^n \left| \sum_{k=1}^{n} a_k x_{ik} x_{m+1k} \geq 0 \right. \right\},
\]
\[
D_n(b) = \left\{ x^{(m+1)} \in \mathbb{R}^n \left| \sum_{k=1}^{n} b_k x_{ik} x_{m+1k} \geq 0 \right. \right\}
\]
and \( D_n(a, b) = D_n(a) \cap D_n(b) \neq \emptyset \), because of \( a_k, b_k > 0, k = 1, \ldots, n \), the following estimate holds for \( G^{n+1}(a, b, X) \):
\[
G^{n+1}(a, b, X) \geq \frac{\left( \prod_{k=1}^{n+1} a_k b_k \right)^{1/4}}{\sqrt{\pi^{n+1}}} \int_{\mathbb{R}_+} \int_{D_n(a, b)} dx_{m+1n+1}
\]
\[
\times \left( \int_{x_{m+1} > 0} \exp \left( -\sum_{k=1}^{n+1} a_k x_{m+1k} \right) \sum_{k=2}^{n+1} \frac{1}{\sqrt{\pi}} \exp(-t_k^2) dt_k \right)^{1/2}
\]
\[
\times \left( \int_{x_{m+1} > 0} \exp \left( -\sum_{k=1}^{n+1} b_k x_{m+1k} \right) \sum_{k=2}^{n+1} \frac{1}{\sqrt{\pi}} \exp(-s_k^2) ds_k \right)^{1/2}
\]
\[
\times \exp \left( \left( \frac{(a_{n+1} x_{1n+1}^2 + x_{m+1n+1}^2)}{1 + \sum_{k=1}^{n+1} a_k x_{ik}^2} - a_{n+1} x_{m+1n+1}^2 \right) + \left( \sum_{k=1}^{n+1} b_k x_{ik}^2 \right) - b_{n+1} x_{m+1n+1}^2 \right)
\]
\[
\times \left( \prod_{k=1}^{n+1} dx_{m+1k} \right)^{1/2} \left( \prod_{k=1}^{n+1} dx_{m+1k} \right)^{-1/4}
\]

Since
\[
\left( \left( 1 + \sum_{k=1}^{n+1} a_k x_{ik} \right) \left( 1 + \sum_{k=1}^{n+1} b_k x_{ik} \right) \right)^{-1/4} \int_{\mathbb{R}_+} \frac{\exp(\frac{1}{2}) \left( (a_{n+1} x_{1n+1}^2 + x_{m+1n+1}^2) + (b_{n+1} x_{m+1n+1}^2) \right)}{1 + \sum_{k=1}^{n+1} b_k x_{ik}^2}
\]
\[
\times \exp \left( \left( \frac{a_{n+1} x_{1n+1}^2 + x_{m+1n+1}^2}{1 + \sum_{k=1}^{n+1} b_k x_{ik}^2} + b_{n+1} x_{m+1n+1}^2 \right) \right)
\]
\[
\times \frac{1}{\sqrt{\pi}} \left( \sum_{k=1}^{n+1} a_k x_{ik}^2 \right)^{1/2}
\]
\[
= \sqrt{\pi} \left( a_{n+1} x_{1n+1}^2 \right)^{1/2} \left( \sum_{k=1}^{n+1} b_k x_{ik}^2 \right)^{1/2}
\]
\[
= \sqrt{\pi} \left( a_{n+1} \right)^{1/2} \left( \sum_{k=1}^{n+1} b_k x_{ik}^2 \right)^{1/2}
\]
\[
= \sqrt{\pi} \left( a_{n+1} \right)^{1/2} \left( \sum_{k=1}^{n+1} b_k x_{ik}^2 \right)^{1/2} > 0
\]
for any collection \((x_1, \ldots, x_n) \in \mathbb{R}^n\), then

\[
\lim_{x_{k+1} \to x_k} G^{a_k+1}(a, b, X) \geq \left( \prod_{k=1}^{n} \frac{a_k b_k}{\sqrt{\pi}} \right)^{1/4} \int_{D(a, b)} \left( \int_{\mathbb{R}^n} \exp \left( -\sum_{k=1}^{n} a_k (x_{m+k} + \sum_{r=2}^{m} l x_{k+r})^2 \right) \otimes \frac{1}{\sqrt{2 \pi}^n} \exp \left( -\frac{1}{2} \sum_{k=1}^{n} b_k x_k^2 \right) dx_k \right)^{1/2} \times \left( \int_{\mathbb{R}^n} \exp \left( -\sum_{k=1}^{n} b_k (x_{m+k} + \sum_{r=2}^{m} l x_{k+r})^2 \right) \otimes \frac{1}{\sqrt{2 \pi}^n} \exp \left( -\frac{1}{2} \sum_{k=1}^{n} a_k x_k^2 \right) dx_k \right)^{-1/2} \times \prod_{k=1}^{n} dx_{m+k} \cdot 2^{-1/2} (a_n + b_n + 1)^{-1/4} \left( 2 + \sum_{k=1}^{n} (a_k + b_k)x_k^2 \right)^{-1/2} > 0.
\]

Thus (3.41) is proved.

Accordingly, \(\theta_k(a, b, X) = 0\) for \(2 \leq k \leq p\), since, in the contrary case, taking account of (3.39) and (3.40), we would arrive at a contradiction with (3.42). This completes the proof of Lemma 3.3.

References

[8] R. S. Ismagilov, Unitary representations of the groups \(G^k(X, G)\), \(G = SU_2\) (Russian), Mat. Sb. 100(142):1 (1976), 117–131. (MR 54 #475)