IRREDUCIBILITY CRITERION FOR REGULAR GAUSSIAN REPRESENTATIONS OF GROUPS OF FINITE UPPER TRIANGULAR MATRICES

A. V. Kosyak

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1. Let E_{kn}, $k, n \in \mathbf{N}$, be matrix units of infinite order. We consider the groups $B(\mathbf{N},\mathbf{R}) = \{I + \sum_{k<n} x_{kn}E_{kn} | x_{kn}$ are finite$\}$, $B^\infty = \{I + \sum_{k<n} x_{kn}E_{kn} | x_{kn}$ are arbitrary$\}$ and the Lie algebra $\mathfrak{b}^\infty = \{\sum_{k<n} x_{kn}E_{kn} | x_{kn}$ are arbitrary$\}$. For matrices of positive numbers $b = (b_{kn})_{k<n}$ (their totality is denoted by $\mathscr{B}$) we define the Gaussian measure μ_b on the space $\mathfrak{b}^\infty$: $d\mu_b(x) = \bigotimes_{k<n} (b_{kn}\pi^{-1})^{1/2} \exp(-b_{kn}x_{kn}^2)\, dx_{kn}$. Let μ_b^π be the measure on B^∞ which is the image of the measure μ_b under the mapping π: $\mathfrak{b}^\infty \ni x \to \pi(x) = I + x \in B^\infty$. We consider the right action R_t and the left action L_t of the group $B(\mathbf{N},\mathbf{R})$ on B^∞. We denote by $(\mu_b^\pi)^{R_t}$, $(\mu_b^\pi)^{L_t}$ the images of the measure μ_b^π under the mappings $R_t, L_t\colon B^\infty \to B^\infty$.

LEMMA 1 [1]. We always have $(\mu_b^\pi)^{R_t} \sim \mu_b^\pi$, $(\mu_b^\pi)^{L_t} \sim \mu_b^\pi$, $t \in B(\mathbf{N},\mathbf{R})$, if and only if $S_{kn}^L(b) = \sum_{m=n+1}^{\infty} b_{km}b_{nm}^{-1} < \infty$, $k, n \in \mathbf{N}$, $k < n$.

We consider the family of right analogues $T^{R,b}$ and of left analogues $T^{L,b}$ of the regular representation of the group $B(\mathbf{N},\mathbf{R})$ into the Hilbert space $\mathscr{H}(b) = L_2(B^\infty, d\mu_b^\pi)$: $(T_{(t)}^{R,b}f)(x) = (d\mu_b^\pi(xt)/d\mu_b^\pi(x))^{1/2} f(xt)$, $(T_{(t)}^{L,b}f)(x) = (d\mu_b^\pi(t^{-1}x)/d\mu_b^\pi(x))^{1/2} f(t^{-1}x)$.

THEOREM 1. The right regular representations $T^{R,b}$, $b \in \mathscr{B}$, are irreducible if and only if no left shifts L_t, $t \in B(\mathbf{N},\mathbf{R})$, are admissible for the measure μ_b^π.

Remarks. 1. The assertion of the theorem has been formulated as a conjecture by R. S. Ismagilov.

2. For $b_{kn} \equiv 1$ the irreducibility of $T^{R,b}$ can be proved by using the method suggested by N. I. Nessonov [2], based on the Fourier transform and the law of large numbers; however, this method does not cover the case of an arbitrary $b \in \mathscr{B}$.

3. Apparently, the first irreducible representations, as subrepresentations of the analogue of a regular representation for the infinite-dimensional current group G^X, have appeared in [3].

Proof of Theorem 1. The necessity is obvious. Sufficiency. Let $(\mu_b^\pi)^{L_t} \perp \mu_b^\pi$, i.e., by Lemma 1, $S_{kn}^L(b) = \infty$, $k, n \in \mathbf{N}$, $k < n$.

LEMMA 2. The measure μ_b^π $B(\mathbf{N},\mathbf{R})$ is ergodic with respect to the right action of $B(\mathbf{N},\mathbf{R})$ on B^∞.

Proof. It is known that each measurable function on $\mathbf{R}^\infty$ with the standard Gaussian measure, not varying under an arbitrary variation of any first coordinates, coincides almost everywhere with a constant [4, Sec. 3, Corollary 1]. Therefore, the proof follows from the fact that the measure μ_b is a tensor product of measures and from the fact that the subgroup $B(n,\mathbf{R})$ of the group $B(\mathbf{N},\mathbf{R})$ acts transitively on itself.

We denote by $\tilde{W}(b)$ the set of selfadjoint or skew-adjoint operators in $\mathscr{H}(b)$, associated to the algebra $W(b) = (T_{(t)}^{R,b}, t \in B(\mathbf{N},\mathbf{R}))''$, and we show that $\{x_{kn}, \partial_{pq} - b_{pq}x_{pq} | k < n,$

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$p < q,\ k, n, p, q \in N\} \subset \tilde{W}(b)$. We shall carry out the proof by induction. Let $A_{kn}{}^R = (d/dt)\, T^{R,b}(I + tE_{kn}|_{t=0}$. A computation yields $A_{kn}{}^R = \sum_{m=1}^{k} x_{mk}(\partial_{mn} - b_{mn}x_{mn})$, $x_{kk} \equiv 1$, $\partial_{mn} = \partial/\partial x_{mn}$.

Induction Base. We show that $\{x_{12},\ \partial_{1k} - b_{1k}x_{1k},\ \partial_{2k+1} - b_{2k+1}x_{2k+1} |\ k \geq 2\} \subset \tilde{W}(b)$. Indeed, $\partial_{1k} - b_{1k}x_{1k} = A_{1k}{}^R \in \tilde{W}(b)$. We approximate $A = x_{12}$ by linear combinations $A_n = \sum_{k=n}^{k_n} c_k A_{1k}{}^R A_{2k}{}^R$. The deviation δ_{12,n,k_n} of $x_{12}1$ from the subspace $\mathscr{H}_{12,n,k_n} = \langle A_{1k}{}^R A_{2k}{}^R 1 |$ $n \leq k \leq k_n\rangle$ is equal to $\delta_{12,n,k_n} = (b_{12})^{-1} \left(2 + \sum_{k=n}^{k_n}\left(1 + \frac{b_{12}b_{2k}}{b_{1k}}\right)^{-1}\right)^{-1}$, and since $S_{12}{}^L(b) = \sum_{k=3}^{\infty} b_{1k}b_{2k}{}^{-1} = \infty$, there exists a sequence $\{k_n\} \subset N$ such that $\lim_{n\to\infty} \delta_{12,n,k_n} = 0$. Let $D = \langle x_{12}^{\alpha_{12}} x_{13}^{\alpha_{13}} x_{23}^{\alpha_{23}} \ldots x_{1k}^{\alpha_{k_1}} x_{2k}^{\alpha_{2k_2}} |\ \alpha_{ij} = 0, 1, \ldots, i < j, k \geq 2\rangle$. Then $A_n\varphi \to A\varphi$, $\varphi \in D$, and, therefore, $A_n \to A$ in the strong resolvent sense [5, Theorem VIII.25], $\partial_{2k} - b_{2k}x_{2k} = A_{2k}{}^R - x_{12}A_{1k}{}^R$, $k \geq 3$.

Induction Hypothesis. Let $\{x_{nm},\ n < m \leq p;\ \partial_{nm} - b_{nm}x_{nm},\ 1 \leq n \leq p,\ m > n\} \subset \tilde{W}(b)$, i.e., we have approximated the operators x_{nm}, $n < m \leq p$, by appropriate combinations of products $(A_{rs}{}^R)$, $1 \leq r \leq p$, $s > r$.

We show that then $\{x_{np+1},\ \partial_{p+1m} - b_{p+1m}k_{p+1m} | n < p + 1 < m\} \subset \tilde{W}(b)$, $m > p + 1$. Indeed, $\partial_{p+1m} - b_{p+1m}x_{p+1m} = A_{p+1m} - \sum_{k=1}^{p} x_{kp+1}(\partial_{km} - b_{km}x_{km})$. We approximate x_{np+1} by the combina- $(\partial_{nk} - b_{nk}x_{nk})\, A_{p+1k}{}^R$. Deviation δ_{np+1,s,k_s} of $x_{np+1}1$ from $\mathscr{H}_{np+1,s,k_s} = \langle(\partial_{nk} - b_{nk}x_{nk})$ $A_{p+1k}{}^R 1 | s \leq k \leq k_s\rangle$ satisfies the estimate

$$C_1 d_{np+1,s,k_s} \leqslant \delta_{np+1,s,k_s} \leqslant C_2 d_{np+1,s,k_s},$$

where $d_{np+1,s,k_s} = (b_{np+1})^{-1} \left(2 + \sum_{k=s}^{k_s}\left(1 + \sum_{r=1,\, r\neq n}^{p} \frac{b_{rk}}{b_{rp+1}} \frac{b_{np+1}}{b_{nk}} + \frac{b_{p+1k}}{b_{nk}} b_{np+1}\right)^{-1}\right)^{-1}$. The conditions $\lim_{s\to\infty} \delta_{np+1,s,k_s} = 0$ are equivalent to the divergence of the series $\sigma_{np+1} = \sum_{k=p+2}^{\infty} b_{nk}\left(\sum_{r=1,\, r\neq n}^{p+1} b_{rk}\right)^{-1}$, one of which diverges, since otherwise we obtain a contradiction with the conditions $S_{kp+1}{}^L(b) = \infty$, $k < p + 1$. Let $\sigma_{n_0 p+1} = \infty$; then $x_{n_0 p+1} \in \tilde{W}(b)$. We denote ${}^{n_0}A_{rs}^R = A_{rs}{}^R - x_{n_0 r}(\partial_{n_0 s} - b_{n_0 s}x_{n_0 s})$, $1 \leq r \leq p + 1$, $n \neq n_0$. In a similar manner one shows that by the operators $(\partial_{nk} - b_{nk}x_{nk})\, {}^{n_0}A_{p+1k}{}^R$ one approximates some operator $x_{n_1 p+1}$, $n_1 \neq n_0$, etc.

Thus, $\{x_{kn},\ \partial_{pq} - b_{pq}x_{pq} | k < n,\ p < q\} \subset \tilde{W}(b)$ and, in particular, $\{x_{kn} | k < n\} \subset \tilde{W}(b)$; therefore, $W(b)$ contains the operators $\{U_{kn}(t) = \exp(itx_{kn}) | t \in \mathbb{R}^1,\ k < n\}$. Assume that the bounded operator $A \in L(\mathscr{H}(b))$ commutes with all the operators $\{T_{(t)}{}^{R,b} | t \in B(\mathbf{N}, \mathbf{R})\}$; then A commutes with all the operators $U_{kn}(t)$ and, therefore, A is an operator of multiplication by an essentially bounded function $f_A(x)$. By virtue of the commutation $[f_A(x), T_{(t)}{}^{R,b}] = 0$, we conclude that $f_A(x) = f_A(xt)$ a.e. By virtue of the ergodicity of the measure $\mu_b{}^{\pi}$, we have $f_A(x) = \text{const}$, i.e., $A = \lambda I$, which is what we intended to prove.

THEOREM 2. The irreducible representations $T^{R,b^{(1)}}$ and $T^{R,b^{(2)}}$ are equivalent if and only if $\mu_{b^{(1)}} \sim \mu_{b^{(2)}}$.

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JORDAN DECOMPOSITION OF MEASURES ON PROJECTIONS IN J-SPACES

M. S. Matveichuk

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In [1] indefinite measures on the logic $\mathscr{P}$ of all J-self-adjoint bounded projections in a J-space have been described. An analog of the Gleason theorem has been obtained. In the present report an analog of the Jordan decomposition for measures on $\mathscr{P}$ has been presented.

Let H be a space with an indefinite metric $[\cdot, \cdot]$, with the canonical decomposition $H = H^+ [\dot{+}] H^-$ and the canonical symmetry J (H is a J-space). With respect to the scalar product $(x, y) \equiv [Jx, y]$ H is a Hilbert space (H.s.). We shall denote by $\mathscr{P}^+$ ($\mathscr{P}^-$) all projectors $p \in \mathscr{P}$, for which the subspace pH is positive (respectively, negative). We shall represent any projector $p \in \mathscr{P}$ in the form $p = p^+ + p^-$, $p^\pm \in \mathscr{P}^\pm$. An orthogonal family of projectors $\{p_i\} \in \mathscr{P}$ will be called a partition of $p \in \mathscr{P}$, if $p = \Sigma p_i$ (the sum is understood in the strong sense).

The mapping $\nu: \mathscr{P} \to R$ will be called a measure, if $\nu(p) = \Sigma \nu(p_i)$ for any partition $p = \Sigma p_i$. Measure ν will be called bounded if $c_\nu \equiv \sup \{|\nu(p)| \backslash \|p\| : p \in \mathscr{P}\} < \infty$; and indefinite, if $\nu/\mathscr{P}^+ \geq 0$ and $\nu/\mathscr{P}^- \leq 0$.

The following theorem holds.

<u>THEOREMS 1.</u> Let H be a J-space of the dimension not less than three, and let $\dim H^+ \leq \dim H^-$ and $\nu: \mathscr{P} \to R$ be a measure. The measure ν is bounded if and only if there exist a (unique) J-self-adjoint nuclear operator A and a number $t \in R$, such that

$$\nu(p) = \mathrm{Sp}(Ap) + t \dim (p^+H), \ \forall p \in \mathscr{P}. \tag{1}$$

Moreover, if $\dim H^+ = \infty$ then $t = 0$ $(0 \cdot \infty \equiv 0)$.

<u>Corollary 1 (an analog of the Jordan decomposition).</u> Each bounded measure on $\mathscr{P}$ in a J-space of dimension not less than three is a linear combination of indefinite measures.

The boundedness of a measure depends essentially on the dimension of H. Indeed, let $\dim H < \infty$, let ν satisfy equality (1), where $A \neq \lambda I$, and let n be a discontinuous endomorphism of the additive group of real numbers. Then $n \circ \nu$ is an unbounded measure. We mention also that an analogical property of measures on orthoprojections has been at first noticed by A. N. Sherstnev [2].

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