EXTENSION OF UNITARY REPRESENTATIONS OF INDUCTIVE LIMITS OF FINITE DIMENSIONAL LIE GROUPS

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We prove that every continuous unitary representation of inductive limit of general linear groups can be extended by continuity to a representation of some Hilbert–Lie group depending on the representation.

1. Introduction

When we study unitary representations of inifinite dimensional groups the problem of extension of concrete representations to some groups involving the initial one arises naturally. This problem seems to have been first recognised by Shale [1], Shale and Stinespring [2]. In their papers the special spinor representation of the group $SO(2\infty)$, i.e. the representation of the group $\mathrm{Spin}(2\infty)$ was constructed, and it was extended to a group of orthogonal operators $1+\sigma_1(\mathscr{H})$, where $\sigma_1(\mathscr{H})$ is the space of nuclear operators in a real Hilbert space $\mathscr{H}$. Analogous questions have been investigated in the paper of Stratila and Voiculescu [3] for the group $U(\infty)=\varinjlim_n U(n)$. Reed [4] and Woods [5] showed that the representation of canonical commutation relations (CCR) with an infinite number of degrees of freedom, i.e. the special representation of the infinite dimensional Heisenberg–Weyl group always can be extended to a representation of some group involving the initial one depending on the representation. In the work of Goodman and Wallach [6] the following Kac's hypothesis dealing with extension of a representation was proved. Let $\mathscr{D}$ be the group of orientation-preserving diffeomorphisms of the circle S^1, $(\mathfrak{D}_\infty)_{\mathbf{R}}$ be its Lie algebra, i.e. the algebra of smooth real vector fields on the circle, $\mathfrak{D}_{\mathbf{R}}$ the subalgebra of vector fields with finite Fourier series. Then every infinitesimally unitary projective positive-energy representation of $\mathfrak{D}_{\mathbf{R}}$ integrates to a continuous projective unitary representation of $\mathscr{D}$.

Extension of representations of infinite dimensional groups is useful for generalization of the Garding construction [7] to infinite dimensional groups. This programme was realized for the group R_0^∞ [8], for the Heizenberg–Weyl group, connected with the representation of CCR [9] and for the group $B(N, R)$ of finite upper triangular matrices of the infinite order [10].

The aim of this paper is to prove that every continuous unitary representation of inductive limit of a general linear group $\mathrm{GL}(Z, R) = \varinjlim_n \mathrm{GL}(2n+1, R)$ can be extended by continuity to the unitary representation of some Hilbert–Lie group $\mathrm{GL}_2(a)$ depending on the representation. We do not know whether an analogous statement holds for every inductive limit of the finite dimensional Lie groups.

In Section 2 the family of "sufficiently close" to GLZ, R) Hilbert–Lie groups $\mathrm{GL}_2(a)$, such that $\mathrm{GL}(Z, R) = \varprojlim_{a \in \mathfrak{A}} \mathrm{GL}_2(a)$, is constructed. In Section 3 we prove Theorem 3.1 about an extension of the representation of the group R_0^∞ using Stone's theorem for the R_0^∞ additive group of finite sequences of real numbers. Of special importance is Section 4, where a theorem about extension of the representation for the subgroup $B(N, R)$ (of the group $\mathrm{GL}(Z, R)$) of upper triangular matrices of infinite order in one direction is proved [11]. The proof is based on studying convergent infinite factors $\prod_{n=1}^{\infty} U_n(t_n)$ of one parameter groups of unitary operators and on the Richter [12] argument concerning the existence of a dense set of C^∞-vectors for the unitary representation of the inductive limite of finite dimensional Lie groups. Transition from the $B(N, R)$ to $B(Z, R)$ group of the finite upper triangular matrices of infinite order in both directions can be shown easily. A theorem about extention of representations for the group $\mathrm{GL}(Z, R)$ based on an analogue of the Gauss decomposition of the group $\mathrm{GL}_2(a)$ in the neighbourhood of identity

$$\mathrm{GL}_2(a) = B_2^*(a)\, D_2^+(a)\, B_2(a),$$

where $B_2^*(a)$ ($B_2(a)$) is the group of lower (upper) triangular matrices and $D_2^+(a)$ is the group of diagonal matrices with positive elements. In the sequal by the term "representation" we mean "continuous unitary representation".

2. Construction of the extended Hilbert–Lie groups $\mathrm{GL}_2(a)$

In this section we construct for the group $\mathrm{GL}(Z, R) = \varinjlim_n \mathrm{GL}(2n+1, R)$ and for its Lie-algebra $\mathfrak{gl}(Z, R) = \varinjlim_n \mathfrak{gl}(2n+1, R)$ the family of Hilbert–Lie groups $\mathrm{GL}_2(a)$ and Hilbert–Lie algebras $\mathfrak{gl}_2(a)$ $a \in \mathfrak{A}$ such that $\varprojlim_{a \in \mathfrak{A}} \mathrm{GL}_2(a) = \mathrm{GL}(Z, R)$ and $\varprojlim_{a \in \mathfrak{A}} \mathfrak{gl}_2(a) = \mathfrak{gl}(Z, R)$.

Consider the sequence of groups $\mathrm{GL}(2n+1, \boldsymbol{R})$ of nondegenerate real matrices of the order $2n+1$. Let E_{km}, $k, m \in \boldsymbol{Z}$ ($E_{km}^{(2n+1)}$, $k, m \in \boldsymbol{Z}$, $-n \leqslant k, m \leqslant n$) be matrix units of infinite order (respectively of the order $2n+1$). Define the imbedding i_{n+1}^n: $\mathrm{GL}(2n+1, \boldsymbol{R}) \to \mathrm{GL}(2n+3, \boldsymbol{R})$,

$$\mathrm{GL}(2n+1, \boldsymbol{R}) \ni t = \sum_{-n \leqslant k,m \leqslant n} t_{km} E_{km}^{(2n+1)} \mapsto i_{n+1}^n(t) = \sum_{-n \leqslant k,m \leqslant n} t_{km} E_{km}^{(2n+3)}$$
$$+ E_{-(n+1),-(n+1)}^{(2n+3)} + E_{n+1,n+1}^{(2n+3)} \in \mathrm{GL}(2n+3, \boldsymbol{R}).$$

Let the appearing inductive limit of groups be denoted by $\mathrm{GL}(\boldsymbol{Z}, \boldsymbol{R}) = \varinjlim_n \mathrm{GL}(2n+1, \boldsymbol{R})$. Let $\mathfrak{gl}(\boldsymbol{Z}, \boldsymbol{R}) = \varinjlim_n \mathfrak{gl}(2n+1, \boldsymbol{R})$ be the inductive limit of corresponding Lie algebras. It is obvious that elements of the group $\mathrm{GL}(\boldsymbol{Z}, \boldsymbol{R})$ (algebra $\mathfrak{gl}(\boldsymbol{Z}, \boldsymbol{R})$) are of the form

$$I + \sum_{k,m \in \boldsymbol{Z}} x_{km} E_{km} \qquad \Big(\sum_{k,m \in \boldsymbol{Z}} x_{km} E_{km}\Big),$$

where only a finite number of x_{km} is nonzero.

Consider the matrix of positive numbers $a = (a_{kn})_{k,n \in \boldsymbol{Z}}$ termed as *weight*. Denote the real Hilbert space

$$\mathfrak{gl}_2(a) = \Big\{x = \sum_{k,n \in \boldsymbol{Z}} x_{kn} E_{kn} \,\Big|\, \|x\|_{\mathfrak{gl}_2(a)}^2 = \sum_{k,n \in \boldsymbol{Z}} |x_{kn}|^2 a_{kn} < \infty\Big\}$$

being the closure of the space $\mathfrak{gl}(\boldsymbol{Z}, \boldsymbol{R})$ in topology generated by the corresponding Hilbert norm. The space $\mathfrak{gl}(\boldsymbol{Z}, \boldsymbol{R})$ is an associative algebra with matrix multiplication.

LEMMA 2.1. *The space $\mathfrak{gl}_2(a)$ is an associative algebra iff the following conditions hold for the weight a:*

$$a_{kn} \leqslant c^2 a_{km} a_{mn}, \qquad k, n, m \in \boldsymbol{Z}, \; c > 0. \tag{2.1}$$

Proof: Let $\mathfrak{gl}_2(a)$ be a Hilbert algebra, i.e. a constant $c > 0$ exists such that

$$\|xy\|_{\mathfrak{gl}_2(a)} \leqslant c\,\|x\|_{\mathfrak{gl}_2(a)} \cdot \|y\|_{\mathfrak{gl}_2(a)}, \qquad x, y \in \mathfrak{gl}_2(a). \tag{2.2}$$

If we substitute $x = E_{km}$, $y = E_{mn}$, $z = xy = E_{kn}$ in inequality (2.2), we get (2.1).

Conversely, let (2.1) hold. Denote

$$x = \sum_{k,m \in \boldsymbol{Z}} x_{km} E_{km}, \qquad y = \sum_{m,n \in \boldsymbol{Z}} y_{mn} E_{mn}, \qquad z = xy = \sum_{k,n \in \boldsymbol{Z}} z_{kn} E_{kn},$$

where $z_{kn} = \sum_{m \in \boldsymbol{Z}} x_{km} y_{mn}$. Let us estimate $|z_{kn}|^2 a_{kn}$. Using (2.1), we get

$$|z_{kn}|^2 a_{kn} = \Big|\sum_{m \in \boldsymbol{Z}} x_{km} y_{mn}\Big|^2 a_{kn} \leqslant c^2 \Big(\sum_{m \in \boldsymbol{Z}} |x_{km}|^2 a_{km}\Big)\Big(\sum_{m \in \boldsymbol{Z}} |y_{mn}|^2 a_{mn}\Big),$$

hence

$$\|xy\|^2_{\mathfrak{gl}_2(a)} = \sum_{k,n\in \mathbf{Z}} |z_{kn}|^2 a_{kn} \leqslant c^2 \sum_{k,n\in \mathbf{Z}} \Big(\sum_{m\in \mathbf{Z}} |x_{km}|^2 a_{km}\Big) \times$$
$$\times \Big(\sum_{m\in \mathbf{Z}} |y_{mn}|^2 a_{mn}\Big) = c^2 \Big(\sum_{k,m\in \mathbf{Z}} |x_{km}|^2 a_{km}\Big)\Big(\sum_{m,n\in \mathbf{Z}} |y_{mn}|^2 a_{mn}\Big)$$
$$= c^2 \|x\|^2_{\mathfrak{gl}_2(a)} \cdot \|y\|^2_{\mathfrak{gl}_2(a)}. \quad \blacksquare$$

Denote the set of weights a for which condition (2.1) holds by $\mathfrak{A}$. Define the bracket $[x, y] = xy - yx$ in $\mathfrak{gl}_2(a)$. Since $\|[x, y]\|_{\mathfrak{gl}_2(a)} \leqslant 2C \|x\|_{\mathfrak{gl}_2(a)} \cdot \|y\|_{\mathfrak{gl}_2(a)}$, the algebra $\mathfrak{gl}_2(a)$ with the bracket becomes a Hilbert–Lie algebra.

Remark 2.2. Choose $m = n$ in conditions (2.1), then we get: $1 \leqslant c^2 a_{nn}$, i.e. $a_{nn} \geqslant c^{-2}$, hence $I = \sum_{n\in \mathbf{Z}} E_{nn}$ does not belong to $\mathfrak{gl}_2(a)$, since $\|I\|^2_{\mathfrak{gl}_2(a)} = \sum_{n\in \mathbf{Z}} a_{nn} = \infty$.

EXAMPLE 2.3. Let $a_{kn} \equiv 1$, $k, n \in \mathbf{Z}$, then conditions (2.1) obviously hold. In this case the algebra $\mathfrak{gl}_2(a)$ coincides with the algebra $\sigma_2(l_2(\mathbf{Z}))$ of Hilbert–Schmidt operators in the space

$$l_2(\mathbf{Z}) = \{x = (x_k)_{k\in \mathbf{Z}} \mid \|x\|^2_{l_2(\mathbf{Z})} = \sum_{k\in \mathbf{Z}} |x_k|^2 < \infty\}.$$

EXAMPLE 2.4. Let $l_2(b)$, $l_2(c)$ be real Hilbert spaces of sequences, $b = (b_k)_{k\in \mathbf{Z}}$, $c = (c_k)_{k\in \mathbf{Z}}$ be two weights. Denote $a_{kn} = c_k b_n^{-1}$, $k, n \in \mathbf{Z}$ then condition (2.1) is equivalent to the following one: $b_m \leqslant c^2 c_m$, $m \in \mathbf{Z}$. In this case the algebra $\mathfrak{gl}_2(a)$ is isomorphic to the algebra of Hilbert–Schmidt operators $\sigma_2(l_2(b), l_2(c))$, acting from $l_2(b)$ into $l_2(c)$.

Remark 2.5. For every weight $c = (c_{kn})_{k,n\in \mathbf{Z}}$ not necessarily belonging to $\mathfrak{A}$ there exists a weight $a = (a_{kn})_{k,n\in \mathbf{Z}}$ such that $\mathfrak{gl}_2(a)$ is an algebra and $\mathfrak{gl}_2(a) \subset \mathfrak{gl}_2(c)$.

Proof: Define $a_{kn} = a_k a_n$, $k, n \in \mathbf{Z}$, where $a_k = a_{-k} = \max_{|p|,|q|\leqslant |k|} \{1, c_{pq}\}$. Then conditions $a_{kn} \leqslant a_{km} a_{mn}$ become $a_k a_n \leqslant a_k a_m a_m a_n$, i.e. $a_m^2 \geqslant 1$, $m \in \mathbf{Z}$ and they hold by definition of a_k. The inclusion $\mathfrak{gl}_2(a) \subset \mathfrak{gl}_2(c)$ is equivalent to the inequalities $a_{kn} \geqslant c_{kn}$, $k, n \in \mathbf{Z}$, which is sufficient to prove for $|k| \geqslant |n|$, taking into consideration the symmetry of a_{kn}. We have

$$a_{kn} = a_k a_n \geqslant a_k = \max_{|p|,|q|\leqslant |k|} \{1, c_{pq}\} \geqslant c_{kn}, \quad k, n \in \mathbf{Z}. \quad \blacksquare$$

Let us construct the Hilbert–Lie group corresponding to the Hilbert–Lie algebra $\mathfrak{gl}_2(a)$. There is no identity in the algebra $\mathfrak{gl}_2(a)$ with multiplication $(x, y) \mapsto xy$ (Remark 2.2). Denote by A the real Hilbert algebra obtained from $\mathfrak{gl}_2(a)$ by adjoining the identity.

The pairs (λ, x), $\lambda \in \boldsymbol{R}^1$, $x \in \mathfrak{gl}_2(a)$ with the natural operations and norm are the elements of the algebra A. Consider the Hilbert algebra of matrices $\tilde{A} = \{\lambda I + x \mid \lambda \in \boldsymbol{R}^1,\ x \in \mathfrak{gl}_2(a)\}$. It is obvious that the mapping $A \ni (\lambda, x) \mapsto \lambda I + x \in \tilde{A}$ is an isomorphism of Hilbert algebras, which in the sequel we will not distinguish.

Consider the set A^* of invertible elements in the algebra A and denote by $\mathrm{GL}_2(a) = \{(\lambda, x) \in A^* \mid \lambda = 1\}$.

LEMMA 2.6. *The set* $\mathrm{GL}_2(a)$ *is a Hilbert–Lie group, for which the Lie algebra is the Hilbert–Lie algebra* $\mathfrak{gl}_2(a)$.

Proof: The set A^* is Hilbert–Lie group [13, Chapter III]. It is obvious that $\mathrm{GL}_2(a)$ is subgroup of A^*. Since $\mathrm{GL}_2(a)$ is a submanifold in A^* [14; 5, 8.3], $\mathrm{GL}_2(a)$ is a Lie subgroup of A^*. It is known [13] that the Lie algebra $L(A^*)$ of the Lie group A^* is isomorphic to the Lie algebra A with the bracket $[x, y] = xy - yx$, hence the Lie subalgebra $A' = \{(\lambda, x) \in A \mid \lambda = 0\}$ of the algebra A, which is isomorphic to the Lie algebra $\mathfrak{gl}_2(a)$, corresponds to the Lie subgroup $\mathrm{GL}_2(a)$ of the group A^*. ■

Consider the set of all Hilbert–Lie algebras $\mathfrak{gl}_2(a)$, $a \in \mathfrak{A}$. Define the relation of partial order on $\mathfrak{A}$. For $a, b \in \mathfrak{A}$ we will write $a < b$ if $a_{km} \leqslant b_{km}$, $k, m \in \boldsymbol{Z}$. If $a < b$, then the algebra $\mathfrak{gl}_2(b)$ is a subalgebra of $\mathfrak{gl}_2(a)$. Let p_b^a: $\mathfrak{gl}_2(a) \to \mathfrak{gl}_2(b)$ be the projection of the algebra $\mathfrak{gl}_2(a)$ on the subalgebra $\mathfrak{gl}_2(b)$. Denote the corresponding projective limit by $\varprojlim_{a \in \mathfrak{A}} \mathfrak{gl}_2(a)$. We show that the algebra $\mathfrak{gl}(\boldsymbol{Z}, \boldsymbol{R})$ is isomorphic to $\varprojlim_{a \in \mathfrak{A}} \mathfrak{gl}_2(a)$. Considered as a set, $\varprojlim_{a \in \mathfrak{A}} \mathfrak{gl}_2(a)$ is equal to $\bigcap_{a \in \mathfrak{A}} \mathfrak{gl}_2(a)$. We prove that $\mathfrak{gl}(\boldsymbol{Z}, \boldsymbol{R}) = \varprojlim_{a \in \mathfrak{A}} \mathfrak{gl}_2(a)$. Indeed, $\mathfrak{gl}(\boldsymbol{Z}, \boldsymbol{R}) \subset \mathfrak{gl}_2(a)$, $a \in \mathfrak{A}$, hence $\mathfrak{gl}(\boldsymbol{Z}, \boldsymbol{R}) \subset \bigcap_{a \in \mathfrak{A}} \mathfrak{gl}_2(a)$. Let $x = \sum_{k,m \in \boldsymbol{Z}} x_{km} E_{km} \in \bigcap_{a \in \mathfrak{A}} \mathfrak{gl}_2(a)$, then only a finite number of x_{km}, $k, m \in \boldsymbol{Z}$ is nonzero otherwise there exists a weight $a = (a_{kn})_{k,n \in \boldsymbol{Z}}$, such that $a_{km} \geqslant x_{km}^{-2}$ for $(k, m) \in \boldsymbol{Z}^2$ with $x_{km} \neq 0$ (Remark 2.5). Then $\|x\|^2_{\mathfrak{gl}_2(a)} = \sum_{k,m \in \boldsymbol{Z}} |x_{km}|^2 a_{km} = \infty$ which contradicts the inclusion $x \in \mathfrak{gl}_2(a)$, hence $x \in \mathfrak{gl}(\boldsymbol{Z}, \boldsymbol{R})$.

The proof of $\varprojlim_{a \in \mathfrak{A}} \mathrm{GL}_2(a) = \mathrm{GL}(\boldsymbol{Z}, \boldsymbol{R})$ is analogous. It is possible to prove that $\mathfrak{gl}(\boldsymbol{Z}, \boldsymbol{R}) = \varinjlim_n \mathfrak{gl}(2n+1, \boldsymbol{R})$ and $\varprojlim_{a \in \mathfrak{A}} \mathfrak{gl}_2(a)$ are isomorphic as topological spaces.

The same is true for the groups $\mathrm{GL}(\boldsymbol{Z}, \boldsymbol{R}) = \varinjlim_n \mathrm{GL}(2n+1, \boldsymbol{R})$ and $\varprojlim_{a \in \mathfrak{A}} \mathrm{GL}_2(a)$.

3. Extension of the representations of the group $\boldsymbol{R}_0^\infty$

Recall that $\boldsymbol{R}_0^\infty = \varinjlim_n \boldsymbol{R}^n$ is an additive group of finite sequences of real numbers with inductive limit topology. Let $R_0^\infty \to t \mapsto u(t) \in U(\mathscr{H})$ be a strongly continuous representation of R_0^∞ in a separable Hilbert space $\mathscr{H}$. Denote

$$G_2(a) = l_2(a) = \{t = (t_k)_{k\in N} \mid \|t\|^2_{l_2(a)} = \sum_{k\in N} |t_k|^2 a_k < \infty\},$$

where $a = (a_k)_{k\in N}$, $a_k > 0$, $k \in N$.

THEOREM 3.1 (*actually proved in* [4]). *Every representation of the group R_0^∞ can be extended by continuity to a representation of some Hilbert–Lie group $G_2(a)$, depending on the representation.*

Proof: Consider the measurable space $(R^\infty, B(R^\infty))$, where $R^\infty = R^1 \times R^1 \times \ldots$, $B(R^\infty)$ is a σ-algebra of subsets in R^∞, generated by cylindric sets with Borel bases. By Stone's theorem for the representation of the group R_0^∞ [15] on a measurable space, there exists a projective valued measure E such that

$$U(t) = \int_{R^\infty} \exp i(t, x)\, dE(x), \qquad t \in R_0^\infty. \tag{3.1}$$

It turns out that for the measure E there exists a Hilbert space $l_2(b) \subset R^\infty$ such that $E(l_2(b)) = I$. Then it is possible to integrate in formula (3.1) not over R^∞ but over $l_2(b) \subset R^\infty$; hence the right part in (3.1) have a sense for $t \in l_2(a) = (l_2(b))'$, $a_n = b_n^{-1}$, $n \in N$

$$U_2(t) = \int_{l_2(b)} \exp i(t, x)\, dE(x), \qquad t \in l_2(a). \tag{3.2}$$

It is obvious that formula (3.2) gives a continuous unitary representation U_2: $l_2(a) \to U(\mathscr{H})$ of a Hilbert–Lie group $l_2(a)$, which is an extension by continuity of the representation U: $R_0^\infty \to U(\mathscr{H})$ of the group R_0^∞ (continuity of the representation U_2 holds by the Lebesgue dominanted convergence theorem). It remains to find the space $l_2(b) \subset R^\infty$ such that $E(l_2(b)) = I$. Let us use the well -known result:

LEMMA 3.2. [4], [16], [17]. *For every probability measure μ on the measurable space $(R^\infty, B(R^\infty))$ there exists a Hilbert space $l_2(b) \subset R^\infty$ such that $\mu(l_2(b)) = 1$.*

Remark 3.3. An analogous statement is true if we substitute a scalar measure μ on the projective valued measure E [17], [10].

4. Extension of the representation of the group $B(N, R)$

Let $B(N, R) = \varinjlim_n B(n, R)$ be a group of finite upper triangular matrices of infinite order with identities on the principal diagonal and $\mathfrak{b}(N, R)$ its Lie algebra. Obviously, $B(N, R)$ $(\mathfrak{b}(N, R))$ is a subgroup (subalgebra) of the group $\mathrm{GL}(Z, R)$ (of the algebra $\mathfrak{gl}(Z, R)$). For given weight $a = (a_{kn})_{k<n}$, $k, n \in N$ let us define the Hilbert–Lie group $B_2(a)$ analogous to $\mathrm{GL}_2(a)$, $a \in \mathfrak{A}$, and let the Hilbert–Lie algebra $\mathfrak{b}_2(a)$ be the closure of the algebra $\mathfrak{b}(N, R)$ in the corresponding Hilbert

norm

$$\mathfrak{b}_2(a) = \{x = \sum_{k<m} x_{km} E_{km} \mid \|x\|^2_{\mathfrak{b}_2(a)} = \sum_{k<m} |x_{km}|^2 a_{km} < \infty\}.$$

Conditions (2.1) here transform into

$$a_{kn} \leqslant c^2 a_{km} a_{mn}, \quad k, n, m \in N, k < m < n, c > 0. \tag{4.1}$$

Let $A = \{(\lambda, x) \mid \lambda \in \boldsymbol{R}^1, x \in \mathfrak{b}_2(a)\}$ be a Hilbert algebra, resulting from $\mathfrak{b}_2(a)$ by adjoining the identity, A^* the set of invertible elements of the algebra A. Then $B_2(a) = \{(\lambda, x) \in A^* \mid \lambda = 1\}$.

LEMMA 4.1. *The set of invertible elements A^* of the algebra A coincides with the set* $\{(\lambda, x) \in A \mid \lambda \neq 0\}$.

Proof (reported by G. I. Olshansky): If $\|x\|_{\mathfrak{b}_2(a)} < |\lambda|$, then the element $y = \sum_{n=1}^{\infty} (-x)^n \lambda^{-n-1}$ belongs to the algebra $\mathfrak{b}_2(a)$, hence $(\lambda^{-1}, y) \in A$. Since $(\lambda, x)(\lambda^{-1}, y) = (1, 0)$, the element $(\lambda, x) \in A$ is invertible in A. Let us take an arbitrary element $(\lambda, x) \in A$ with $\lambda \neq 0$, $x = \sum_{k<n} x_{kn} E_{kn}$. Define $x_m = \sum_{k<n\leqslant m} x_{kn} E_{kn}$, $x^{(m)} = x - x_m$, $m \in N$. Since $\|x - x_m\|_{\mathfrak{b}_2(a)} = \|x^{(m)}\|_{\mathfrak{b}_2(a)} \to 0$ as $m \to \infty$ there exists $m_0 \in N$ such that $\|x^{(m_0)}\| < \lambda$. The element $(1, x^{(m_0)} \lambda^{-1})$ is invertible in A by finiteness of x_{m_0}, *the element* $(1, x^{(m_0)}\lambda^{-1})$ is invertible by the previous proof, so the equality $(\lambda, x) = \lambda(1, x\lambda^{-1}) = \lambda(1, x^{(m_0)} \lambda^{-1})(1, x_{m_0} \lambda^{-1})$ completes the proof. ■

COROLLARY 4.2. *Every element $(1, x) \in A$ is invertible, i.e. the group $B_2(a)$ consists of elements* $\{(\lambda, x) \in A \mid \lambda = 1\}$.

THEOREM 4.3. *Every unitary representation U: $B(N, \boldsymbol{R}) \to U(\mathscr{H})$ of the group $B(N, \boldsymbol{R})$ can be extended by continuity to a unitary representation U_2: $B_2(a) \to U(\mathscr{H})$ of some Hilbert–Lie group, depending on the representation.*

The proof is based on the construction of a *convergent infinite product* of unitary operators.

In the sequel we will need an explicit expression for $t^{-1} \in B_2(a)$, the inverse element of $t = I + \sum_{k<n} t_{kn} E_{kn} \in B_2(a)$. Use the equality $t^{-1} t = I$, defining $t^{-1} = \omega = I + \sum_{k<n} \omega_{kn} E_{kn}$. We have $(I + \sum_{k<r} \omega_{kr} E_{kr})(I + \sum_{r<n} t_{rn} E_{rn}) = I$. Transform the left part, denoting $\omega_{kk} = t_{kk} = 1$, $k \in N$. We get

$$(I + \sum_{k<r} \omega_{kr} E_{kr})(I + \sum_{r<n} t_{rn} E_{rn}) = (\sum_{k\leqslant r} \omega_{kr} E_{kr})(\sum_{r\leqslant n} t_{rn} E_{rn})$$

$$= (\sum_{k\leqslant n} (\sum_{r=k}^{n} \omega_{kr} t_{rn}) E_{kn}).$$

Since $I = \sum_{k \leqslant n} \delta_{kn} E_{kn}$,

$$\sum_{r=k}^{n} \omega_{kr} t_{rn} = \delta_{kn}, \quad k \leqslant n,\ k,\ n \in N, \tag{4.2}$$

or

$$\omega_{kn} = -\sum_{r=k}^{n-1} \omega_{kr} t_{rn}, \quad k < n,\ k,\ n \in N. \tag{4.3}$$

Having applied the correlation (4.3), we get:

$$\omega_{kk+1} = -t_{kk+1}, \quad \omega_{kn} = -t_{kn} + \sum_{r=1}^{n-k-1} (-1)^{r+1} \sum_{k<n_1<n_2<\ldots<n_r<n} t_{kn_1} t_{n_1 n_2} \ldots t_{n_r n},$$
$$n > k+1. \tag{4.4}$$

Now let $(U_n(t))_{n \in N}$ be an arbitrary family of strongly continuous one parameter groups of the unitary operators in a Hilbert space $\mathscr{H}$. Define on $\boldsymbol{R}_0^\infty$ (see Section 3) an operator valued function $\boldsymbol{R}_0^\infty \ni t \mapsto U(t) = U_1(t_1) \ldots U_n(t_n) \in U(\mathscr{H})$, here $t = (t_1, \ldots, t_n, 0, \ldots) \in \boldsymbol{R}_0^\infty$. The function $U(t)$ is continuous on $\boldsymbol{R}_0^\infty$. For $t = (t_1, \ldots, t_n, t_{n+1}, \ldots) \in \boldsymbol{R}^\infty$ define $t^{(n)} = (t_1, \ldots, t_n, 0, \ldots) \in \boldsymbol{R}_0^\infty$ and denote $U(t^{(0)}) = I$. Fix the dense countable set $\mathscr{F} = \{f_m\}_{m \in N} \subset \mathscr{H}$, denote

$$\gamma_{1,m}(t) = \sum_{n=0}^{\infty} \|(U(t^{n+1)}) - U(t^{(n)})) f_m\|, \tag{4.5}$$

$$\gamma_{2,m}(t) = \sum_{n=0}^{\infty} \|(U^{-2}(t^{(n+1)}) - U^{-1}(t^{(n)})) f_m\| \tag{4.6}$$

and define the topological space $\boldsymbol{R}(U) = \{t \in \boldsymbol{R}^\infty | \gamma_{k,m}(t) < \infty,\ k = 1, 2;\ m \in N\}$ in which for the base of the topology we set the family of finite intersections of subsets of $B = \{W(t, k, m, \varepsilon) = \{s \in \boldsymbol{R}(U) | \gamma_{k,m}(t-s) < \varepsilon\},\ k = 1, 2;\ m \in N,\ \varepsilon > 0\}$.

LEMMA 4.5. *The function $\boldsymbol{R}_0^\infty \ni t \mapsto U(t) \in U(\mathscr{H})$ can be extended by continuity to a continuous at zero function on the topological space $\boldsymbol{R}(U)$ strictly containing $\boldsymbol{R}_0^\infty$*

$$\boldsymbol{R}(U) \ni t \mapsto \tilde{U}(t) \in U(\mathscr{H}).$$

Proof: Consider an arbitrary sequence of unitary operators $(U_n)_{n \in N}$. We show that if for every $f \in \mathscr{H}$

$$\sum_{n \in N} \|(U_{n+1} - U_n) f\| = \sum_{n \in N} \|(U_n^{-1} U_{n+1} - I) f\| < \infty, \tag{4.7}$$

$$\sum_{n \in N} \|(U_{n+1}^{-1} - U_n^{-1}) f\| = \sum_{n \in N} \|(U_n U_{n+1}^{-1} - I) f\| < \infty, \tag{4.8}$$

then there exists an operator $U = s\lim_n U_n \in U(\mathscr{H})$. Indeed, then for every $f \in \mathscr{H}$ the following estimate holds:

$$\|(U_{m+p} - U_m) f\| = \Big\| \sum_{n=m}^{m+p-1} (U_{n+1} - U_n) f \Big\| \leqslant \sum_{n=m}^{m+p-1} \|(U_{n+1} - U_n) f\|,$$

hence the sequence $(U_n f)_{n\in N}$ is fundamental and there exists a vector $Uf = s\lim_n U_n f$. We show that $\mathscr{H} \ni f \mapsto Uf \in \mathscr{H}$ is a linear unitary operator. Linearity is obvious, U is isometric: $\|Uf\| = \lim_n \|U_n f\| = \|f\|$, hence to prove unitarity it suffices to show that $R(U) = \mathscr{H}$, ($R(U)$ is the image of the operator U). In other words, it is necessary to show that for every $g \in \mathscr{H}$ there exists $f \in \mathscr{H}$ such that $Uf = g$. By (4.6) there exists a linear isometric operator $V = s\lim_n U_n^{-1}$. Put $f = Vg = \lim_n U_n^{-1} g$. Since $U_n^{-1} g \to f_0\ Uf = \lim_n U_n f = g$. Simultaneously we have constructed the operator $U^{-1} = V$.

It is enough to demand the fulfilment of the conditions (4.5), (4.6) not for all $f \in \mathscr{H}$ but for the dense countable set $\mathscr{F} = \{f_m\}_{m\in N} \subset \mathscr{H}$ only. Indeed, on the dense set $\mathscr{F}$ the isometric operator U: $Uf_m = \lim_n U_n f_m$ is defined. By linearity and continuity we extend the operator U to the whole space. The same procedure should be repeated with the operator V. As a result we get the unitary operator U.

The previous discussion gives the next sufficient conditions for existence of the limit operator $\tilde{U}(t) = s\lim_n U(t^{(n)})$ and for its unitary: for $t \in \boldsymbol{R}^\infty$ the series (4.5) and (4.6) should converge. Obviously, $\boldsymbol{R}(u)$ strictly contains $\boldsymbol{R}_0^\infty$ and on $\boldsymbol{R}(u)$ the mapping $\boldsymbol{R}(u) \ni t \mapsto \tilde{u}(t) \in U(\mathscr{H})$ is defined.

We prove that the function $\tilde{u}(t)$ is continuous at zero. For $f_m \in \mathscr{F}$ we have

$$\|(\tilde{U}(t) - I) f_m\| \leqslant \|(\tilde{U}(t) - U(t^{(n)})) f_m\| + \|(U(t^{(n)}) - I) f_m\|$$

$$\leqslant \|(U(t) - U(t^{(n)})) f_m\| + \sum_{k=0}^{n-1} \|(U(t^{(k+1)}) - U(t^{(k)})) f_m\|.$$

Passing to the limit as $n \to \infty$ we get $\|(\tilde{U}(t) - I) f_m\| \leqslant \gamma_{1,m}(t)$. For every $f \in \mathscr{H}$, the estimate $\|(\tilde{U}(t) - I) f\| \leqslant \gamma_{1,m}(t) + 2\|f_m - f\|$, which implies continuity of $\tilde{U}(t)$, holds. ■

Remark 4.6. If it is the existence of the strong limit $U = s\lim_n U_n$ that is required, the limit operator U is not necessarily unitary, while being isometric. For example, in the space $\mathscr{H} = l_2$ let us consider an operator of the right-shift: $l_2 \ni a = (a_1, a_2, \ldots) \mapsto Va = (0, a_1, a_2, \ldots) \in l_2$ for which $\|Va\| = \|a\|$ but $R(V) \neq l_2$. Consider the sequence of operators U_n: $l_2 \ni a = (a_1, \ldots, a_{n-1}, a_n, a_{n+1}, \ldots) \mapsto U_n a = (a_n, a_1, \ldots, a_{n-1}, a_{n+1}, \ldots) \in l_2$. Obviously, U_n are unitary operators and

$V = s\lim_n U_n$. Indeed, for every $a \in l_2$

$$\lim_n \|(U_n - V)a\|^2 = \lim_n \left(|a_n|^2 + \sum_{k=n}^{\infty} |a_{k+1} - a_k|^2\right) = 0. \qquad \blacksquare$$

Let us return to the group $B(N, R)$. Let $B^{(\infty)}$ be the group of all upper triangular matrices of infinite order with identities on the principal diagonal. Denote for $t = I + \sum_{k<n} t_{kn} E_{kn} \in B^{\infty}$ by $t^{(n)}$, t_n, t_{kn}, $k < n$, $k, n \in N$ respectively the matrices:

$$t^{(n)} = I + \sum_{k<r\leq n} t_{kr} E_{kr},\ t_n = I + \sum_{k=1}^{n-1} t_{kn} E_{kn},\ t_{(kn)} = I + t_{kn} E_{kn},\ t_1 = I.$$

Then strict calculation gives

$$t^{(n)} = t_n t_{n-1} \dots t_2 \cdot t_1, \qquad n \in N. \tag{4.9}$$

Note that the set of all matrices $t_n(t_{(kn)})$ if n is fixed ((k, n) respectively) forms the group isomorphic to the additive group R^{n-1} (R^1 respectively).

Denote by $U_n(t_n) = U(t_n)$, $U_{kn}(t_{(kn)}) = U(t_{(kn)})$, A_{kn} the generators of the one parameter group $U_{kn}(t_{(kn)})$. It is known [12] that for every strongly continuous unitary representation U of inductive limit of connected finite dimensional Lie groups $G = \lim_{\vec{n}} G_n$ there exists a dense set $\mathscr{H}^{\infty}(U)$ of smooth vectors. We have $B(N, R) = \lim_{\vec{n}} B(n, R)$, where $B(n, R)$ is the connected group of all upper triangular $n \times n$ matrices with identities on the principal diagonal, hence $\mathscr{H}^{\infty}(U) \subset \mathscr{H}$ is a dense set. Choose the dense set $\mathscr{F} = \{f_m\}_{m\in N}$, consisting of smooth vectors. To prove the theorem it suffices to find the weight $a = (a_{kn})_{k<n}$ such that the series (4.5) and (4.6) converge for $t \in B_2(a)$.

Let $t \in B^{\infty}$, then conditions (4.5), (4.6) take the form

$$\gamma_{1,m}(t) = \sum_{n\in N} \left\|\left(U(t^{(n+1)}) - U(t^{(n)})\right) f_m\right\| = \sum_{n\in N} \left\|\left(U\left((t^{(n)})^{-1} t^{(n+1)}\right) - I\right) f_m\right\|, \tag{4.10}$$

$$\gamma_{2,m}(t) = \sum_{n\in N} \left\|\left(U^{-1}(t^{(n+1)}) - U^{-1}(t^{(n)})\right) f_m\right\| = \sum_{n\in N} \left\|\left(U\left(t^{(n)}(t^{(n+1)})^{-1}\right) - I\right) f_m\right\|$$

$$= \sum_{n\in N} \left\|\left(U\left(t_n \dots t_1 (t_1)^{-1} \dots (t_n)^{-1} (t_{n+1})^{-1}\right) - I\right) f_m\right\| = \sum_{n\in N} \left\|\left(U\left((t_{n+1})^{-1}\right) - I\right) f_m\right\|. \tag{4.11}$$

Using (4.2), calculate

$$(t^{(n)})^{-1} t^{(n+1)} = \left((t^{(n+1)})^{-1} t^{(n)}\right)^{-1}$$

$$= \left(\left(I + \sum_{k<p\leq n+1} \omega_{kp} E_{kp}\right)\left(I + \sum_{p<m\leq n} t_{pm} E_{pm}\right)\right)^{-1} = \left(I + \sum_{k<m\leq n} \left(\sum_{p=k}^{m} \omega_{kp} t_{pm}\right)\right.$$

$$+\sum_{k=1}^{n}\omega_{kn+1}E_{kn+1}\Big)^{-1}=\Big(I+\sum_{k<m\leqslant n}\delta_{km}E_{km}+\sum_{k=1}^{n}\omega_{kn+1}E_{kn+1}\Big)^{-1}$$

$$=\Big(I-\sum_{k=1}^{n}\omega_{kn+1}E_{kn+1}\Big)=(\omega_{n+1})^{-1}. \tag{4.12}$$

Let $U(t)$ be a one parameter group of unitary operators, A its generator, E the spectral measure of the operator A. Then the inequality holds

$$\|(U(t)-I)f\|\leqslant|t|\,\|Af\|,\qquad t\in\boldsymbol{R}^1, f\in\mathscr{D}(A). \tag{4.13}$$

($\mathscr{D}(A)$ is the domain of the operator A).
Indeed,

$$\|(U(t)-I)f\|^2=\int|\exp i(tx)-1|^2\,d(E(x)f,f)$$
$$=\int 4\sin^2(tx/2)\,d(E(x)f,f)\leqslant\int|tx|^2\,d(E(x)f,f)=\|tAf\|^2.$$

Choose now the weight $a=(a_{kn})_{k<n}$ such that conditions (4.1) hold and

$$B_m^2=\sum_{k<n}\|A_{kn}f_m\|^2a_{kn}^{-1}<\infty,\qquad m\in\boldsymbol{N}. \tag{4.14}$$

Then by (4.12)–(4.14) we get

$$\gamma_{2,m}(t)=\sum_{n\in N}\|(U((t_{n+1})^{-1})-I)f_m\|\leqslant\sum_{n=2}^{\infty}\Big\|\Big(\sum_{k=1}^{n-1}t_{kn}A_{kn}\Big)f_m\Big\|$$
$$\leqslant\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}|t_{kn}|\,\|A_{kn}f_m\|\leqslant\Big(\sum_{k<n}|t_{kn}|^2a_{kn}\Big)^{1/2}\Big(\sum_{k<n}\|A_{kn}f_m\|^2a_{kn}^{-1}\Big)^{1/2}$$
$$=\|t-I\|_{b_2(a)}\cdot B_m<\infty,$$

$$\gamma_{1,m}(t)=\sum_{n\in N}\|(U((t^{(n)})^{-1}t^{(n+1)})-I)f_m\|$$
$$\leqslant\sum_{n=2}^{\infty}\Big\|\Big(\sum_{k=1}^{n-1}\omega_{kn}A_{kn}\Big)f_m\Big\|\leqslant\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}|\omega_{kn}|\,\|A_{kn}f_m\|$$
$$\leqslant\Big(\sum_{k<n}|\omega_{kn}|^2a_{kn}\Big)^{1/2}\Big(\sum_{k<n}\|A_{kn}f_m\|^2a_{kn}^{-1}\Big)^{1/2}=\|t^{-1}-I\|_{b_2(a)}\cdot B_m<\infty,$$

hence the representation can be extended to a unitary representation of the group $B_2(a)$. By Lemma 4.5 this representation is continuous at the identity, hence it is continuous everywhere.

The existence of the weight with the property (4.14) results from the following considerations. Let us take the infinite matrix of positive numbers $(b_{mp})_{m,p\in N}$. Then there exists a sequence of positive numbers $(a_p)_{p\in N}$ such that $\sum_{p\in N}b_{mp}a_p^{-1}<\infty$,

$m \in N$. Indeed, define $a_p = p^2 \max_{k \leqslant p} b_{kp}$, then

$$\sum_{p=m}^{\infty} b_{mp} a_p^{-1} \leqslant \sum_{p=m}^{\infty} p^{-2} < \infty, \qquad m \in N.$$

In our case we put $a_{kn} = (kn)^2 \max_{m<k+n} \|A_{kn} f_m\|^2$, $k < n$, then

$$\sum_{\substack{k<n \\ k+n>m}} \|A_{kn} f_m\|^2 a_{kn}^{-1} \leqslant \sum_{\substack{k<n \\ k+n>m}} (kn)^{-2} < \infty.$$

If conditions (4.1) do not hold we define $\bar{a}_{kn} = a_k \cdot a_n$, $k < n$, where $a_k = \max_{p<q\leqslant k} \{1, a_{pq}\}$, $k \in N$ and use Remark 2.5.

We show that $B_2(a) \ni t \mapsto U_2(t) \in U(\mathscr{H})$ is a representation. Indeed let $t, s \in B_2(a)$, then $t^{(n)} \to t$, $s^{(n)} \to s$ and $(ts)^{(n)} = t^{(n)} s^{(n)} \to ts$, $n \to \infty$ in $B_2(a)$. Hence

$$U_2(ts) = s\lim_n U\big((ts)^{(n)}\big) = s\lim_n U(t^{(n)} s^{(n)})$$

$$= s\lim_n U(t^{(n)})\, U(s^{(n)}) = U_2(t)\, U_2(s). \qquad \blacksquare$$

5. Extension of the representation of the group $B(\boldsymbol{Z}, \boldsymbol{R})$

Let $B(\boldsymbol{Z}, \boldsymbol{R})$ be group of finite upper triangular matrices of infinite order in both directions with identities on the principal diagonal, $\mathfrak{b}(\boldsymbol{Z}, \boldsymbol{R})$ its Lie algebra. We define the Hilbert–Lie group for the weight $a = (a_{kn})_{(k,n)\in \boldsymbol{Z}_+^2}$, where $\boldsymbol{Z}_+^2 = \{(k, n) \in \boldsymbol{Z}^2 \mid k < n\}$, analogously to $\mathrm{GL}_2(a)$, $a \in \mathfrak{A}$.

Conditions (2.1) in this case become

$$a_{kn} \leqslant c^2 a_{km} a_{mn}; \quad k, n, m \in \boldsymbol{Z},\ k < m < n,\ c > 0.$$

THEOREM 5.1. *Every unitary representation $U\colon B(\boldsymbol{Z}, \boldsymbol{R}) \to U(\mathscr{H})$ of the group $B(\boldsymbol{Z}, \boldsymbol{R})$ can be extended by continuity to a unitary representation $U_2\colon B_2(a) \to U(\mathscr{H})$ of some Hilbert–Lie group depending on the representation.*

The proof is analogous to the proof of Theorem 4.4. We note that elements of group $B(\boldsymbol{Z}, \boldsymbol{R})$, respectively $\mathfrak{b}(\boldsymbol{Z}, \boldsymbol{R})$, are finite matrices:

$$t = I + \sum_{(k,m)\in \boldsymbol{Z}_+^2} t_{km} E_{km}, \qquad x = \sum_{(k,m)\in \boldsymbol{Z}_+^2} x_{km} E_{km}. \tag{5.0}$$

Denote by $B^{\infty} = B^{\infty}(\boldsymbol{Z}, \boldsymbol{R})$ $\big(\mathfrak{b}^{\infty} = \mathfrak{b}^{\infty}(\boldsymbol{Z}, \boldsymbol{R})\big)$ the group (algebra) of all upper triangular matrices of infinite order in both directions with identities on the principal diagonal. Arbitrary matrices of the type (5.0) are elements of the group B^{∞} (respectively of the algebra $\mathfrak{b}^{\infty}$).

For every $t \in B^{\infty}$ denote by $t^{(n)} \in B(2n+1, \boldsymbol{R})$ the element

$$t^{(n)} = I + \sum_{-n \leqslant k,m \leqslant n} t_{km} E_{km}.$$

We will show that for a fixed representation $U\colon B(\boldsymbol{Z}, \boldsymbol{R}) \to U(\mathcal{H})$ we can find the weight a such that for $t \in B_2(a)$ there exists the limit

$$U_2(t) = s \lim_n U(t^{(n)}) \in U(\mathcal{H}) \tag{5.1}$$

being a unitary operator. Moreover, (5.1) gives a representation of the group $B_2(a)$, $U_2\colon B_2(a) \to U(\mathcal{H})$.

Choose the dense set $\mathcal{F} = \{f_m\}_{m \in N} \subset \mathcal{H}$ consisting of smooth vectors for the representation U. By the proof of Theorem 4.3 the limit (5.1) exists if for $t \in B_2(a)$ the series converge:

$$\gamma_{1,m}(t) = \sum_{n \in N} \left\|\left(U(t^{(n+1)}) - U(t^{(n)})\right) f_m\right\| = \sum_{n \in N} \left\|\left(U\left((t^{(n)})^{-1} t^{(n+1)}\right) - I\right) f_m\right\| < \infty,$$

$$\gamma_{2,m}(t) = \sum_{n \in N} \left\|\left(U^{-1}(t^{(n+1)}) - U^{-1}(t^{(n)})\right) f_m\right\| = \sum_{n \in N} \left\|\left(U\left(t^{(n)}(t^{(n+1)})^{-1}\right) - I\right) f_m\right\| < \infty.$$

For $\quad t = I + \sum_{(k,m) \in \boldsymbol{Z}_+^2} t_{km} E_{km} \in B^{\infty} \quad$ denote $\quad \omega = t^{-1} = I + \sum_{(k,m) \in \boldsymbol{Z}_+^2} \omega_{km} E_{km} \in B^{\infty}$.

Obviously, for ω_{km}, $(k, m) \in \boldsymbol{Z}_+^2$ holds

$$\sum_{r=k}^{n} \omega_{kr} t_{rn} = \delta_{kn}, \qquad k, n \in \boldsymbol{Z}, k \leqslant n.$$

$$\omega_{kk+1} = -t_{kk+1}, \qquad \omega_{kn} = -t_{kn} + \sum_{r=1}^{n-k-1} (-1)^{r+1} \sum_{k < n_1 < n_2 < \dots < n_r < n} t_{kn_1} t_{n_1 n_2} \dots t_{n_r n}, \quad n > k+1 \tag{5.2}$$

(see (4.4)). By (5.2) we have

$$(t^{-1})^{(n)} = (t^{(n)})^{-1}, \qquad t \in B^{\infty}. \tag{5.3}$$

For $t = I + \sum_{(k,m) \in \boldsymbol{Z}_+^2} t_{km} E_{km} \in B^{\infty}$ denote also

$$t_{\rightarrow}^{(n+1)} = I + \sum_{-(n+1) < m \leqslant (n+1)} t_{-(n+1)m} E_{-(n+1)m},$$

$$t_{\downarrow}^{(n+1)} = I + \sum_{-(n+1) < k < (n+1)} t_{k(n+1)} E_{k(n+1)}.$$

Then strict calculation gives us

$$t^{(n+1)} = t_{\downarrow}^{(n+1)} t^{(n)} t_{\rightarrow}^{(n+1)} = t_{\downarrow}^{(n+1)} t_{\downarrow}^{(n)} \dots t_{\downarrow}^{(1)} t_{\rightarrow}^{(1)} \dots t_{\rightarrow}^{(n)} t_{\rightarrow}^{(n+1)}. \tag{5.4}$$

We note that the equality (4.12) can be rewritten in the form

$$(t^{(n)})^{-1}\, t^{(n+1)}_{\downarrow}\, t^{(n)} = (\omega^{(n+1)}_{\downarrow})^{-1} \tag{5.5}$$

and the equality used in (4.11) can be presented in the form

$$t^{(n)}\,\omega^{(n+1)}_{\downarrow}\,\omega^{(n)} = (t^{(n+1)}_{\downarrow})^{-1}. \tag{5.6}$$

By (5.3)–(5.6) we have

$$(t^{(n)})^{-1}\, t^{(n+1)} = (t^{-1})^{(n)}\, t^{(n+1)}_{\downarrow}\, t^{(n)}\, t^{(n+1)}_{\rightarrow} = \omega^{(n)}\, t^{(n+1)}_{\downarrow}\, t^{(n)}\, t^{(n+1)}_{\rightarrow} = (\omega^{(n+1)}_{\downarrow})^{-1}\, t^{(n+1)}. \tag{5.7}$$

$$\begin{aligned} t^{(n)}\,(t^{(n+1)})^{-1} &= t^{(n)}\,(t^{-1})^{(n+1)} = t^{(n)}\,\omega^{(n+1)} \\ &= t^{(n)}\,\omega^{(n+1)}_{\downarrow}\,\omega^{(n)}\,\omega^{(n+1)}_{\rightarrow} = (t^{(n+1)}_{\downarrow})^{-1}\,\omega^{(n+1)}_{\rightarrow}. \end{aligned} \tag{5.8}$$

Denote

$$Z^2_{\rightarrow} = \{(k, m) \in Z^2_+ \,|\, k+m \leqslant 0\},$$
$$Z^2_{\downarrow} = \{(k, m) \in Z^2_+ \,|\, k+m > 0\}.$$

Obviously, $Z^2_{\rightarrow} \cup Z^2_{\downarrow} = Z^2_+$, $Z^2_{\rightarrow} \cap Z^2_{\downarrow} = \emptyset$. By (5.7)–(5.8) we have

$$\begin{aligned} \gamma_{1,m}(t) &= \sum_{n\in N} \left\|\left(U\left((t^{(n)})^{-1}\, t^{(n+1)}\right) - I\right) f_m\right\| = \sum_{n\in N} \left\|\left(U\left((\omega^{(n+1)}_{\downarrow})^{-1}\, t^{(n+1)}_{\rightarrow}\right) - I\right) f_m\right\| \\ &= \sum_{n\in N} \left\|\left(U\left((\omega^{(n+1)}_{\downarrow})^{-1}\, t^{(n+1)}\right) - U\left((\omega^{(n+1)}_{\downarrow})^{-1}\right) + U\left((\omega^{(n+1)}_{\downarrow})^{-1}\right) - I\right) f_m\right\| \\ &\leqslant \sum_{n\in N} \left\|\left(U\left(t^{(n+1)}\right) - I\right) f_m\right\| + \sum_{n\in N} \left\|\left(U\left((\omega^{(n+1)}_{\downarrow})^{-1}\right) - I\right) f_m\right\| \\ &\leqslant \sum_{n\in N} \Big\|\Big(\sum_{-(n+1)<k\leqslant n+1} t_{-(n+1)k}\, A_{-(n+1)k}\Big) f_m\Big\| + \sum_{n\in N} \Big\|\Big(-\sum_{-(n+1)<k<n+1} \omega_{kn+1}\, A_{kn+1}\Big) f_m\Big\| \\ &\leqslant \sum_{(k,n)\in Z^2_{\rightarrow}} |t_{kn}|\, \|A_{kn} f_m\| + \sum_{(k,n)\in Z^2_{\downarrow}} |\omega_{kn}|\, \|A_{kn} f_m\| < \infty. \end{aligned} \tag{5.9}$$

Analogously,

$$\gamma_{2,m}(t) \leqslant \sum_{(k,n)\in Z^2_{\rightarrow}} |\omega_{kn}|\, \|A_{kn} f_m\| + \sum_{(k,n)\in Z^2_{\downarrow}} |t_{kn}|\, \|A_{kn} f_m\| < \infty. \tag{5.10}$$

Obviously, conditions (5.9)–(5.10) are equivalent to the following:

$$\alpha_{1,m}(t) = \sum_{(k,n)\in Z^2_+} |t_{kn}|\, \|A_{kn} f_m\| < \infty,$$

$$\alpha_{2,m}(t) = \sum_{(k,n)\in Z^2_+} |\omega_{kn}|\, \|A_{kn} f_m\| < \infty,$$

and the last inequalities are valid by the estimates

$$\sum_{(k,n)\in Z_+^2} |t_{kn}|\,\|A_{kn} f_m\| \leqslant \Big(\sum_{(k,n)\in Z_+^2} |t_{kn}|^2 a_{kn}\Big)^{1/2}\Big(\sum_{(k,n)\in Z_+^2} \|A_{kn} f_m\|^2 a_{kn}^{-1}\Big)^{1/2}$$
$$= \|t-I\|_{\mathfrak{b}_2(a)}\cdot B_m < \infty,$$

$$\sum_{(k,n)\in Z_+^2} |\omega_{kn}|\,\|A_{kn} f_m\| \leqslant \Big(\sum_{(k,n)\in Z_+^2} |\omega_{kn}|^2 a_{kn}\Big)^{1/2}\Big(\sum_{(k,n)\in Z_+^2} \|A_{kn} f_m\|^2 a_{kn}^{-1}\Big)^{1/2}$$
$$= \|t^{-1}-I\|_{\mathfrak{b}_2(a)}\cdot B_m < \infty,$$

where the weight $a = (a_{kn})_{(k,n)\in Z_+^2}$ is chosen so that

$$B_m^2 = \sum_{(k,n)\in Z_+^2} \|A_{kn} f_m\|^2 a_{kn}^{-1} < \infty, \qquad m\in N.$$

and the space $\mathfrak{b}_2(a)$ is a Hilbert–Lie algebra. By Remark 2.5 it can be always chosen.

6. Extension of the representation of the group $\mathrm{GL}(Z, R)$

THEOREM 6.1. *Every unitary representation of the group* $\mathrm{GL}(Z, R)$ *can be extended by continuity to a unitary representation* U_2: $\mathrm{GL}_2(a)\to U(\mathcal{H})$ *of some Hilbert–Lie group* $\mathrm{GL}_2(a)$ *depending on the representation.*

The proof is based on Theorem 5.2 and on the analogy of Gauss decomposition for the group $\mathrm{GL}_2(a)$ (see also [18]). Let $D_2^+(a)$, $B_2(a)$, $B_2^*(a)$ be subgroups of $\mathrm{GL}_2(a)$ consisting respectively of diagonal matrices with positive elements, upper triangular and lower triangular matrices with units on the principal diagonal.

LEMMA 6.2. *There exists neighbourhood of the identity of the group* $W\subset \mathrm{GL}_2(a)$ *where the decomposition holds*:

$$t = t_1 t_2 t_3, \quad t\in W, \quad t_1\in D_2^+(a), \quad t_2\in B_2^*(a), \quad t_3\in B_2(a).$$

Proof: Let $\exp\colon A\to A^*$ be the exponential map of the algebra $A = \{(\lambda, x)\,|\,\lambda\in R^1,\ x\in \mathfrak{gl}_2(a)\}$ into the Lie group A^* [13, Chapter III]. Since A is a complete normed algebra (relative to multiplication $(x, y)\mapsto xy$) with identity, therefore the exponential mapping has form: $A\ni x\mapsto \exp x = \sum_{n=0}^{\infty} x^n/n!\in A^*$ [13, Chapter III].

Consider Lie subalgebras of the Lie algebra $\mathfrak{gl}_2(a)$:

$$\mathfrak{D}_2(a) = \{x = \sum_{k\in Z} x_{kk} E_{kk}\,|\,\|x\|^2_{\mathfrak{D}_2(a)} = \sum_{k\in Z} |x_{kk}|^2 a_{kk} < \infty\},$$

$$\mathfrak{b}_2^*(a) = \{x = \sum_{(k,n)\in \boldsymbol{Z}_-^2} x_{kn} E_{kn} | \|x\|^2_{\mathfrak{b}^*_2(a)} = \sum_{(k,n)\in \boldsymbol{Z}_+^2} |x_{kn}|^2 a_{kn} < \infty\},$$

$$\mathfrak{b}_2(a) = \{x = \sum_{(k,n)\in \boldsymbol{Z}_+^2} x_{kn} E_{kn} | \|x\|^2_{\mathfrak{b}_2(a)} = \sum_{(k,n)\in \boldsymbol{Z}_+^2} |x_{kn}|^2 a_{kn} < \infty\},$$

where $\boldsymbol{Z}_-^2 = \{(k, n) \in \boldsymbol{Z}^2 | k > n\}$.

Obviously, $\mathfrak{gl}_2(a) = \mathfrak{D}_2(a) \oplus \mathfrak{b}_2^*(a) \oplus \mathfrak{b}_2(a)$ is the direct sum of vector subspaces. Let $x = (x_1, x_2, x_3)$ be the corresponding decomposition of the element $x \in \mathfrak{gl}_2(a)$.

Consider the corresponding subgroups of the group $\mathrm{GL}_2(a)$:

$$D_2^+(a) = \{\exp x \in \mathrm{GL}_2(a) | x \in \mathfrak{D}_2(a)\},$$

$$B_2^*(a) = \{I + x \in \mathrm{GL}_2(a) | x \in \mathfrak{b}_2^*(a)\},$$

$$B_2(a) = \{I + x \in \mathrm{GL}_2(a) | x \in \mathfrak{b}_2(a)\}.$$

We will show that there exists a neighbourhood of the identity in $\mathrm{GL}_2(a)$ where the decomposition holds

$$t = t_1 \cdot t_2 \cdot t_3, \quad t \in W, \quad t_1 \in D_2^+(a), \quad t_2 \in B_2^*(a), \quad t_3 \in B_2(a).$$

Denote by θ: $\mathfrak{gl}_2(a) \to \mathrm{GL}_2(a)$ the mapping

$$\mathfrak{gl}_2(a) \ni x = (x_1, x_2, x_3) \mapsto \theta(x) = \exp x_1 \exp x_2 \exp x_3 \in \mathrm{GL}_2(a).$$

Restriction of the mapping θ to a sufficiently small neighbourhood of zero $V \subset \mathfrak{gl}_2(a)$ has an open image $W \subset \mathrm{GL}_2(a)$ and is an isomorphism of the manifold V on the manifold $W = \theta(V)$ [13, Chapter III]. On W the inverse mapping θ^{-1}: $W \to V$ is defined. Then for every element $t \in W$ the decomposition holds $t = t_1 \cdot t_2 \cdot t_3$, where $t_i = \theta((\theta^{-1}(t))_i)$, $i = 1, 2, 3$. ■

Let us now take the unitary representation U: $\mathrm{GL}(\boldsymbol{Z}, \boldsymbol{R}) \to U(\mathscr{H})$ of group $\mathrm{GL}(, \boldsymbol{R})$. We denote by A the generators of the one parameter groups $U(I + tE_{kn})$, $k, n \in \boldsymbol{Z}$, $k \neq n$, $U(I + (\exp t - 1) E_{nn})$, $n \in \boldsymbol{N}$, $t \in \boldsymbol{R}^1$.

Let $U^{(1)}$, $U^{(2)}$, $U^{(3)}$ be restrictions of the representation U on subgroups $D^+(\boldsymbol{Z}, \boldsymbol{R})$, $B^*(\boldsymbol{Z}, \boldsymbol{R})$, $B(\boldsymbol{Z}, \boldsymbol{R})$ respectively, consisting of diagonal matrices with positive elements, lower triangular and upper triangular.

We consider the algebra $\mathfrak{D}(\boldsymbol{Z}, \boldsymbol{R}) = \varinjlim_n \mathfrak{D}(2n+1, \boldsymbol{R})$ consisting of finite diagonal matrices, with additive group isomorphic to $\boldsymbol{R}_0^\infty \simeq \boldsymbol{R}_0^{\boldsymbol{Z}}$. Since the group $D^+(\boldsymbol{Z}, \boldsymbol{R})$ is isomorphic to the additive group of the algebra $\mathfrak{D}(\boldsymbol{Z}, \boldsymbol{R})$:

$$\mathfrak{D}(\boldsymbol{Z}, \boldsymbol{R}) \ni x = \sum_{k \in \boldsymbol{Z}} x_{kk} E_{kk} \mapsto \exp x = \sum_{k \in \boldsymbol{Z}} \exp x_{kk} E_{kk} \in D^+(\boldsymbol{Z}, \boldsymbol{R}),$$

there exists a matrix $a^{(1)} = (a_{kk}^{(1)})_{k\in \boldsymbol{Z}}$ such that the representation $U^{(1)}\colon D^+(\boldsymbol{Z}, \boldsymbol{R}) \to U(\mathscr{H})$ can be extended to a strongly continuous unitary representation $U_2^{(1)}\colon D_2^+(a^{(1)}) \to U(\mathscr{H})$ of the Hilbert–Lie group $D_2^+(a^{(1)})$ (Theorem 3.1).

By Theorem 5.1 there exists a weight $a^{(3)} = (a_{kn}^{(3)})_{(k,n)\in \boldsymbol{Z}_+^2}$ such that the representation $U^{(3)}\colon B(\boldsymbol{Z}, \boldsymbol{R}) \to U(\mathscr{H})$ can be extended to a unitary representation $U_2^{(3)}\colon B_2(a^{(3)}) \to U(\mathscr{H})$ of the Hilbert–Lie group $B_2(a^{(3)})$. Since the group $B^*(\boldsymbol{Z}, \boldsymbol{R})$ is obtained from the group $B(\boldsymbol{Z}, \boldsymbol{R})$ by means of transposition, a theorem analogous to Theorem 5.1 holds, i.e. there exists a weight $a^{(2)} = (a_{kn}^{(2)})$ $(k, n) \in \boldsymbol{Z}_-^2$, such that the representation $U^{(2)}\colon B^*(\boldsymbol{Z}, \boldsymbol{R}) \to U(\mathscr{H})$ can be extended to a unitary representation $U_2^{(2)}\colon B_2^*(a^{(2)}) \to U(\mathscr{H})$ of the Hilbert–Lie group $B_2^*(a^{(2)})$.

We now define the weight

$$\tilde{a}_{kn} = \begin{cases} a_{kk}^{(1)}, & k = n, \\ a_{kn}^{(2)}, & k > n, \\ a_{kn}^{(3)}, & k < n. \end{cases}$$

If for the weight $\tilde{a}$ conditions (2.1) hold, then $\mathrm{GL}_2(\tilde{a})$ is the desired Hilbert–Lie group, i.e. the representation $U\colon \mathrm{GL}(\boldsymbol{Z}, \boldsymbol{R}) \to U(\mathscr{H})$ can be extended to a representation of $\mathrm{GL}_2(a)$. Otherwise, by Remark 2.5 there exists an appropriate larger weight $a = (a_{kn})_{k,n\in \boldsymbol{Z}}$, $a_{kn} \geqslant \tilde{a}_{kn}$, $k, n \in \boldsymbol{Z}$. Obviously, in this case $D_2^+(a) \subset D_2^+(a^{(1)})$, $B_2^*(a) \subset B_2^*(a^{(2)})$, $B_2(a) \subset B_2(a^{(3)})$, hence the representations are defined by $U_2^{(1)}\colon D_2^+(a) \to U(\mathscr{H})$, $U_2^{(2)}\colon B_2^*(a) \to U(\mathscr{H})$, $U_2^{(3)}\colon B_2(a) \to U(\mathscr{H})$. In the neighbourhood of identity W of the group $\mathrm{GL}_2(a)$ we define the continuous mapping

$$\mathrm{GL}_2(a) \supset W \ni t = t_1 t_2 t_3 \mapsto U_2(t) = U_2^{(1)}(t_1)\, U_2^{(2)}(t_2)\, U_2^{(3)}(t_3) \in U(\mathscr{H}).$$

Continuity $U_2\colon W \to U(\mathscr{H})$ follows from the continuity of representations $U_2^{(1)}$, $U_2^{(2)}$, $U_2^{(3)}$, of the mappings $\theta\colon V \to W$, $\theta^{-1}\colon W \to V$ and of projections $\mathfrak{gl}_2(a) \ni x \mapsto x_1 \in \mathfrak{D}_2(a)$, $\mathfrak{gl}_2(a) \ni x \mapsto x_2 \in \mathfrak{b}_2^*(a)$, $\mathfrak{gl}_2(a) \ni x \mapsto x_3 \in \mathfrak{b}_2(a)$. For $t = \sum_{k,m\in \boldsymbol{Z}} t_{km} E_{km} \in \mathrm{GL}_2(a)$ we denote by

$$t^{(n)} = I + \sum_{-n\leqslant k,m\leqslant n} t_{km} E_{km} \in \mathrm{GL}(\boldsymbol{Z}, \boldsymbol{R}) \subset \mathrm{GL}_2(a),\ n \in \boldsymbol{N}.$$

Obviously, $\lim_n t^{(n)} = t$ and $\lim_n t^{(n)} s^{(n)} = ts = \lim_n (ts)^{(n)}$ in $\mathrm{GL}_2(a)$. Since $U_2\colon W \to U(\mathscr{H})$ is continuous and $\lim_n t^{(n)} = t = t_1 t_2 t_3 = \lim_n t_1^{(n)} t_2^{(n)} t_3^{(n)}$ in $\mathrm{GL}_2(a)$ for $t \in W$, therefore

$$\begin{aligned} U_2(t) &= U_2^{(1)}(t_1)\, U_2^{(2)}(t_2)\, U_2^{(3)}(t_3) = s\lim_n U(t_1^{(n)})\, U(t_2^{(n)})\, U(t_3^{(n)}) \\ &= s\lim_n U(t_1^{(n)} t_2^{(n)} t_3^{(n)}) = s\lim_n U(t^{(n)}), \end{aligned}$$

hence the mapping can be extended to the whole group $GL_2(a)$

$$GL_2(a) \ni t \mapsto U_2(t) = s\lim_n U(t^{(n)}) \in U(\mathscr{H}). \tag{6.1}$$

Formula (6.1) gives a strongly continuous unitary representation of the group. ■

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