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Irreducibility Criterion for Quasiregular Representations of the Group of Finite Upper Triangular Matrices

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Abstract. An analog of the quasiregular representation is defined for the group of infinite-order finite upper triangular matrices. It uses G-quasi-invariant measures on some G-spaces. The criterion for the irreducibility and equivalence of the constructed representations is given. This criterion allows us to generalize Ismagilov's conjecture on the irreducibility of an analog of regular representations of infinite-dimensional groups.

Key words: Ismagilov's conjecture, quasiregular representation, infinite-dimensional group.

1. Let $G = B_0^{\mathbb{N}}$ be the group of matrices of the form $I + x$, where x is an infinite-order finite upper triangular matrix, let $\widetilde{G} = B^{\mathbb{N}} = \{I + x = I + \sum_{1\leqslant k<n} x_{kn}E_{kn}\}$ be the group of arbitrary upper triangular matrices, where E_{kn}, $k, n \in \mathbb{N}$, are infinite-order matrix units, and let

$$\mu = \mu_{(b,0)} = \bigotimes_{k<n} (b_{kn}/\pi)^{1/2} \exp(-b_{kn}x_{kn}^2)\, dx_{kn}$$

be a centered Gaussian product measure on $B^{\mathbb{N}}$. Let $R_t(x) = xt^{-1}$ and $L_s(x) = sx$ be the right and the left actions of the group G on $\widetilde{G}$ and let $\mu^{L_s}(\,\cdot\,) = \mu(L_s^{-1}(\,\cdot\,))$. We define an analog of the right regular representation $T^{R,\mu}$, $G \ni t \mapsto T_t^{R,\mu} \in U(L^2(\widetilde{G}, d\mu))$, of the group G by the formula $(T_t^{R,\mu}f)(x) = (d\mu(xt)/d\mu(x))^{1/2}f(xt)$, $f \in L^2(\widetilde{G}, d\mu)$. In 1985, R. S. Ismagilov put forward the following conjecture.

Conjecture 1. *The analog of the right regular representation*

$$T^{R,\mu}\colon G \to U(L^2(\widetilde{G}, d\mu))$$

of the group G is irreducible if and only if
1) $\mu^{L_s} \perp \mu$ *for all* $s \in G \setminus \{e\}$,
2) *the measure μ is G-right-ergodic.*

It is clear that Ismagilov's conjecture is of interest for an arbitrary infinite-dimensional group G and a G-quasi-invariant measure μ on the topological group $\widetilde{G}$ containing the original group G as a dense subgroup. In the case $G = B_0^{\mathbb{N}}$, $\widetilde{G} = B^{\mathbb{N}}$ and $\mu = \mu_{(b,0)}$, Ismagilov's conjecture was proved by the author in [1, 2]. For arbitrary product measures on the group $B^{\mathbb{N}}$ with some condition on the moments, it was proved in [3]. In the case of the diffeomorphism groups for the interval and the circle, it was proved in [4]. An open question is whether this conjecture holds in the general case.

An analog of the right regular representation for infinite-dimensional groups (current groups) was first defined and studied in [5–8]. For arbitrary infinite-dimensional groups it was defined in [4].

2. Let us consider a G-space X, where G is some group. With every G-space X and an arbitrary G-quasi-invariant measure μ on X, one can associate the unitary representation $G \ni t \mapsto \pi_t^{\alpha,\mu,X} \in U(L^2(X, d\mu))$ of the group G by the formula $(\pi_t^{\alpha,\mu,X}f)(x) = (d\mu(\alpha_{t^{-1}}(x))/d\mu(x))^{1/2}f(\alpha_{t^{-1}}(x))$, $f \in L^2(X, d\mu)$, where $G \ni t \mapsto \alpha_t \in \mathrm{Aut}(X)$ is the action of G in the space X and $\mathrm{Aut}(X)$ is the group of measurable automorphisms of X.

Let us consider the subgroup $\alpha(G)' = \{g \in \mathrm{Aut}(X) \mid \{g, \alpha_t\} = g\alpha_t g^{-1}\alpha_t^{-1} = e$ for all $t\}$ in $\mathrm{Aut}(X)$. Necessary conditions for the irreducibility of the representation $\pi^{\alpha,\mu,X}$ are the following:

1) $\mu^g \perp \mu$ for all $g \in \alpha(G)' \setminus \{e\}$,

2) the measure μ is G-ergodic.

With regard to the description of the commutant of the right regular representation for a locally-compact group (the commutant is generated by operators of the left regular representation [9]) and Conjecture 1, it can be expected that these conditions turn out to be sufficient too.

Conjecture 2. *The representation $\pi^{\alpha,\mu,X} \colon G \to U(L^2(X, d\mu))$ is irreducible if and only if*

1) $\mu^g \perp \mu$ *for all* $g \in \alpha(G)' \setminus \{e\}$,

2) *the measure μ is G-ergodic.*

3. Examples. We will show that Conjecture 2 holds for the group $G = B_0^{\mathbb{N}}$, for some G-spaces $X^{\{p\}}$ that are sets of cosets of the group $\widetilde{G} = B^{\mathbb{N}}$ with respect to some subgroups, and for noncentered Gaussian product measures $\mu^{\{p\}}_{(b,a)} = \bigotimes_{k\in\{p\}} \bigotimes_{n=k+1}^{\infty} \mu_{(b_{kn},a_{kn})}$ on $X^{\{p\}}$. For the case of centered Gaussian product measures $\mu^{\{p\}}_{(b,0)}$, see [10]. The proof is also valid for arbitrary product measures with some conditions on the moments. Conjecture 2 also holds for the group $G = SL_0(\infty, \mathbb{R})$, for some G-spaces that are subspaces of the space $\operatorname{Mat}(\infty, \mathbb{R})$ of real infinite-order matrices, and for arbitrary product measures with some condition on the moments. This problem will be considered in a separate publication.

Let us consider the following subgroups $X^{\{p\}}$ and $X_{\{p\}}$ of the group $B^{\mathbb{N}}$, where $\{p\} = (p_1, p_2, \dots)$ is a finite subset of the set of positive integers:

$$X^{\{p\}} = \left\{ I + x \in B^{\mathbb{N}} \;\middle|\; I + x = I + \sum_{k\in\{p\}} \sum_{n=k+1}^{\infty} x_{kn} E_{kn} \right\},$$

$$X_{\{p\}} = \left\{ I + x \in B^{\mathbb{N}} \;\middle|\; I + x = I + \sum_{k\in\mathbb{N}\setminus\{p\}} \sum_{n=k+1}^{\infty} x_{kn} E_{kn} \right\}.$$

Obviously, the right action of the group $G = B_0^{\mathbb{N}}$ is well defined on the set of left cosets $X_{\{p\}} \setminus B^{\mathbb{N}}$, and we have $X_{\{p\}} \setminus B^{\mathbb{N}} \simeq X^{\{p\}}$. In the case where $X = X^{\{p\}}$, the group $\alpha(G)' \subset \operatorname{Aut}(X)$ obviously contains the image of the following group $B(\{p\}, \mathbb{R})$ with respect to the left action $L\colon B(\{p\}, \mathbb{R}) \to \operatorname{Aut}(X^{\{p\}})$:

$$B(\{p\}, \mathbb{R}) = \left\{ I + x \in B^{\mathbb{N}} \;\middle|\; I + x = I + \sum_{k,n\in\{p\},\, k<n} x_{kn} E_{kn} \right\}.$$

Let us set $B(m, \mathbb{R}) = B((1, \dots, m), \mathbb{R})$.

On the group $X^{\{p\}}$, we define the noncentered Gaussian product measure

$$d\mu^{\{p\}}(x) = d\mu^{\{p\}}_{(b,a)}(x) = \bigotimes_{k\in\{p\}} \bigotimes_{n=k+1}^{\infty} (b_{kn}/\pi)^{1/2} \exp(-b_{kn}(x_{kn} - a_{kn})^2)\, dx_{kn},$$

where $b = (b_{kn})_{k<n}$, $b_{kn} > 0$, and $a = (a_{kn})_{k<n}$, $a_{kn} \in \mathbb{R}^1$. Let us write $T^{R,\mu,\{p\}} = \pi^{R,\mu^{\{p\}},X^{\{p\}}}$. The representations $T^{R,\mu,\{p\}}$ will naturally be called an *analog of quasiregular representations.*

Theorem 1. *The representation $T^{R,\mu,\{p\}} \colon B_0^{\mathbb{N}} \to U(L^2(X^{\{p\}}, d\mu^{\{p\}}_{(b,a)}))$ is irreducible if and only if*

1) $(\mu^{\{p\}}_{(b,a)})^{L_s} \perp \mu^{\{p\}}_{(b,a)}$ *for all* $s \in B(\{p\}, \mathbb{R}) \setminus \{e\}$,

2) *the measure $\mu^{\{p\}}_{(b,a)}$ is $B_0^{\mathbb{N}}$-right-ergodic.*

Remark 1. Condition 2) holds for all product measures. It may not hold for some other measures.

Remark 2. In the case $\{p\} = n$, $n \in \mathbb{N}$, the group $B(\{p\}, \mathbb{R})$ is trivial, and Condition 1) disappears. But the representation $T^{R,\mu,\{p\}}$ is irreducible in this case as well.

Theorem 2. *Two irreducible representations $T^{R,\mu,\{p\}}$ and $T^{R,\mu',\{p'\}}$ are equivalent, $T^{R,\mu,\{p\}} \sim T^{R,\mu',\{p'\}}$, if and only if*

1) $\{p\} = \{p'\}$,
2) $\mu^{\{p\}} \sim (\mu')^{\{p'\}}$.

For the measure $\mu^{(1,\dots,m)}_{(b,a)}$, $m \in \mathbb{N}$, the assertion below holds.

Lemma 1. *The following conditions are equivalent*:

1) $(\mu^{(1,\dots,m)}_{(b,a)})^{L_t} \perp \mu^{(1,\dots,m)}_{(b,a)}$ *for all* $t \in B(m,\mathbb{R}) \setminus \{e\}$,
2) $S^L_{pq}(\mu) = S^{L_0}_{pq}(\mu) + S^{L_1}_{pq}(\mu) = \infty$ *for all* $p < q \leqslant m$, *where*

$$S^{L_0}_{pq}(\mu) = \frac{1}{4}\sum_{n=q+1}^{\infty}\frac{b_{pn}}{b_{qn}}, \qquad S^{L_1}_{pq}(\mu) = \frac{1}{2}\sum_{n=q+1}^{\infty} b_{pn}a^2_{qn}.$$

The idea of the proof of the irreducibility for $\{p\} = (1,\dots,m)$. Let us denote by $\mathfrak{A}^m$ the von Neumann algebra generated by the representation $T^{R,\mu,m} := T^{R,\mu,(1,\dots,m)}$, $\mathfrak{A}^m = (T^{R,\mu,m}_t \mid t \in B^{\mathbb{N}}_0)''$. Moreover, let $\langle f_n \mid n = 1,2,\dots\rangle$ be the closure of the linear space generated by vectors $\{f_n\}_{n=1}^{\infty}$ in a Hilbert space H.

Now let Condition 1) of Theorem 1 hold. By Lemma 1, this is equivalent to the divergence of the series $S^L_{pq}(\mu)$ for all $p < q \leqslant m$. Therefore, using Lemmas 2 and 3 (see below), it is possible to show that the operators of multiplication by the independent variables x_{kn}, $k < n$, $1 \leqslant k \leqslant m$, can be approximated in the strong resolvent sense by the generators $A_{kn} = \frac{d}{dt}T^{R,\mu,m}_{I+tE_{kn}}|_{t=0}$, i.e., that the operator x_{kn} is affiliated to the von Neumann algebra $\mathfrak{A}^m$. This is denoted by $x_{kn}\ \eta\ \mathfrak{A}^m$ in [9]. In this case, an $A \in (\mathfrak{A}^m)'$ must be the operator of multiplication by some essentially bounded function $a\colon X^m \to \mathbb{C}$. The commutation

$[A, T^{R,\mu,m}_t] = 0$ for all $t \in B^{\mathbb{N}}_0$ implies that $a(xt) = a(x)$ for all $t \in B^{\mathbb{N}}_0$. Therefore, by the ergodicity of the measure $\mu^{(1,\dots,m)}_{(b,a)}$ on the space $X^m = X^{(1,\dots,m)}$, we have $A = a = \text{const}$.

Let us set $D_{kn} = \partial/\partial x_{kn} - b_{kn}(x_{kn} - a_{kn})$. The generators A_{kn}, $k < n$, have the form

$$A_{kn} = \sum_{r=1}^{k-1} x_{rk}D_{rn} + D_{kn}, \quad k \leqslant m, \qquad A_{kn} = \sum_{r=1}^{m} x_{rk}D_{rn}, \quad k > m.$$

Let $\Delta_p = \{1,\dots,p\}$. For an arbitrary subset $d_p \subset \Delta_p$, $1 \leqslant p \leqslant m$, we set $A^{d_p}_{pn} = \sum_{r\in d_p} x_{rp}D_{rn}$, $p < n$, $x_{rr} = 1$, $r \in \mathbb{N}$.

Lemma 2. $x_{pq} \in \langle A^{d_p}_{pn}A^{d_q}_{qn}\mathbf{1} \mid q < n\rangle$, $1 \leqslant p \leqslant m$, $p < q$, *if and only if*

$$\Sigma^{d_p,d_q}_{pq} = \sum_{n=q+1}^{\infty}\frac{b^2_{pn}}{\sum_{r\in d_p} b_{rn}\sum_{s\in d_q} b_{sn}} = \infty.$$

Moreover, in this case, we have $x_{pq}\ \eta\ \mathfrak{A}^m$.

Let the inverse element of $I + x = I + \sum x_{kn}E_{kn} \in X^{\{p\}}$ have the form $(I+x)^{-1} = I + \sum x^{-1}_{kn}E_{kn} \in X^{\{p\}}$.

Lemma 3. $x^{-1}_{pr} \in \langle (x^{-1}_{pk} + a_{pk})x_{rk} \mid m < k\rangle$, $p+1 \leqslant r \leqslant m$, *if and only if*

$$\sigma_{pr}(\mu) = \sum_{k=m+1}^{\infty}\frac{\frac{1}{2b_{rk}} + a^2_{rk}}{\frac{1}{2b_{pk}} + \sum_{s=p+1, s\neq r}^{m}\left(\frac{1}{2b_{sk}} + a^2_{sk}\right) + \frac{1}{2b_{rk}}} = \infty.$$

Moreover, in this case, we have $x^{-1}_{pr}\ \eta\ \mathfrak{A}^m$.

The particular cases $m = 1,2,3$ were considered in [11].

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