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MATHEMATICS

Regular Representations of the Central Extension of the Group of Diffeomorphisms of a Circle[1]

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We define an analog of the regular representations of the Virasoro–Bott group Vir, which is the central extension of the group $\mathrm{Diff}_+(S^1)$ of orientation preserving diffeomorphisms of the circle, with the use of quasi-invariant measures on Vir. The decomposition of these representations gives a family of nonisomorphic representations $T^{L,\sigma,n,m}$, where $\sigma > 0$ and $n, m \in \mathbf{Z}$. In [1], a similar result is obtained for the group $\mathrm{Diff}_+(S^1)$.

The Kac–Moody groups and the central extension of the group of diffeomorphisms of the circle are important for quantum physics (see [2, 3]). The difference between them is that, for the Kac–Moody groups, the cocycles are defined only locally [4], while for the group of diffeomorphisms of the circle, they are defined globally [5, 6].

Our goal is to define regular representations of the Virasoro–Bott group with the use of quasi-invariant measures on some completion of this group. These measures extend the Shavgulidze–Malliavin measure [7, 8].

Apparently, the first regular representations for noncommutative infinite-dimensional groups were considered in [9–11]. The first criterion for the irreducibility of the regular representations of some infinite-dimensional groups was given in [12] (see also reference [11] in [12]). Book [5] is also concerned with the representation theory of infinite-dimensional groups, in particular, with representations of the group of diffeomorphisms of the circle.

A unitary representation for the Kac–Moody groups was constructed in [13, 14]; it generalizes the Albeverio–Hoegh-Krohn representation for loop groups [9].

1. REGULAR REPRESENTATIONS

Let $\mathrm{Diff}_+(S^1)$ be the group of orientation preserving C^∞-diffeomorphisms of the circle $S^1 = \{z \in \mathbf{C}^1 : |z| = 1\} = \{e^{2\pi i\theta} | \theta \in [0, 1]\} \cong \mathbf{R}^1/2\pi\mathbf{Z}$. Recall [5] (see also [6]) that the group Vir is the central extension of the group $G = \mathrm{Diff}_+(S^1)$; i.e., $\mathrm{Vir} = G \times \mathbf{R}$, and its multiplication operation is defined by

$$(\alpha_1, t_1) \circ (\alpha_2, t_2) = (\alpha_1 \circ \alpha_2, t_1 + t_2 + B(\alpha_1, \alpha_2)), \quad (1)$$

where B is the Bott cocycle, i.e.,

$$B(\alpha_1, \alpha_2) = \int_{S^1} \ln(\alpha_1 \circ \alpha_2)' d\ln\alpha_2'. \quad (2)$$

Our further considerations employ the quotient group $\mathrm{Vir}/2\pi\mathbf{Z}$, which is the set $G \times S^1$ with the multiplication

$$(\alpha_1, \tau_1) \circ (\alpha_2, \tau_2) = (\alpha_1 \circ \alpha_2, \tau_1\tau_2 \exp(iB(\alpha_1, \alpha_2))) \quad (3)$$

for this group, we use the same notation Vir.

Let us define some quasi-invariant measures on the group Vir. By $\mathrm{Diff}^n_+(S^1)$, where $n = 1, 2, \ldots$, we denote the group of C^n-diffeomorphisms of the circle, and by $\mathrm{Diff}^n_0(S^1)$, its subgroup of diffeomorphisms leaving the initial point $1 = \exp(i0)$ fixed. We have $\mathrm{Diff}^1_+(S^1) = S^1 \cdot \mathrm{Diff}^1_0(S^1)$. This means that any element $\alpha \in \mathrm{Diff}^1_+(S^1)$ can be uniquely represented as the product $\alpha = \theta\varphi$, where $\theta \in S^1$, $\varphi \in \mathrm{Diff}^1_0(S^1)$, $\theta = \alpha(0)$, and $\varphi = (\alpha(0))^{-1}\alpha$.

In [8] (see also [1] for more details), a measure ν_σ with $\sigma > 0$ is constructed on the group $\mathrm{Diff}^1_+(S^1) = S^1 \cdot \mathrm{Diff}^1_0(S^1)$; it has the form $\nu_\sigma = mb_\sigma$, where m is the Haar measure on S^1 and $b_\sigma = B_\sigma^{A^{-1}}$ is the measure on $\mathrm{Diff}^1_0(S^1)$

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corresponding to the Brownian Bridge B_σ on ${}_0C_0[0, 1]$ in accordance with the Shavgulidze mapping (see [7]) A: $\mathrm{Diff}_0^1(S^1) \to {}_0C_0[0, 1]$,

$$\begin{aligned}\mathrm{Diff}_0^1(S^1) \ni \;\; &\varphi(t) \mapsto (A\varphi)(t)\\ &= \ln\varphi'(t) - \ln\varphi'(0) \in {}_0C_0[0, 1],\end{aligned} \tag{4}$$

where

$${}_0C_0[0, 1] = \{x \in C[0, 1] : x(0) = x(1) = 0\}.$$

We set $\mathrm{Vir}^n = S^1 \cdot \mathrm{Diff}_0^n(S^1) \cdot S^1$ for $n = 1, 2, \ldots$. Note that Vir^n is a group for $n = 2, 3, \ldots$, but Vir^1 is not a group (see (2)). Nevertheless, by virtue of (2), the right and left actions of the group Vir^3 are well-defined on the manifold Vir^1; they act by the rules $R_gh = hg^{-1}$ and $L_gh = gh$ for $g \in \mathrm{Vir}^3$, $h \in \mathrm{Vir}^1$. Indeed, the stochastic integral (2) is well-defined in this case. We define a measure on the manifold $\mathrm{Vir}^1 = S^1 \cdot \mathrm{Diff}_0^1(S^1) \cdot S^1$ as the product, i.e., by $\mu_\sigma = m \cdot b_\sigma \cdot m$.

Theorem 1. *The measure* μ_σ *on the manifold* Vir^1 *is quasi-invariant with respect to the left action of the group* Vir^3, *i.e.*, $\mu_\sigma^{L_g} \sim \mu_\sigma$ *for any* $\forall g \in \mathrm{Vir}^3$.

The proof is based on Lemma 8 from [1, p. 525] (see also [15, p. 324]). Now, we can define an analog $T^{L,\sigma}$: $\mathrm{Vir}^3 \to U(H_\sigma)$ of the left regular representation of the group Vir^3 in the space $H_\sigma = L^2(\mathrm{Vir}^1, \mu_\sigma)$ in a natural way as

$$(T_g^{L,\sigma}f)(h) = \left(\frac{d\mu_\sigma(g^{-1}h)}{d\mu_\sigma(h)}\right)^{\frac{1}{2}} f(g^{-1}h),$$
$$f \in H_\sigma, \quad g \in \mathrm{Vir}^3.$$

2. A DECOMPOSITION OF THE REGULAR REPRESENTATION

To prove the reducibility of the left regular representation, we show that the measure μ_σ is invariant with respect to the right action of the torus $\mathbf{T}^2$. By $\mathbf{T}^2$, we denote the subgroup $S^1 \cdot e \cdot S^1 \cong S^1 \times S^1$ of the group $S^1 \cdot \mathrm{Diff}_0^3(S^1) \cdot S^1$, where e is the identity element in $\mathrm{Diff}_0^3(S^1)$.

Theorem 2. *(i) The measure* μ_σ *is invariant with respect to the right action of the group* $\mathbf{T}^2 = S^1 \cdot e \cdot S^1$, i.e., $\mu_\sigma^{L_s} = \mu_\sigma$ *for any* $\forall s = (\xi, e, \tau) \in \mathbf{T}^2$;

(ii) the image $\mu_\sigma^{R_g}$ *of the measure* μ_σ *under the right action of the group* $\mathrm{Diff}_0^3(S^1) \cong e \cdot \mathrm{Diff}_0^3(S^1) \cdot e$ *is orthogonal to the initial measure, i.e.*,

$$\mu_\sigma^{R_g} \perp \mu_\sigma, \forall g = (e, \varphi, e) \in e \cdot \mathrm{Diff}_0^3(S^1) \cdot e,$$
$$\varphi \neq e.$$

The proof is based on Lemmas 9 and 10 from [1, p. 528].

Thus, we can construct a right representation of the group $\mathbf{T}^2$ in the space H_σ, which is defined by the rule $(T_s^{R,\sigma}f)(h) = f(hs)$ for $s = (\xi, e, \tau) \in \mathbf{T}^2$ and commutes with the left representation $T^{L,\sigma}$, i.e., $[T_g^{L,\sigma}, T_s^{R,\sigma}] = 0$ for any $\forall g \in \mathrm{Vir}^3$ and $s \in \mathbf{T}^2$.

Setting

$$H_{n,m,\sigma} = \{f \in H_\sigma\colon T^{R,\sigma}_{(\xi,e,\tau)}f = \xi^n\tau^m f\},$$
$$n, m \in \mathbf{Z},$$

we obtain

$$H_\sigma = \bigoplus_{n,m \in \mathbf{Z}} H_{n,m,\sigma}, \tag{5}$$

$$T^{L,\sigma} = \bigoplus_{n,m \in \mathbf{Z}} T^{L,n,m,\sigma}, \tag{6}$$

where $T^{L,n,m,\sigma}$ is the restriction of the representation $T^{L,\sigma}$ to the invariant subspace $H_{n,m,\sigma}$.

Conjecture. *(i) Decomposition* (6) *is a decomposition of the representation* $T^{L,\sigma}$ *into irreducible representations* $T^{L,n,m,\sigma}$ *with* $n, m \in \mathbf{Z}$;

(ii) $T^{L,n,m,\sigma} \sim T^{L,n',m',\sigma'} \Leftrightarrow (n, m, \sigma) = (n', m', \sigma')$.

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