Quasiregular representations of the infinite-dimensional nilpotent group

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Received 7 February 2006; accepted 13 March 2006

Communicated by Paul Malliavin

Abstract

In the present work an analog of the quasiregular representation which is well known for locally-compact groups is constructed for the nilpotent infinite-dimensional group $B_0^\mathbb{N}$ and a criterion for its irreducibility is presented. This construction uses the infinite tensor product of arbitrary Gaussian measures in the spaces $\mathbb{R}^m$ with $m > 1$ extending in a rather subtle way previous work of the second author for the infinite tensor product of one-dimensional Gaussian measures.

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Keywords: Infinite-dimensional groups; Nilpotent groups; Quasiregular representations; Irreducibility; Infinite tensor products; Gaussian measures; Ismagilov conjecture

1. Introduction

1.1. The setting and the main results

Let $(X, \mathcal{B})$ be a measurable space and let $\text{Aut}(X)$ denote the group of all measurable automorphisms of the space $X$. With any measurable action $\alpha: G \to \text{Aut}(X)$ of a group
G on the space X and a G-quasi-invariant measure μ on X one can associate a unitary representation \( \pi^{\alpha,\mu,X} : G \to U(L^2(X,\mu)) \), of the group G by the formula
\[
(\pi^{\alpha,\mu,X} t f)(x) = \left(\frac{d\mu(\alpha_t^{-1}(x))}{d\mu(x)}\right)^{1/2} f(\alpha_t^{-1}(x)),
\]
\( f \in L^2(X,\mu) \). Let us set \( \alpha(G) = \{ \alpha_t \in \text{Aut}(X) \mid t \in G \} \).

Let \( \alpha(G)' \) be the centralizer of the subgroup \( \alpha(G) \) in \( \text{Aut}(X) \):
\[
\alpha(G)' = \{ g \in \text{Aut}(X) \mid \{ g, \alpha_t \} = g \alpha_t g^{-1} \alpha_t^{-1} = e \forall t \in G \}.
\]
The following conjecture has been discussed in [23–25].

**Conjecture 1.** The representation \( \pi^{\alpha,\mu,X} : G \to U(L^2(X,\mu)) \) is irreducible if and only if:

1. \( \mu^g \perp \mu \forall g \in \alpha(G)' \setminus \{ e \} \) (where \( \perp \) stands for singular),
2. the measure \( \mu \) is G-ergodic.

We recall that a measure \( \mu \) is G-ergodic if \( f(\alpha_t(x)) = f(x) \forall t \in G \) implies \( f(x) = \text{const} \mu \) a.e. for all functions \( f \in L^1(X,\mu) \).

In this paper we shall prove Conjecture 1 in the case where G is the infinite-dimensional nilpotent group \( G = B_{0N}^N \) of finite upper-triangular matrices of infinite order with unities on the diagonal, the space \( X = X^m \) being the set of left cosets \( G_m \setminus B_{0N}^N \), \( G_m \) being suitable subgroups of the group \( B_{0N}^N \) of all upper-triangular matrices of infinite order with unities on the diagonal, and \( \mu \) an infinite tensor product of Gaussian measures on the spaces \( \mathbb{R}^m \) with some fixed \( m > 1 \).

A more detailed explanation of the concepts used here is given in the following sections.

### 1.2. Regular and quasiregular representations of locally compact groups

Let G be a locally compact group. The right \( \rho \) (respectively left \( \lambda \)) regular representation of the group G is a particular case of the representation \( \pi^{\alpha,\mu,X} \) with the space \( X = G \), the action \( \alpha \) being the right action \( \alpha = R \) (respectively the left action \( \alpha = L \)), and the measure \( \mu \) being the right invariant Haar measure on the group G (see, for example, [8,16,17,37]).

A quasiregular representation of a locally compact group G is also a particular case of the representation \( \pi^{\alpha,\mu,X} \) (see, for example, [37, p. 27]) with the space \( X = H \setminus G \), where H is a subgroup of the group G, the action \( \alpha \) being the right action of the group G on the space X and the measure \( \mu \) being some quasi-invariant measure on the space X (this measure is unique up to a scalar multiple). We remark that in [16,17] this representation has also been called geometric representation.

### 1.3. Analogs of the regular and quasiregular representations of infinite-dimensional groups and the Ismagilov conjecture

In the present article we will consider the approach which deals with analogs for infinite-dimensional groups of the regular and quasiregular representations of finite-dimensional groups. Let G be an infinite-dimensional topological group. To define an analog of the regular representation, let us consider some topological group \( \tilde{G} \), containing the initial group G as a dense subgroup, i.e. \( \tilde{G} = \overline{G} \) (G being the closure of G). Suppose we have some quasi-invariant measure \( \mu \) on \( X = \tilde{G} \) with respect to the right action of the group G, i.e. \( \alpha = R \), \( R_t(x) = xt^{-1} \). In this case we shall call the representation \( \pi^{\alpha,\mu,\tilde{G}} \) an analog of the regular representation. We shall denote this representation by \( T^{R,\mu} \), and the Conjecture 1 is reduced to the following Ismagilov conjecture.
Conjecture 2. (Ismagilov, 1985) The right regular representation \( T_{R,\mu} : G \rightarrow U(L^2(\tilde{G}, \mu)) \) is irreducible if and only if:

1. \( \mu^{Lt} \perp \mu \forall t \in G \setminus \{e\} \),
2. the measure \( \mu \) is \( G \)-ergodic.

Remark 3. In the case of the right regular representation, the group \( \alpha(G)' = R(G)' \subset Aut(\tilde{G}) \) obviously contains the group \( L(G) \), the image of the group \( G \) with respect to the left action.

The work [11] initiated the study of representations of current groups, i.e. groups \( C(X, U) \) of continuous mappings \( X \mapsto U \), where \( X \) is a finite-dimensional Riemannian manifold and \( U \) is a finite-dimensional Lie group.

The regular representation of infinite-dimensional groups, in the case of current groups, was studied firstly in [1,4,5,14] (see also the book [6]). An analog of the regular representation for an arbitrary infinite-dimensional group \( G \), using a \( G \)-quasi-invariant measure on some completion \( \tilde{G} \) of such a group, is defined in [18,20].

For \( X = S^1 \), \( U \) a compact or non-compact connected Lie group, Wiener measures on the loop groups \( \tilde{G} = C(X, U) \) were constructed and their quasi-invariance were proved in [1,4–6,28–32].

Conjecture 2 was formulated by R.S. Ismagilov for the group \( G = B^0_N \) and the measure \( \mu \) being the product of arbitrary one-dimensional centered Gaussian measures on the group \( \tilde{G} = B^N \) and was proved for this case in [18,19].

The first result in this direction was proved in [33]. For the complex infinite-dimensional Borel group \( Bor_{0,c}^e,\mathbb{N} \) and the standard Gaussian measure on its completion \( Bor_{0,c}^e,\mathbb{N} \) the irreducibility of the corresponding regular representation was proved there. Here \( Bor_{c,0}^e,\mathbb{N} \) (respectively \( Bor_{r,c,0}^{e,\mathbb{N}} \)) is the group of matrices of the form \( x = \exp t + s \) where \( t \) is a diagonal matrix with a finite number of nonzero real elements (respectively arbitrary real elements) and \( s \) is a finite (respectively arbitrary) complex strictly upper-triangular matrix.

For the product of arbitrary one-dimensional measures on the group \( B^N \) Conjecture 2 was proved in [21] under some technical assumptions on the measure.

In [20] Conjecture 2 was proved for the groups of the interval and circle diffeomorphisms. For the group of the interval diffeomorphisms the Shavgulidze measure [35] was used, the image of the classical Wiener measure with respect to some bijection. For the group of circle diffeomorphisms the Malliavin measure [30] was used.

Whether Conjecture 2 holds in the general case is an open problem.

In [25] it was shown that Conjecture 1 holds for the inductive limit \( G = SL_0(2\infty, \mathbb{R}) = \lim_{\rightarrow} SL(2n - 1, \mathbb{R}) \), of the special linear groups (simple groups) acting on a strip of length \( m \in \mathbb{N} \) in the space of real matrices which are infinite in both directions, the measure \( \mu \) being a product Gaussian measure.

Let us consider the special case of a \( G \)-space, namely the homogeneous space \( X = H \setminus \tilde{G} \), where \( H \) is a subgroup of the group \( \tilde{G} \) and \( \mu \) is some quasi-invariant measure on \( X \) (if it exists) with respect to the right action \( R \) of the group \( G \) on the homogeneous space \( H \setminus \tilde{G} \). In this case we call the corresponding representation \( \pi_{R,\mu,H}^{\tilde{G},\tilde{G}} \) an analog of the quasiregular or geometric representation of the group \( G \) (see [22]).

In [2] Conjecture 1 was proved for the solvable infinite-dimensional real Borel group \( G = Bor^r_{0,\mathbb{N}} \) acting on \( G \)-spaces \( X^m \), \( m \in \mathbb{N} \), where \( X^m \) is the set of left cosets \( G_m \setminus Bor^r_{0,\mathbb{N}} \), and \( G_m \) is some subgroups of the group \( Bor^N_{0,\mathbb{N}} \) of all upper-triangular matrices of infinite order with non-
zero elements on the diagonal. The measure $\mu$ on $X^m$ is the product of infinitely many one-dimensional Gaussian measures on $\mathbb{R}$.

In [23,24] Conjecture 1 was proved for the nilpotent group $G = B_0^\mathbb{N}$ and some $G$-spaces $X^m$, $m \in \mathbb{N}$, being the set of left cosets $G_m \setminus B^\mathbb{N}$, where $G_m$ are some subgroups of the group $B^\mathbb{N}$. Here the measure $\mu$ on $X^m$ is the infinite product of arbitrary one-dimensional Gaussian measures on $\mathbb{R}$. In this case the variables $x_{pq}$, $1 \leq p < q \leq m$, can be approximated by linear combinations of the expressions $A_{pn}A_{qn}$, $q < n$, where $A_{kn}$ are generators of one-parameter groups $\exp(t E_{kn})$, $k < n$, $t \in \mathbb{R}$.

In [3], using results of [21], we extended the results of [22–24] to the case of an infinite tensor product of arbitrary centered Gaussian measures.

In the present article we generalize results of [22–24] in another direction. Namely we prove Conjecture 1 for the same nilpotent infinite-dimensional group $G = B_0^\mathbb{N}$ and the same $G$-spaces $X^m$, $m \in \mathbb{N}$, but with a measure $\mu$ which is the infinite tensor product of arbitrary centered Gaussian measures on $\mathbb{R}^m$, for any arbitrary fixed $m \in \mathbb{N}$. More precisely, the measure $\mu$ on $X^m \simeq \mathbb{R}^1 \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^{m-1} \times \mathbb{R}^m \times \mathbb{R}^m \times \cdots$ is the infinite tensor product of arbitrary Gaussian measures:

$$\mu = \mu_B^m = \bigotimes_{n=2}^\infty \mu_{B(n)},$$

where $\mu_{B(n)}$ is a Gaussian measure on the space $\mathbb{R}^{n-1}$ for $2 \leq n \leq m$ and $\mu_{B(n)}$ is a Gaussian measure on the space $\mathbb{R}^m$ for $n > m$. In this case for the approximation of the variables $x_{pq}$, $1 \leq p < q \leq m$, we also use the commutative family of the generators $A_{kn}$, $1 \leq k \leq m < n$, but the corresponding expressions are much more complicated. In fact the extensions of [22–24] to the present case are not at all simple, the above expressions are no longer polynomials in the generators $A_{kn}$ they rather involve, next to the generators, also the one-parameter groups

$$T_{\exp(t E_{kn})}^R \mu_{B(n)}^m = \exp(t A_{kn}), \quad t \in \mathbb{R},$$

their derivatives and very special suitable chosen combinations that allow to approximate in an appropriate way the variables involved (see Lemmas 12 and 15).

2. Main objects

Let us consider the group $\widetilde{G} = B^\mathbb{N}$ of all upper-triangular real matrices of infinite order with unities on the diagonal

$$\widetilde{G} = B^{\mathbb{N}} = \left\{ I + x \mid x = \sum_{1 \leq k < n} x_{kn} E_{kn} \right\},$$

and its subgroup

$$G = B_0^{\mathbb{N}} = \left\{ I + x \in B^\mathbb{N} \mid x \text{ is finite} \right\},$$

where $E_{kn}$ is an infinite-dimensional matrix with 1 at the place $k, n \in \mathbb{N}$ and zeros elsewhere, $x = (x_{kn})_{k<n}$ is finite means that $x_{kn} = 0$ for all $(k, n)$ except for a finite number of indices $k, n$. 
Obviously, $B^0_n = \lim_{n \to \infty} B(n, \mathbb{R})$ is the inductive limit of the group $B(n, \mathbb{R})$ of real upper-triangular matrices with units on the principal diagonal

$$B(n, \mathbb{R}) = \left\{ I + \sum_{1 \leq k < r \leq n} x_{kr} E_{kr} \mid x_{kr} \in \mathbb{R} \right\}$$

with respect to the natural imbedding $B(n, \mathbb{R}) \subset B(n + 1, \mathbb{R})$. For $m \in \mathbb{N}$ we also define the subgroups $G_m$, respectively $G^m$, of the group $B^0_n$ as follows:

$$G_m = \left\{ I + x \in B^0_n \mid x = \sum_{m < k < n} x_{kn} E_{kn} \right\},$$

$$G^m = \left\{ I + x \in B^0_n \mid x = \sum_{1 \leq k \leq m, k < n} x_{kn} E_{kn} \right\}.$$ 

Since $B^0_n = G_m \cdot G^m$ the space $X^m$ of left cosets $X^m = G_m \setminus B^0_n$ is isomorphic to the group $G^m$. We use the notation $X^m \simeq G^m$. By construction, the right action $R$ of the group $G$ is well defined on the space $X^m$. More precisely if we define the decomposition $x = x_m \cdot x^m$:

$$B^0_n \ni x \mapsto x_m \cdot x^m \in G_m \cdot G^m,$$

the right action $R$ of the group $B^0_n$ on the space $X^m$ is defined as follows:

$$R_t(x^m) = (x^m t^{-1})^m, \quad x^m \in G^m, \ t \in B^0_n.$$ 

Define the measure $\mu^m := \mu^m_B$ on the space $X^m \simeq G^m$

$$X^m \simeq \mathbb{R}^1 \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^{m-1} \times \mathbb{R}^m \times \mathbb{R}^m \times \cdots$$

by the formula $\mu^m_B = \otimes_{n=2}^{\infty} \mu_{B(n)}$, where $\mu_{B(n)}$ is the Gaussian measure on the space $\mathbb{R}^m$ for $n > m$ (respectively on the space $\mathbb{R}^{n-1}$ for $2 \leq n \leq m$) defined by

$$d\mu_{B(n)}(x) = \frac{1}{\sqrt{(2\pi)^m \det B(n)}} \exp\left( -\frac{1}{2} (B(n)^{-1} x, x) \right) dx \leqno(1)$$

$$= \sqrt{\frac{\det C(n)}{(2\pi)^m} \exp\left( -\frac{1}{2} (C(n)^{-1} x, x) \right)} dx,$$

where $B(n)$ are positive-definite operators in the space $\mathbb{R}^m$ (or $\mathbb{R}^{n-1}$), $x = (x_{1n}, x_{2n}, \ldots, x_{mn})$, $dx$ is a Lebesgue measure on $\mathbb{R}^m$ and $C(n) = (B(n))^{-1}$.

**Lemma 4.** For the measure $\mu^m_B$ we have

$$\left( \mu^m_B \right)^R_t \sim \mu^m_B \quad \forall t \in B^0_n$$

(with $\sim$ meaning equivalence).
Proof. The right action $R_t$ for $t \in B_0^N$ changes linearly only a finite number of coordinates of the point $x \in X^m$. □

Now we can define the representation associated with the right action $T^R,\mu_B^n : B_0^N \rightarrow U\left(L^2\left(X^m, \mu_B^n\right)\right)$ in the natural way, i.e.

$$(T^R_t f)(x) = \left(\frac{d\mu_B^n(R_t^{-1}(x))}{d\mu_B^n(x)}\right)^{1/2} f\left(R_t^{-1}(x)\right).$$

Theorem 5. For the measure $\mu_B^m$ the following four statements are equivalent:

(i) the representation $T^{R,\mu_B^n}$ is irreducible;
(ii) $(\mu_B^m)^L \perp \mu_B^m \forall t \in B(m, \mathbb{R}) \setminus \{e\}$;
(iii) $(\mu_B^m)^{L_{\exp(tEpq)}} \perp \mu_B^m \forall t \in \mathbb{R} \setminus \{0\}, 1 \leq q < m$;
(iv) $S_{pq}^L(\mu_B^m) = \sum_{n=q+1}^{\infty} \frac{c(n)_{pp}b(n)_{qq}}{c(n)_{rr}} = \infty \forall 1 \leq p < q \leq m$,

where $B(n) = (b(n))_{k,r=1}^m$, $C(n) = (c(n))_{k,r=1}^m$ and $C(n) = (B(n))^{-1}$.

The proof of Theorem 5 is given in Sections 3–5 and Appendices A–C.

Lemma 6. The measure $\mu_B^m$ on the space $X^m$ is ergodic with respect to the right action $R$ of the group $B_0^N$ on the space $X^m$.

Proof. It is well known that any measurable function on $\mathbb{R}^\infty = \mathbb{R} \times \mathbb{R} \times \cdots$ with the standard Gaussian measure $\mu_\infty = \bigotimes_{n=1}^{\infty} \mu_{I_n}$, where $I_n \equiv I$ (see (1)) which is invariant under any change of the first coordinates (i.e. with respect to the additive action of the group $\mathbb{R}^\infty$) coincides almost everywhere with a constant function (see [36, Section 3, Corollary 1]). The proof works also in the case where we replace $\mathbb{R}$ by $\mathbb{R}^m$, $m > 1$, and the standard Gaussian measure $\mu_1$ on $\mathbb{R}$ with any probability measure $\mu_{B(n)}$ on $\mathbb{R}^m$ equivalent with the Lebesgue measure on $\mathbb{R}^m$. To prove this it is sufficient to see that any function $f \in L^1(\mathbb{R}^m)^\infty \otimes_{n=1}^{\infty} \mu_{B(n)}$ is the limit of $\mu_k$-a.e. constant functions $f^k$: $f = \lim_k f^k$, where $\mu_k = \bigotimes_{n=1}^{k} \mu_{B(n)}$.

$$f^k = \int_{(\mathbb{R}^m)^\infty} f(x) d\mu_k(x) \quad \text{and} \quad \mu_k = \bigotimes_{n=k+1}^{\infty} \mu_{B(n)}.$$  

Therefore the proof follows from the fact that the measure $\mu_B^m = \bigotimes_{n=2}^{\infty} \mu_{B(n)}$ on the space $X^m = \mathbb{R}^1 \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^{m-1} \times \mathbb{R}^m \times \cdots$ is the infinite tensor product of Gaussian measures $\mu_{B(n)}$ on the space $\mathbb{R}^m$ (for $n > m$), from the fact that the right action $R_t$ for $t \in B_0^N$ changes only a finite number of coordinates of the point $x \in X^m$, and that the group $G_0^m = G^m \cap B_0^N \subset X^m$ acts transitively on itself. In fact it is shown that the measure is ergodic with respect to the action of the subgroup $G_0^m \subset B_0^N$. □
3. Idea of the proof of irreducibility

Proof of Theorem 5. The proof of Theorem 5 is organized as follows:

\[(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i).\]

The parts \((i) \Rightarrow (ii) \Rightarrow (iii)\) are evident. The part \((iii) \Leftrightarrow (iv)\) follows from Lemma 8, which is based on the Kakutani criterion \([15]\).

The idea of the proof of irreducibility, i.e. the part \((iv) \Rightarrow (i)\). Let us denote by \(\mathfrak{A}^m\) the von Neumann algebra generated by the representation \(T^{R,\mu^m_B}\)

\[\mathfrak{A}^m = \left( T^{R,\mu^m_B} | t \in G \right)''\]

We show that \((iv) \Rightarrow [\mathfrak{A}^m]' \subset L^\infty(X^m, \mu^m_B) \Rightarrow (i)\). Let the inclusion \([\mathfrak{A}^m]' \subset L^\infty(X^m, \mu^m_B)\) holds. Using the ergodicity of the measure \(\mu^m_B\) (Lemma 6) this proves the irreducibility. Indeed in this case an operator \(A \in [\mathfrak{A}^m]'\) should be the operator of multiplication (since \([\mathfrak{A}^m]' \subset L^\infty(X^m, \mu^m_B)\)) by some essentially bounded function \(a \in L^\infty(X^m, \mu^m_B)\). The commutation relation \(\left[ A, T^{R,\mu^m_B}_t \right] = 0 \forall t \in B^\mathbb{N}_0\) implies \(a(R^{-1}_t(x)) = a(x) \pmod{\mu^m_B} \forall t \in B^\mathbb{N}_0\), so by ergodicity of the measure \(\mu^m_B\) with respect to the right action of the group \(B^\mathbb{N}_0\) on the space \(X^m\) we conclude that \(A = a = \text{const} \pmod{\mu^m_B}\). This then proves the irreducibility in Theorem 5, i.e. the part \([\mathfrak{A}^m]' \subset L^\infty(X^m, \mu^m_B) \Rightarrow (i)\).

The proof of the remaining part, i.e. the implication \((iv) \Rightarrow [\mathfrak{A}^m]' \subset L^\infty(X^m, \mu^m_B)\) is based on the fact that the operators of multiplication by independent variables \(x_{pq}, 1 \leq p \leq m, p < q\), may be approximated in the strong resolvent sense by some functions of the generators

\[A_{kn}^{R,m} = \frac{d}{dt} T^{R,\mu^m_B}_{t+iE_{kn}} \bigg|_{t=0}, \quad k, n \in \mathbb{N}, k < n,\]

i.e. that the operators \(x_{pq}\) are affiliated with the von-Neumann algebra \(\mathfrak{A}^m\). See Lemma 15 and Corollary 17.

Definition. Recall (cf., e.g., \([9]\)) that a non-necessarily bounded self-adjoint operator \(A\) in a Hilbert space \(H\) is said to be affiliated with a von Neumann algebra \(M\) of operators in this Hilbert space \(H\), if \(\exp(itA) \in M\) for all \(t \in \mathbb{R}\). One then writes \(A \in M\).

Since the algebra \((\exp(itx_{pq}) | t \in \mathbb{R}, 1 \leq p \leq m, p < q)'''\) is the maximal abelian subalgebra in the von Neumann algebra \(B(H)\) of all bounded operator in the Hilbert space \(H = L^2(X^m, \mu^m_B)\) we conclude that \((\exp(itx_{pq}) | t \in \mathbb{R}, 1 \leq p \leq m, p < q)''' = L^\infty(X^m, \mu^m_B)\). The inclusion \((\exp(itx_{pq}), 1 \leq p \leq m, p < q) \subset \mathfrak{A}^m\) implies \([\mathfrak{A}^m]' \subset L^\infty(X^m, \mu^m_B)\).

To finish the proof of Theorem 5 it remains to prove the implication

\[(iv) \Rightarrow (x_{pq} \in \mathfrak{A}^m, 1 \leq p \leq m, p < q) \iff (\exp(itx_{pq}) \in \mathfrak{A}^m, 1 \leq p \leq m, p < q)\]

(see Section 5). It is sufficient to prove that \(\Sigma_m > CS_m\), for some \(C > 0\), where

\[S_m := \sum_{1 \leq p < q \leq m} S_{pq}^L(\mu^m_B), \quad \Sigma_m := \sum_{1 \leq r \leq p < q \leq m} \Sigma_{pq}^r(m),\]
and the series $S_{pq}^L(\mu^m)$ and $\Sigma_{pq}^r(\mu^m)$ are defined in Lemmas 8 and 15 (see also (18)). This is done in Appendices A–C.

In Appendix A we define the generalization of the characteristic polynomial for matrix $C$ and establish some its properties. These properties are used then in Appendices B and C. For a matrix $C \in \text{Mat}(k, \mathbb{C})$ we set

$$G_k(\lambda) = \det C_k(\lambda), \quad \text{where } C_k(\lambda) = C + \sum_{r=1}^{k} \lambda_r E_{rr}, \quad \lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k.$$

**Lemma A.** (See Appendix A, Lemma A.7) For a positive definite matrix $C \in \text{Mat}(k, \mathbb{C}), \lambda \in \mathbb{R}^k$ with $\lambda_r \geq 0, r = 1, \ldots, k$, we have

$$\frac{\partial}{\partial \lambda_p} G_k(\lambda) \geq 0,$$

where $G_l(\lambda) = M_{12 \ldots l}^{12 \ldots l}(C_k(\lambda))$ and $1 \leq p \leq l \leq k$.

The proof of Lemma A is based on the following inequality (see Lemma A.6).

**Lemma B.** (Hadamard–Ficher’s inequality [12,13], see also [27]) Let $C \in \text{Mat}(m, \mathbb{R})$ be a positive definite matrix and $\emptyset \subseteq \alpha, \beta \subseteq \{1, \ldots, m\}$. Then

$$\left| \begin{array}{cc} \det C_\alpha & \det C_{\alpha \cap \beta} \\ \det C_{\alpha \cup \beta} & \det C_\beta \end{array} \right| = \left| \begin{array}{cc} M(\alpha) & M(\alpha \cap \beta) \\ M(\alpha \cup \beta) & M(\beta) \end{array} \right| \geq 0,$$

where $C_\alpha$ for $\alpha = \{\alpha_1, \ldots, \alpha_s\}$ denotes the matrix which entries lie on the intersection of $\alpha_1, \ldots, \alpha_s$ rows and $\alpha_1, \ldots, \alpha_s$ columns of the matrix $C$ and $M(\alpha) = M_\alpha^\alpha(C) = \det C_\alpha$ are corresponding minors of the matrix $C$.

The “best” approximation of $x_{pq}$ by the generators $A_{kn}^{R,m}$ is based on the exact computation of the matrix elements

$$\phi_p(t) = (T_t^{R,m} \mu^m_B 1, 1), \quad t = I + \sum_{r=1}^{p} t_r E_{rn}, \quad (t_r)_{r=1}^{p} \in \mathbb{R}^P,$$

of the representation $T_t^{R,m} \mu^m_B$ and their generalization (see Appendix B, Lemma B.1), and on the finding the appropriate combinations of operator functions of the generators $A_{kn}^{R,m}$ (see Remark 13) to approximate the operators of multiplication by $x_{pq}$.

Finally the proof of the inequality $\Sigma_m > CS_m$, is based on Lemmas A, B and 16 dealing with some inequalities involving the generalized characteristic polynomials. Lemma 16 is proved in Appendix C.
Remark 7. We shall firstly prove the approximation of \( x_{kn} \) in the above sense for the one vector \( 1 \in L^2(X^m, \mu_B^m) \). Secondly, the approximation also holds for some dense set \( D \) of analytic vectors in the space \( L^2(X^m, \mu_B^m) \)

\[
D = \left\{ X^\alpha = \prod_{1 \leq k \leq m, k < n} x_{kn}^\alpha \mid \alpha \in \Lambda \right\},
\]

for the corresponding operators, where \( \Lambda = \{ \alpha = (\alpha_{kn})_{1 \leq k \leq m, k < n} \} \) is the set of finite (i.e. \( \alpha_{kn} = 0 \) for a large \( n \)) multi-indices \( \alpha \) with \( \alpha_{kn} = 0, 1, \ldots \) and \( \{ f_n \mid n \in \mathbb{N} \} \) means the closure of the linear space generated by the set of vectors \( \{ f_n \mid n \in \mathbb{N} \} \). So using [34, Theorem VIII.25] we conclude that the convergence holds in the strong resolvent sense. (We observe that the proof of approximation in the strong resolvent sense is the same as the one given in [19, Lemma 2.2, p. 250].) Since the generators \( A_R^{m,n} \) are affiliated with the von Neumann algebra \( \mathfrak{A}^m \) the limit \( x_{kn} \) is also affiliated with \( \mathfrak{A}^m \).

We prove the part (iii) \( \iff \) (iv). The proof is an immediate consequence of the following:

**Lemma 8.** For the measure \( \mu_B^m \) we have the equivalence of

(iii) \( (\mu_B^m)^{L^{\exp(tEpq)}} \perp \mu_B^m \quad \forall t \in \mathbb{R} \setminus \{0\} \quad \forall 1 \leq p < q \leq m \) and

(iv) \( \mathcal{L}_{pq}(\mu_B^m) = \sum_{n=q+1}^{\infty} c_{pp} b_{qq}^{(n)} = \sum_{n=q+1}^{\infty} \frac{c_{pq} A_q^{(n)}(C^{(n)})}{\det(C^{(n)})} = \infty \quad \forall 1 \leq p < q \leq m, \)

where \( B^{(n)} = (b_{kr})_{k,r=1}^m \), \( C^{(n)} = (c_{kr})_{k,r=1}^m \) and \( C^{(n)} = (B^{(n)})^{-1} \).

**Proof.** The proof is based on the Kakutani criterion [15] and on the exact formula for the Hellinger integral

\[
H(\mu, \nu) = \int_X \sqrt{\frac{d\mu}{d\rho} \frac{d\nu}{d\rho}} \, d\rho,
\]

for two Gaussian measure \( \mu = \mu_{B_1} \) and \( \nu = \mu_{B_2} \) (see [26]):

\[
H(\mu_{B_1}, \mu_{B_2}) = \left( \frac{\det B_1 \det B_2}{\det^2 \frac{1}{2} B_1 + \frac{1}{2} B_2} \right)^{-1/4} = \left( \frac{\det C_1 \det C_2}{\det^2 \frac{1}{2} C_1 + \frac{1}{2} C_2} \right)^{1/4},
\]

where \( C_i = (B_i)^{-1}, i = 1, 2. \)

Let us consider the one-parameter subgroup \( \exp(tE_{pq}) = I + tE_{pq} \in B(m, \mathbb{R}) \), \( 1 \leq p < q \leq m, t \in \mathbb{R} \). Using (1) we have for the positive definite operator \( B = B^{(n)} \) in \( \mathbb{R}^m \):

\[
d\mu_B^{L_{t+tE_{pq}}}(x) = \sqrt{\frac{\det C}{(2\pi)^m}} \exp\left( -\frac{1}{2} \left( C \exp(tE_{pq})x, \exp(tE_{pq})x \right) \right) d\exp(tE_{pq})x
\]
\[
= \sqrt{\frac{\text{det}\ C}{(2\pi)^m}} \exp\left(-\frac{1}{2} (\exp(tE_{pq})^* C \exp(tE_{pq})x, x)\right) dx = d\mu_{Bpq(t)}(x),
\]

where \((B_{pq}(t))^{-1} = C_{pq}(t) = \exp(tE_{pq})^* C \exp(tE_{pq})\) (we note that \(\text{det}\ C = \text{det}\ C_{pq}(t)\)). Hence, using (2) we get

\[
H(\mu_{B}^{L_{1+tE_{pq}}}, \mu_{B}) = \left(\frac{\det C_{pq}(t) \det C}{\det^2 \frac{C_{pq}(t)+C}{2}}\right)^{1/4} = \left(\frac{\det C}{\det^2 \frac{C_{pq}(t)+C}{2}}\right)^{1/2}. \tag{3}
\]

We shall prove that

\[
\det \frac{C_{pq}(t) + C}{2} = \det C + \frac{t^2}{4} c_{pp} A_{q}^p (C), \tag{4}
\]

where \(A_{q}^p (C), 1 \leq p, q \leq m,\) denote the cofactors of the matrix \(C\) corresponding to the row \(p\) and the column \(q\). We have

\[
\frac{\det \frac{C_{pq}(t) + C}{2}}{\det C} = \frac{\det C + \frac{t^2}{4} c_{pp} A_{q}^p (C)}{\det C} = 1 + \frac{t^2}{4} c_{pp} b_{qq},
\]

hence

\[
\left(\frac{\det C}{\det \frac{C_{pq}(t) + C}{2}}\right)^{1/2} = \left(1 + \frac{t^2}{4} c_{pp} b_{qq}\right)^{-1/2}
\]

and finally, using (3) we get

\[
H((\mu_{B}^{m})^{L_{1+tE_{pq}}}, \mu_{B}^{m}) = \prod_{n=q+1}^{\infty} H(\mu_{B(n)}^{L_{1+tE_{pq}}}, \mu_{B(n)}) = \prod_{n=q+1}^{\infty} \left(1 + \frac{t^2}{4} c_{pp} b_{qq}^{(n)}\right)^{-1/2},
\]

where

\[
B^{(n)} = \sum_{1 \leq r,s \leq m} b_{rs}^{(n)} E_{rs} \quad \text{and} \quad C^{(n)} := (B^{(n)})^{-1} = \sum_{1 \leq r,s \leq m} c_{rs}^{(n)} E_{rs}.
\]

So using the properties of the Hellinger integral for two Gaussian measures we conclude that

\[
(\mu_{B}^{m})^{L_{1+tE_{pq}}} \perp \mu_{B}^{m} \quad \forall t \in \mathbb{R} \setminus \{0\} \iff \prod_{n=q+1}^{\infty} \left(1 + \frac{t^2}{4} c_{pp} b_{qq}^{(n)}\right)^{-1/2} = 0 \iff S_{pq}^{L}(\mu_{B}^{m}) = \infty.
\]
To prove (4) we set $C_{pq}(t) = \exp(tE_{pq})^* C \exp(tE_{pq})$. We have for $m \in \mathbb{N}$ and $1 \leq p < q \leq m$ using the identity $\exp(tE_{pq}) = I + tE_{pq}$, $t \in \mathbb{R}$,

$$
C_{pq}(t) = \begin{vmatrix}
    c_{11} & \ldots & c_{1p} & \ldots & c_{1q} + tc_{1p} & \ldots & c_{1m} \\
    \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    c_{1p} & \ldots & c_{pp} & \ldots & c_{pq} + tc_{pp} & \ldots & c_{pm} \\
    \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    c_{1q} + tc_{1p} & \ldots & c_{pq} + tc_{pp} & \ldots & c_{qq} + 2tc_{pq} + t^2c_{pp} & \ldots & c_{qm} + tc_{pm} \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    c_{1m} & \ldots & c_{pm} & \ldots & c_{qm} + tc_{pm} & \ldots & c_{mm}
\end{vmatrix},
$$

hence

$$
\det \frac{C_{pq}(t) + C}{2} = \begin{vmatrix}
    c_{11} & \ldots & c_{1p} & \ldots & c_{1q} + \frac{1}{2}c_{1p} & \ldots & c_{1m} \\
    c_{1p} & \ldots & c_{pp} & \ldots & c_{pq} + \frac{1}{2}c_{pp} & \ldots & c_{pm} \\
    c_{1q} + tc_{1p} & \ldots & c_{pq} + \frac{1}{2}c_{pp} & \ldots & c_{qq} + 2tc_{pq} + \frac{1}{4}t^2c_{pp} & \ldots & c_{qm} + \frac{1}{2}c_{pm} \\
    c_{1m} & \ldots & c_{pm} & \ldots & c_{qm} + \frac{1}{2}c_{pm} & \ldots & c_{mm}
\end{vmatrix} = \det C + \frac{t^2}{4}c_{pp}A^q_q(C).
$$

This ends the proof of Lemma 8, and thus also of (iii) $\Leftrightarrow$ (iv). □

4. Approximation of the variables $x_{pq}$

**Remark 9.** In what follows we shall omit the upper and lower index $n \in \mathbb{N}$ in all the expressions $c_{kr}^{(n)}, b_{kr}^{(n)}, B^{(n)}, C^{(n)}, \Xi_{pq}^{n}, g^{kn}, g_{kr}^{(n)}$, etc.

We first prove Lemmas 12 and 15, which give a suitable approximation of $x_{pq}$ only on the vector $f = 1 \in L^2(X^m, \mu_B^m)$ (cf. Remark 7).

We shall also use the well-known result (see, for example, [7, Chapter I, Section 52])

$$
\min_{x \in \mathbb{R}^n} \left( \sum_{k=1}^n a_k x_k^2 \right) \left( \sum_{k=1}^n x_k = 1 \right) = \left( \sum_{k=1}^n \frac{1}{a_k} \right)^{-1}, \quad a_k > 0, \quad k = 1, 2, \ldots, n.
$$

We use the same result in a slightly different form with $b_k \neq 0, \quad k = 1, 2, \ldots, n$,

$$
\min_{x \in \mathbb{R}^n} \left( \sum_{k=1}^n a_k x_k^2 \right) \left( \sum_{k=1}^n x_k b_k = 1 \right) = \left( \sum_{k=1}^n \frac{b_k^2}{a_k} \right)^{-1}. \quad (5)
$$
The minimum is realized for
\[ x_k = \frac{b_k}{a_k} \left( \sum_{k=1}^{n} \frac{b_k^2}{a_k} \right)^{-1}. \]

For any subset \( I \subset \mathbb{N} \) let us denote as before by \( \langle f_n \mid n \in I \rangle \) the closure of the linear space generated by the set of vectors \( \{ f_n \mid n \in I \} \) in a Hilbert space \( H \).

We note that the distance \( d(f_{n+1}; \langle f_1, \ldots, f_n \rangle) \) of the vector \( f_{n+1} \) in \( H \) from the hyperplane \( \langle f_1, \ldots, f_n \rangle \) may be calculated in terms of the Gram determinants \( \Gamma(f_1, f_2, \ldots, f_k) \) corresponding to the set of vectors \( f_1, f_2, \ldots, f_k \) (see [10]):

\[
d(f_{n+1}; \langle f_1, \ldots, f_n \rangle) = \min_{t=(t_k) \in \mathbb{R}^n} \left\| f_{n+1} + \sum_{k=1}^{n} t_k f_k \right\|^2 = \frac{\Gamma(f_1, f_2, \ldots, f_{n+1})}{\Gamma(f_1, f_2, \ldots, f_n)}, \quad (6)
\]

where the Gram determinant is defined by \( \Gamma(f_1, f_2, \ldots, f_n) = \det \gamma(f_1, f_2, \ldots, f_n) \) and \( \gamma(f_1, f_2, \ldots, f_n) =: \gamma_n \) is the Gram matrix

\[
\gamma(f_1, f_2, \ldots, f_n) = \begin{pmatrix}
(f_1, f_1) & (f_1, f_2) & \cdots & (f_1, f_n) \\
(f_2, f_1) & (f_2, f_2) & \cdots & (f_2, f_n) \\
\vdots & \vdots & \ddots & \vdots \\
(f_n, f_1) & (f_n, f_2) & \cdots & (f_n, f_n)
\end{pmatrix}.
\]

**Lemma 10.** We have

\[
d(f_{n+1}; \langle f_1, \ldots, f_n \rangle) = \frac{\det \gamma_{n+1}}{\det \gamma_n} = (f_{n+1}, f_{n+1}) - (\gamma_n^{-1} d_{n+1}, d_{n+1}),
\]

where \( d_{n+1} = ((f_1, f_{n+1}), (f_2, f_{n+1}), \ldots, (f_n, f_{n+1})) \in \mathbb{R}^n \).

**Proof.** We may write

\[
\left\| \sum_{k=1}^{n} t_k f_k - f_{n+1} \right\|^2 = \sum_{k, m=1}^{n} t_k t_m (f_k, f_k) - 2 \sum_{k=1}^{n} t_k (f_k, f_{n+1}) + (f_{n+1}, f_{n+1})
\]

\[
= (\gamma_n t, t) - 2(t, d_{n+1}) + (f_{n+1}, f_{n+1}),
\]

where \( t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n \). Using (58) for \( A_n = \gamma_n \) we get

\[
(\gamma_n t, t) - 2(t, d_{n+1}) = (\gamma_n (t - t_0), (t - t_0)) - (\gamma_n^{-1} d_{n+1}, d_{n+1}),
\]

where \( t_0 = \gamma_n^{-1} d_n \). Hence we get (see (6))

\[
\min_{t=(t_k) \in \mathbb{R}^n} \left\| f_{n+1} - \sum_{k=1}^{n} t_k f_k \right\|^2 = \min_{t=(t_k) \in \mathbb{R}^n} ((\gamma_n t, t) - 2(t, d_{n+1}) + (f_{n+1}, f_{n+1}))
\]
\[ = (f_{n+1}, f_{n+1}) - (\gamma_n^{-1} d_{n+1}, d_{n+1}) + \min_{t = (t_k) \in \mathbb{R}^n} (\gamma_n(t - t_0), (t - t_0)) \]
\[ = (f_{n+1}, f_{n+1}) - (\gamma_n^{-1} d_{n+1}, d_{n+1}). \]

**Remark 11.** In fact a more general result holds. Let us denote by \( A_{n+1} \) the real non-necessarily symmetric matrix in \( \mathbb{R}^{n+1} \) and by \( A_n \) its \( n \times n \) block after crossing the element in the last column and row, by \( v_{n+1} = (a_{1n+1}, a_{2n+1}, \ldots, a_{nn+1}) \), \( h_{n+1} = (a_{n+11}, a_{n+12}, \ldots, a_{n+1n}) \) vectors \( v_{n+1, h_{n+1}} \in \mathbb{R}^n \). If \( \det A_n \neq 0 \) then we have

\[
\frac{a_{n+1n+1} - (A_n^{-1} v_{n+1}, h_{n+1})}{\det A_n}. \]

**Proof.** It is sufficient to use the identity (Schur–Frobenius decomposition)

\[
A_{n+1} = \begin{pmatrix} A_n & v_{n+1}' \\ h_{n+1} & a_{n+1n+1} \end{pmatrix} = \begin{pmatrix} A_n & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \text{Id} & A_n^{-1} v_{n+1}' \\ h_{n+1} & a_{n+1n+1} \end{pmatrix}. \]

The generators

\[
A_{kn} := A_{kn}^{R, m} = d \frac{d}{dt} T_{I + tE_{kn}}^{R, \mu_m^B} \bigg|_{t=0}
\]

of the one-parameter groups \( I + t E_{kn} \) have the following form (on smooth functions of compact support):

\[
A_{kn} = \sum_{r=1}^{k-1} x_{rk} D_{rn} + D_{kn}, \quad 1 \leq k \leq m, k < n, \quad A_{kn} = \sum_{r=1}^{m} x_{rk} D_{rn}, \quad m < k < n,
\]

where

\[
D_{kn} = \partial/\partial x_{kn} - \frac{1}{2} (x, (B^{(n)})^{-1} E_{kn}), \quad 1 \leq k < n.
\]

To simplify the further computations let us consider the corresponding Fourier transforms \( F_m \) in the variables \( x_{kn}, 1 \leq k \leq m, m < n, \)

\[
F_m : L^2(X^m, \mu_B^m) \rightarrow L^2(X^m, \mu_C^m).
\]

We have

\[
F_m D_{kn} F_m^{-1} = iy_{kn} \text{ for } (k, n), \quad 1 \leq k \leq m, m < n, \quad \text{and} \quad F_m 1 = 1.
\]

Let us set \( \mu_C = \bigotimes_{n=2}^\infty \mu_C^{(n)} \) with \( C^{(n)} = B^{(n)} \) for \( 2 \leq n \leq m \) and \( C^{(n)} = (B^{(n)})^{-1} \) for \( n > m \). We define the Fourier transform \( F_m \) as the infinite tensor product \( F_m = \bigotimes_{n=m+1}^\infty F_{mn} \) where

\[
F_{mn} : L^2(\mathbb{R}^m, \mu_B^{(n)}) \rightarrow L^2(\mathbb{R}^m, \mu_C^{(n)}).
\]
is the image of the standard Fourier transform $F^m$ in the space $L^2(\mathbb{R}^m, dx)$, i.e. $F_{mn} = U(C^{(n)})^{-1} F^m U(B^{(n)})$, where

$$U(B^{(n)}) = \left(\frac{d\mu_B^{(n)}(x)}{dx}\right)^{1/2} \xrightarrow{F^m} L^2(\mathbb{R}^m, \mu_B^{(n)}) \xrightarrow{F_{mn}} L^2(\mathbb{R}^m, \mu_C^{(n)})$$

$$U(C^{(n)}) = \left(\frac{d\mu_C^{(n)}(x)}{dx}\right)^{1/2}$$

Since the standard Fourier transform $F^m$ is defined as follows:

$$(F^m f)(y) = \frac{1}{\sqrt{(2\pi)^m}} \int_{\mathbb{R}^m} \exp(i(y,x)f(x)) dx,$$

and, for $D = B^{(n)}$ respectively $D = C^{(n)}$

$$U(D) = \left(\frac{d\mu_D(x)}{dx}\right)^{1/2} = \frac{1}{((2\pi)^m \det D)^{1/4}} \exp\left(-\frac{1}{4} (D^{-1}x, x)\right),$$

we have finally for $F_{mn}$:

$$(F_{mn} f)(y) = \left(U(C^{(n)})^{-1} F^m U(B^{(n)}) f\right)(y)$$

$$= \frac{1}{((2\pi)^m \det C^{(n)})^{1/4}} \exp\left(\frac{1}{4} ((C^{(n)})^{-1}y, y)\right) \frac{1}{\sqrt{(2\pi)^m}} \times \int_{\mathbb{R}^m} \exp(i(y,x) f(x)) ((2\pi)^m \det B^{(n)})^{1/4} \exp\left(-\frac{1}{4} ((B^{(n)})^{-1}x, x)\right) dx$$

$$= \frac{\exp\left(\frac{1}{4} ((C^{(n)})^{-1}y, y)\right)}{\sqrt{(2\pi)^m \det C^{(n)}}} \int_{\mathbb{R}^m} \exp\left(i(y,x) - \frac{1}{4} ((B^{(n)})^{-1}x, x)\right) f(x) dx.$$

Using Fourier transform $F_m$ we obtain for $\tilde{A}_{kn} = F_m A_{kn}(F_m)^{-1}$:

$$\tilde{A}_{kn} = i \left(\sum_{r=1}^{k-1} x_r y_k + y_k \right), \quad 1 \leq k \leq m < n, \quad \tilde{A}_{kn} = \sum_{r=1}^{m} D_{rk}(y) y_r, \quad m < k < n, \quad (9)$$

where

$$D_{kn}(y) = \frac{\partial}{\partial y_{kn}} - \frac{1}{2} (x, (C^{(n)})^{-1} E_{kn}), \quad 1 \leq k < n.$$

Let us set for $s = (s_1, \ldots, s_r) \in \mathbb{R}^r$ and $1 \leq r \leq p < q \leq m$

$$\xi_n^{rp}(s) = F_m \left(D_{pn} \exp\left(\sum_{l=1}^{r} s_l A_{ln}\right)\right) 1 = i y_{pn} \exp\left(\sum_{l=1}^{r} s_l \tilde{A}_{ln}\right) 1. \quad (10)$$
For a function $f : X^m \to \mathbb{C}$ we set
\[
Mf = \int_{X^m} f(x) \, d\mu_B^m(x).
\]

To approximate the variables $x_{pq}, 1 \leq p < q \leq m$, we use

**Lemma 12.** Let $1 \leq r \leq p < q \leq m$. For any \( s^{(n)} = (s_1^{(n)}, \ldots, s_r^{(n)}) \in \mathbb{R}^r \), and for any \( \alpha^{(n)} = (\alpha_1^{(n)}, \ldots, \alpha_m^{(n)}) \in \mathbb{R}^m \), $n \in \mathbb{N}$, we have

\[
x_{pq} \in \left\{ \exp \left( \sum_{l=1}^{r} s_l^{(n)} A_{ln} \right) \left( \sum_{k=1}^{m} \alpha_k^{(n)} A_{kn} \right) 1 \mid n \in \mathbb{N}, m < n \right\} \Leftrightarrow \Sigma_{pq}^r (s, \alpha, m) = \infty,
\]

where \( s = (s^{(n)})_{n=m+1}^{\infty}, \alpha = (\alpha^{(n)})_{n=m+1}^{\infty}, \alpha_q^{(n)} = 1 \) and

\[
\Sigma_{pq}^r (s, \alpha, m) = \sum_{n=m+1}^{\infty} \frac{|M^{x_{pq}} (s^{(n)})|^2}{c_{pp}^{(n)} - |M^{x_{pq}} (s^{(n)})|^2 + \| (A_{qn} - x_{pq} D_{pn} + \sum_{k=1,k \neq p}^{m} \alpha_k^{(n)} A_{kn}) 1 \|^2}.
\]

(11)

Before proving Lemma 12 let us make some comments about the procedure for arriving at the expressions used for the approximation of the variables $x_{pq}$ on the left-hand side of the equivalence in Lemma 12.

**Remark 13.**

1. The operator $A_{qn} = \sum_{r=1}^{q-1} x_{rq} D_{rn} + D_{qn}$ contains $x_{pq}$ for $r = p$.

2. Since $MD_{pn} 1 = 0$ and $MD_{pn} \exp(s A_{pn}) 1 \neq 0$ we may first think of considering $\exp(s A_{pn}) A_{qn} 1$, $1 \leq p < q \leq m$ (similarly as in [23,24] where the linear combinations of $A_{pn} A_{qn}$ were used). But this is not sufficient for the approximation. We might then try to consider the expression

\[
\exp(s A_{pn}) \left( \sum_{k=1}^{m} \alpha_k A_{kn} \right), \quad 1 \leq p < m < n,
\]

with $\alpha_q = 1$. The calculations show again that these combinations are still not sufficient to approximate $x_{pq}$. We arrive then at the suggestion to take

\[
\exp \left( \sum_{l=1}^{r} s_l A_{ln} \right) \left( \sum_{k=1}^{m} \alpha_k A_{kn} \right), \quad 1 \leq r \leq p < q \leq m < n,
\]

which is the choice made in Lemma 12.

3. For approximation of the variable $x_{pq}$ we use $p$ different combinations, corresponding to $\Sigma_{pq}^r (s, \alpha, m), 1 \leq r \leq p$. All these combinations are essential, i.e. none of them can be omitted. This can be seen by constructing corresponding counterexamples and is in a contrast to the previous cases considered in [23,24] where only one combination of $A_{pn} A_{qn}$ were used to approximate $x_{pq}$.
4. To make the expression \( \Sigma_{pq}^r(s, \alpha, m) \) in (11) larger (to apply then the criterium in Lemma 12) we chose \( s^{(n)} \in \mathbb{R}^r \) such that
\[
|M_{\xi n}^{rp}(s^{(n)})|^2 = \max_{s \in \mathbb{R}^r} |M_{\xi n}^{rp}(s)|^2
\]
(which is possible, \( |M_{\xi n}^{rp}(s)|^2 \) being continuous and bounded).

5. With the same aim we chose \( \alpha^{(n)}_k \) in such a way that
\[
\left\| \left( A_{qn} - x_{pq} D_{pn} + \sum_{k=1, k \neq q}^m \alpha^{(n)}_k A_{kn} \right) \right\|_2^2 = \min_{(t_k) \in \mathbb{R}^{m-1}} \left\| \left( A_{qn} - x_{pq} D_{pn} + \sum_{k=1, k \neq q}^m t_k A_{kn} \right) \right\|_2^2.
\]

6. The right-hand side of the previous expression is equal (see (6)) to
\[
\frac{\Gamma(g_1, g_2, \ldots, g_q^p, \ldots, g_m)}{\Gamma(g_1, g_2, \ldots, g_{q-1}, g_{q-1}, \ldots, g_m)},
\]
where
\[
g_k := g_{kn} := A_{kn} 1, \quad 1 \leq k \leq m, \ k \neq q, \quad g_q^p := g_{qn} := (A_{qn} - x_{pq} D_{pn}) 1. \quad (12)
\]

**Proof of Lemma 12.** If we put \( \sum_n t_n M_{\xi n}^{rp}(s^{(n)}) = 1 \) we get
\[
\left\| \left[ \sum_{n} t_n \exp \left( \sum_{l=1}^r s^{(n)}_l A_{ln} \right) \left( \sum_{k=1}^m \alpha^{(n)}_k A_{kn} \right) - x_{pq} \right] \right\|_2^2
\]
\[
= \left\| \left[ \sum_{n} t_n \exp \left( \sum_{l=1}^r s^{(n)}_l A_{ln} \right) \left( A_{qn} - x_{pq} D_{pn} + x_{pq} D_{pn} + \sum_{k=1, k \neq q}^m \alpha^{(n)}_k A_{kn} \right) - x_{pq} \right] \right\|_2^2
\]
\[
= \left\| \sum_{n} t_n \left[ x_{pq} \left( D_{pn} \exp \left( \sum_{l=1}^r s^{(n)}_l A_{ln} \right) - M_{\xi n}^{rp}(s^{(n)}) \right) \right.ight.
\]
\[
\left. + \exp \left( \sum_{l=1}^r s^{(n)}_l A_{ln} \right) \left( A_{qn} - x_{pq} D_{pn} + \sum_{k=1, k \neq q}^m \alpha^{(n)}_k A_{kn} \right) \right] \right\|_2^2
\]
\[
= \sum_{n} t_n^2 \left\| x_{pq} \right\|_2^2 \left\| \left( D_{pn} \exp \left( \sum_{l=1}^r s^{(n)}_l A_{ln} \right) - M_{\xi n}^{rp}(s^{(n)}) \right) \right\|_2^2
\]
\[
+ \left\| \exp \left( \sum_{l=1}^r s^{(n)}_l A_{ln} \right) \left( A_{qn} - x_{pq} D_{pn} + \sum_{k=1, k \neq q}^m \alpha^{(n)}_k A_{kn} \right) \right\|_2^2
\]
\[
= \sum_{n} t_n^2 \left\| x_{pq} \right\|_2^2 \left( c_{pp}^{(n)} - |M_{\xi n}^{rp}(s^{(n)})|^2 \right)
\]
where we have used the equality \( \| \xi - M \xi \|^2 = \| \xi \|^2 - |M \xi|^2 \):

\[
\left\| D_{pn} \exp \left( \sum_{l=1}^{r} s_l^{(n)} A_{ln} \right) - M \xi_{n}^{r,p} (s^{(n)}) \right\|^2 \leq \left\| D_{pn} \right\|^2 - |M \xi_{n}^{r,p} (s^{(n)})|^2 = c_{pp}^{(n)} - |M \xi_{n}^{r,p} (s^{(n)})|^2.
\]

**Definition.** We shall say that two series \( \sum_{n} a_{n} \) and \( \sum_{n} b_{n} \) with positive members are *equivalent* and shall denote this by \( \sum_{n} a_{n} \sim \sum_{n} b_{n} \) if they are convergent or divergent simultaneously. We note that if \( a_{n} > 0, b_{n} > 0, n \in \mathbb{N} \), then we have

\[
\sum_{n} a_{n} + b_{n} \sim \sum_{n} a_{n}.
\]  

Using (5) we get, setting \( b = (M \xi_{n}^{r,p} (s^{(n)}))_{n=m+1}^{m+1+N} \in \mathbb{R}^{N} \), \( N \in \mathbb{N} \),

\[
\min_{t \in \mathbb{R}^{N}} \left( \left\| \sum_{n=m+1}^{m+1+N} \sum_{l=1}^{r} t_{n} \exp \left( \sum_{l=1}^{r} s_l^{(n)} A_{ln} \right) \left( \sum_{k=1}^{m} \alpha_{k}^{(n)} A_{kn} \right) - x_{pq} \right\| \right)^{2} \bigg| (t, b) = -1 \sim \left( \sum_{n=m+1}^{m+1+N} c_{pp}^{(n)} - |M \xi_{n}^{r,p} (s^{(n)})|^2 + \| (A_{qn} - x_{pq} D_{pn} + \sum_{k=1, k \neq p}^{m} \alpha_{k}^{(n)} A_{kn}) 1 \|^2 \right)^{-1}.
\]

Due to Remark 9 we shall write \( C \) (respectively \( \hat{C} \)) instead of \( C^{(n)} \) (respectively \( \hat{C}^{(n)} \)), where

\[
C^{(n)} = \begin{pmatrix}
    c_{11}^{(n)} & c_{12}^{(n)} & \cdots & c_{1m}^{(n)} \\
    c_{12}^{(n)} & c_{22}^{(n)} & \cdots & c_{2m}^{(n)} \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{1m}^{(n)} & c_{2m}^{(n)} & \cdots & c_{mm}^{(n)}
\end{pmatrix},
\]

\[
\hat{C}^{(n)} = \begin{pmatrix}
    c_{11}^{(n)} & c_{12}^{(n)} & \cdots & c_{1m}^{(n)} \\
    c_{12}^{(n)} & c_{11}^{(n)} + c_{22}^{(n)} & \cdots & c_{2m}^{(n)} \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{1m}^{(n)} & c_{2m}^{(n)} & \cdots & c_{11}^{(n)} + c_{22}^{(n)} + \cdots + c_{mm}^{(n)}
\end{pmatrix}.
\]

**Remark.** To simplify the further computations we assume that the measures \( \mu_{B(n)} \) for \( 2 \leq n \leq m \) are standard: \( B^{(n)} = I \). Since \( \mu_{B}^{m} = \bigotimes_{n=2}^{\infty} \mu_{B(n)} \) this assumption, which only concerns finitely many of the \( \mu_{B(n)} \)'s, does not change the equivalence class of the initial measure \( \mu_{B}^{m} \) and the equivalence class of the corresponding representation \( T^{R, \mu_{B}^{m}} \).
Using this remark, notation (12) and Fourier transforms we conclude that
\[ \Gamma(g_1, g_2, \ldots, g_m) = \det \hat{C}, \quad \text{i.e.} \quad \Gamma(g_{1n}, g_{2n}, \ldots, g_{mn}) = \det \hat{C}^{(n)}, \quad (14) \]
since \((g_q, g_p) = (\hat{C})_{pq}, 1 \leq p, q \leq m\). Indeed for \(p \neq q\) we have
\[ (g_{qn}, g_{pn}) = (g_{pn}, g_{qn}) = \left( \sum_{r=1}^{p-1} x_{rp} y_{rn} + y_{pn}, \sum_{s=1}^{q-1} x_{sq} y_{sn} + y_{qn} \right) = (y_{pn}, y_{qn}) = c_{pq}^{(n)}, \]
\[ (g_{pn}, g_{pn}) = \left\| \sum_{r=1}^{p-1} x_{rp} y_{rn} + y_{pn} \right\|^2 = \sum_{r=1}^{p-1} \| x_{rp} \|^2 \| y_{rn} \|^2 + \| y_{pn} \|^2 = \sum_{r=1}^{p} c_{rr}^{(n)} = (\hat{C}^{(n)})_{pp} \]
(we reinserted here the upper index \(n\) in \(c_{pq}^{(n)}\) for clarity).

In the following we shall need a variant of Lemma 12 using Remark 13 replacing the \(|M_{\xi_{np}}(s)|\) by its maximum \(\xi^{f_{np}}(s)\). Let us set (see (10) for definition of \(\xi_{np}^{f_{np}}(s)\))
\[ \xi_n^{f_{np}} = \max_{s \in \mathbb{R}} |M_{\xi_{np}}^{f_{np}}(s)|^2. \quad (15) \]

Now we see that using \(s\) and \(\alpha\) as in parts 4 and 5 of Remark 13 we have
\[ \Sigma_{pq}^{r}(s, \alpha, m) \]
\[ = \sum_{n} \max_{x \in \mathbb{R}} |M_{\xi_{np}}^{f_{np}}(s^{(n)})|^2 - \max_{x \in \mathbb{R}} |M_{\xi_{np}}^{f_{np}}(s^{(n)})|^2 + \| (A_{qn} - x_{pq} D_{pn} + \sum_{k=1, k \neq p}^{m} \alpha_k^{(n)} A_{kn}) 1 \|^2 \]
\[ \stackrel{(13)}{=} \sum_{n} \max_{s^{(n)} \in \mathbb{R}} |M_{\xi_{np}}^{f_{np}}(s^{(n)})|^2 + \| (A_{qn} - x_{pq} D_{pn} + \sum_{k=1, k \neq p}^{m} \alpha_k^{(n)} A_{kn}) 1 \|^2 \]
\[ \stackrel{(15)}{=} \sum_{n} \frac{\xi_n^{f_{np}}}{\Gamma(g_{1n}, g_{2n}, \ldots, g_{1n}, \ldots, g_{mn})} - \frac{\Gamma(g_{1n}, g_{2n}, \ldots, g_{1n}, \ldots, g_{mn})}{\Gamma(g_{1n}, g_{2n}, \ldots, g_{1n}, \ldots, g_{mn})} \]
\[ \text{Remark 9} \]
\[ = \sum_{n} \frac{\xi_n^{f_{np}} \Gamma(g_1, g_2, \ldots, g_{q-1}, g_{q+1}, \ldots, g_m)}{\Gamma(g_1, g_2, \ldots, g_{q-1}, g_{q+1}, \ldots, g_m)} = \sum_{n} \frac{\xi_n^{f_{np}} A_{q}^{f_{np}}(\hat{C}^{(n)})}{\det \hat{C}^{(n)}}. \quad (14) \]

For the latter equality we have used the fact that
\[ c_{pp} \Gamma(g_1, g_2, \ldots, g_{q-1}, g_{q+1}, \ldots, g_m) + \Gamma(g_1, g_2, \ldots, g_{q-1}, g_{q+1}, \ldots, g_m) = \Gamma(g_1, g_2, \ldots, g_m), \]
which follows from (26). Indeed it is sufficient to take in (26) \(C = \hat{C} - c_{pp} E_{qq}\) and \(\lambda_q = c_{pp}\). Then we have
\[ \Gamma(g_1, g_2, \ldots, g_m) = \det \hat{C} = \det(\hat{C} - c_{pp} E_{qq} + c_{pp} E_{qq}) \]
\[ = \det(\hat{C} - c_{pp} E_{qq}) + c_{pp} A_q^q(\hat{C} - c_{pp} E_{qq}) \]
\[ = \Gamma(g_1, g_2, \ldots, g_q, \ldots, g_m) + c_{pp} \Gamma(g_1, g_2, \ldots, g_{q-1}, g_{q+1}, \ldots, g_m). \]

So we have proved the following lemma.

**Lemma 15.** Let \( 1 \leq r \leq p < q \leq m \). Then for some \( s_l = (s_l^{(n)})_{n=m+1}^{\infty}, \alpha_k = (\alpha_k^{(n)})_{n=m+1}^{\infty} \), where \( s_l^{(n)}, \alpha_k^{(n)} \in \mathbb{R}, 1 \leq l \leq r, 1 \leq k \leq m \), we have

\[
x_{pq} \in \left\{ \exp \left( \sum_{l=1}^{r} s_l^{(n)} A_{ln} \right) \left( \sum_{k=1}^{m} \alpha_k^{(n)} A_{kn} \right) 1 \mid n \in \mathbb{N}, m < n \right\} \]

\[
\Leftrightarrow \quad \Sigma_{pq}^r(m) = \sum_n \frac{\Sigma_{pq}^r A_{q}^{q}(\hat{C}^{(n)})}{\det \hat{C}^{(n)}} = \infty. \tag{16} \]

5. The proof of (iv) \( \Rightarrow (x_{pq} \eta \mathfrak{A}^m, 1 \leq p \leq m, p < q) \) in Theorem 5

**Idea.** We prove firstly that \( x_{pq} \eta \mathfrak{A}^m \) for some \( (p, q) \): \( 1 \leq p < q \leq m \) if conditions (iv) are valid. Further we prove that this also holds for all such \( (p, q) \). For this it is sufficient to prove that

\[ \Sigma_m > C S_m \quad \text{for some } C > 0, \tag{17} \]

where

\[
S_m := \sum_{1 \leq p < q \leq m} S_{pq}^L(\mu_m), \quad \text{and} \quad \Sigma_m := \sum_{1 \leq r \leq p < q \leq m} \Sigma_{pq}^r(m) \tag{18} \]

(see (16) for the definition of \( \Sigma_{pq}^r(m) \)). Indeed, in this case \( S_m = \infty \) since \( S_{pq}^L(\mu_m) = \infty \) \( \forall p, q : 1 \leq p < q \leq m \) by Lemma 8 hence \( \Sigma_m = \infty \) by (17) and finally we conclude that \( \Sigma_{pq}^r(m) = \infty \) for some \( r, p, q : 1 \leq r \leq p < q \leq m \). By Lemma 15 we get that \( x_{pq} \eta \mathfrak{A}^m \).

The proof of (17) is based on Appendices A–C. In Appendix A we define the generalization of the characteristic polynomial for matrix \( C \in \text{Mat}(m, \mathbb{C}) \) and establish some of its properties. These properties are used then in Appendices B and C.

In Appendix B we estimate \( \Xi_{pq}^n = \max_{t \in \mathbb{R}^p} \left| M_{pq}^n(t) \right|^2 \). This estimation is based on Lemma B.1 which gives us the exact formula for

\[ M_{pq}^n(t) = \left( D_{qn} T_{\exp(\sum_{r=1}^{p} t_r E_{rn})}^{R, \mu_m^R} \frac{1}{1}, 1 \right), \quad t = (t_1, t_2, \ldots, t_p) \in \mathbb{R}^p, 1 \leq p \leq m \]

(see (44)), where \( D_{kn} \) is defined in (8). The latter formula is based on the exact formulas for the matrix elements

\[ \phi_p(t) := \phi_p^{(n)}(t) = \left( T_{\exp(\sum_{r=1}^{p} t_r E_{rn})}^{R, \mu_m^R} \frac{1}{1}, 1 \right), \quad t = (t_r)_{r=1}^{p} \in \mathbb{R}^p, 1 \leq p \leq m \]
(see (40)) and theirs generalizations (see (42)). We cannot calculate explicitly
\[ \varSigma_{n}^{pq} = \max_{t \in \mathbb{R}} |M_{n}^{pq}(t)|^2, \]
but we are able by Lemmas B.1 and B.2 to obtain the estimation \( \varSigma_{n}^{pq} > \Psi_{n}^{pq} \),
\[ \Psi_{n}^{pq} := \frac{(M_{12...p-1}^{12...p-1q} (C_{p,q}^{(n)})^2 \exp(-1))}{M_{12...p-1}^{12...p-1} (C_{p,q}^{(n)})^2 + \sum_{k=2}^{p} \hat{\lambda}_{k} (A_{k}^{p}(C_{p}^{(n)}))^2} \]
(see (47) and (48)). The crucial for proving (17) is Lemma 16 dealing with some inequalities involving the generalized characteristic polynomials. This lemma is proved in Appendix C.

We use the notations of Lemma 8 (see Remark 9):
\[ S_{pq}^{L}(\mu_{B}^{m}) = \sum_{n=q+1}^{\infty} c_{pp}^{(n)} b_{qq}^{(n)} = \sum_{n=q+1}^{\infty} \frac{c_{pp}^{(n)} A_{q}^{q}(C_{m}^{(n)})}{\det C_{m}^{(n)}} = \sum_{n=q+1}^{\infty} \frac{c_{pp} A_{q}^{q}(C_{m})}{\det C_{m}}. \]
Let
\[ \lambda = (\lambda_{k})_{k=1}^{m} \in \mathbb{R}^{m}, \quad \hat{\lambda} = (\hat{\lambda}_{k})_{k=1}^{m}, \quad \hat{\lambda}_{1} = 0, \quad \hat{\lambda}_{k} = \sum_{r=1}^{k-1} c_{rr}, \quad 2 \leq k \leq m, \quad (19) \]
\[ f_{q} = e \sum_{1 \leq r \leq p < q} \Psi_{rp}^{r}, \quad 2 \leq q \leq m, \quad f_{2} = e \Psi_{11}^{11} = c_{11}, \]
\[ f_{3} = e(\Psi_{11}^{11} + \Psi_{12}^{12} + \Psi_{22}^{22}), \quad \ldots, \quad (20) \]
\[ C_{m} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1m} \\ c_{12} & c_{22} & \cdots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1m} & c_{2m} & \cdots & c_{mm} \end{pmatrix}, \]
\[ \hat{C}_{m} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1m} \\ c_{12} & c_{11} + c_{22} & \cdots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1m} & c_{2m} & \cdots & c_{11} + \cdots + c_{mm} \end{pmatrix}. \quad (21) \]

Obviously, we have \( \hat{C}_{m} = C_{m}(\hat{\lambda}) \), where \( \hat{\lambda} \in \mathbb{C}^{m} \), is defined in (19) and we use the notation \( C_{m}(\lambda) := C_{m} + \sum_{k=1}^{m} \lambda_{k} E_{kk} \).

We have the following expressions for \( S_{m} \) and \( \Sigma_{m} \):
\[ S_{m} := \sum_{1 \leq r < k \leq m} S_{rk}^{L}(\mu_{B}^{m}) \sim \sum_{n=m+1}^{\infty} \frac{\sum_{k=2}^{m} (\sum_{r=1}^{k-1} c_{rr}) A_{k}^{k}(C_{m})}{\det C_{m}} = \sum_{n=m+1}^{\infty} \frac{\sum_{k=2}^{m} \hat{\lambda}_{k} A_{k}^{k}(C_{m})}{\det C_{m}}. \]
We have replaced the series
\[ S_{pq}^L (\mu_B) = \sum_{n=q+1}^{\infty} c_{pp}^{(n)} p_{qq}^{(n)} \]
with the equivalent one
\[ S_{pq}^L (\mu_B) \sim \sum_{n=m+1}^{\infty} c_{pp}^{(n)} p_{qq}^{(n)}. \]
If we use the equality \( \hat{C}_m = C_m (\hat{\lambda}) \), we get
\[
\Sigma_m := \sum_{1 \leq r \leq p < q \leq m} \Sigma_{pq}^r (m) = \sum_{2 \leq q \leq m} \Sigma_{pq} (m) = \sum_{2 \leq q \leq m} \sum_{1 \leq r \leq p < q} \sum_{n} \frac{\Xi_{pq}^{r p} A_{q}^{(n)} (\hat{C}_m^{(n)})}{\det \hat{C}_m^{(n)}}
\]
\[
\geq \sum_{n} \frac{e^{-1} \sum_{q=2}^{m} f_q A_{q}^{(n)} (C_m (\hat{\lambda}))}{\det C_m + \sum_{q=2}^{m} \hat{\lambda}_q A_{q}^{(n)} (C_m (\hat{\lambda} [q]))}
\]
\[
\geq \sum_{n} \frac{e^{-1} \sum_{q=2}^{m} f_q A_{q}^{(n)} (C_m (\hat{\lambda}))}{\det C_m}
\]
\[
\geq \sum_{n} \frac{\sum_{q=2}^{m} \hat{\lambda}_q A_{q}^{(n)} (C_m (\hat{\lambda} [q]))}{\det C_m} = \Sigma_m. \tag{22}
\]

The implications \( S_m = \infty \Rightarrow \Sigma_m = \infty \) is based on the equality (see (26))
\[
A_k^k (C_m (\hat{\lambda} [k])) = A_k^k (C_m) + \sum_{r=1}^{m-k} \sum_{1 \leq i_1 < \cdots < i_r \leq m} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_r} A_{k i_1 i_2 \cdots i_r}^k (C_m) \tag{23}
\]
and on the following lemma.

**Lemma 16.** For \( \hat{\lambda} = (\hat{\lambda}_r)_{r=1}^{m} \in \mathbb{R}^m, \hat{\lambda}_1 = 0, \hat{\lambda}_k = \sum_{r=1}^{k-1} c_{rr}, 2 \leq k \leq m, \) we have
\[
I_m^k := f_k A_k^k (C_m (\hat{\lambda})) - \hat{\lambda}_k A_k^k (C_m (\hat{\lambda} [k])) \geq 0, \quad 2 \leq k \leq m. \tag{24}
\]

Let us suppose that Lemma 16 holds. Using (13), (22)–(24) we have
\[
\Sigma_m \geq \sum_{n} \frac{e^{-1} \sum_{q=2}^{m} f_q A_{q}^{(n)} (C_m (\hat{\lambda}))}{\det C_m + \sum_{q=2}^{m} \hat{\lambda}_q A_{q}^{(n)} (C_m (\hat{\lambda} [q]))} \geq \sum_{n} \frac{\sum_{q=2}^{m} \hat{\lambda}_q A_{q}^{(n)} (C_m (\hat{\lambda} [q]))}{\det C_m}
\]
\[
\geq \sum_{n} \frac{\sum_{q=2}^{m} \hat{\lambda}_q A_{q}^{(n)} (C_m)}{\det C_m} = S_m. \tag{23}
\]

Finally we have \( \Sigma_m > S_m \).
Corollary 17. If $S^L_{kn}(\mu^m_B) = \infty$ for some $1 \leq k < n \leq m$ then one of the series $\Sigma^r_{pq}(m)$, $1 \leq r \leq p < q \leq m$, is divergent and hence by Lemma 15 we can approximate the corresponding variable $x_{pq}$.

Remark 18. The approximation of other variables $x_{pq}$, $1 \leq p < q \leq m$, follows the schema used in [23]. For the particular case $1 \leq m \leq 4$ see also the schema used in [22].

Further we can approximate the remaining variables $x_{kn}$, $1 \leq k \leq m < n$, as in [23]. This implies the inclusion $(\mathfrak{M}^m)^{'} \subset L^\infty(X^m, \mu^m_B)$ and so the irreducibility of the representation (see “The idea of the proof of irreducibility” at the beginning of Section 3).

Acknowledgments

The authors thank Prof. Mark Malamud for the references [27] and [12], dealing with Hadamard–Ficher’s inequality. The second author would like to thank the Institute of Applied Mathematics, University of Bonn for the hospitality. The financial support by the DFG project 436 UKR 113/72 is gratefully acknowledged.

Appendix A. The generalized characteristic polynomial and its properties

We define $G_m(\lambda)$ the generalization of the characteristic polynomial $p_C(t) = \det(tI - C)$, $t \in \mathbb{C}$, of the matrix $C \in \text{Mat}(m, \mathbb{C})$:

$$G_m(\lambda) = \det C_m(\lambda), \quad \lambda \in \mathbb{C}^m, \quad \text{where } C_m(\lambda) = C + \sum_{k=1}^{m} \lambda_k E_{kk}. \quad (25)$$

We denote by

$$M^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_r}(C) \quad (\text{respectively } A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_r}(C)), \quad 1 \leq i_1 < \ldots < i_r \leq m, \quad 1 \leq j_1 < \ldots < j_r \leq m,$$

the minors (respectively the cofactors) of the matrix $C$ with $i_1, i_2, \ldots, i_r$ rows and $j_1, j_2, \ldots, j_r$ columns. By definition

$$A^{1}_{12 \ldots m}(C) = M^{0}_{0}(C) = 1 \quad \text{and} \quad M^{12 \ldots m}_{12 \ldots m}(C) = A^{0}_{0}(C) = \det C.$$

Lemma A.1. For the generalized characteristic polynomial $G_m(\lambda)$ of $C \in \text{Mat}(m, \mathbb{C})$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{C}^m$ we have:

$$G_m(\lambda) = \det \left( C + \sum_{k=1}^{m} \lambda_k E_{kk} \right) = \det C + \sum_{r=1}^{m} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq m} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_r} A^{i_1 i_2 \ldots i_r}_{i_1 i_2 \ldots i_r}(C). \quad (26)$$

Remark A.2. If we set $\lambda_\alpha = \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_r}$ where $\alpha = (i_1, i_2, \ldots, i_r)$ and $A^\alpha_{\emptyset}(C) = A^{i_1 i_2 \ldots i_r}_{i_1 i_2 \ldots i_r}(C)$, $\lambda_\emptyset = 1$, $A^\emptyset_{\emptyset}(C) = \det C$ we may write (26) as follows:

$$G_m(\lambda) = \det C_m(\lambda) = \sum_{\emptyset \subseteq \alpha \subseteq \{1,2,\ldots,m\}} \lambda_\alpha A^\alpha_{\alpha}(C). \quad (27)$$
Proof. Probably Lemma A.1 is known in the literature, but since we do not know any precise reference, we provide here a direct proof. Obviously $G_m(\lambda)$ is a polynomial in the variables $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{C}^m$ of order $m$. A direct calculation gives us

$$\frac{\partial^r G_m(\lambda)}{\partial \lambda_1 \partial \lambda_2 \ldots \partial \lambda_r} = \begin{vmatrix}
1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\
& & \ddots & \ddots & \ddots & \ddots & \ddots \\
c_{1r+1} & c_{2r+1} & \ldots & c_{rr+1} + \lambda_{r+1} & \ldots & c_{r+1m} \\
& & & \ddots & \ddots & \ddots & \ddots \\
c_{1m} & c_{2m} & \ldots & c_{rm} & c_{r+1m} & \ldots & c_{mm} + \lambda_m
\end{vmatrix},$$

hence

$$\left. \frac{\partial^r G_m}{\partial \lambda_1 \partial \lambda_2 \ldots \partial \lambda_r} \right|_{\lambda = 0} = A_{12 \ldots r}^{12 \ldots r}(C).$$

Similarly we have for $1 \leq i_1 < i_2 < \cdots < i_r \leq m$

$$\left. \frac{\partial^r G_m}{\partial \lambda_{i_1} \partial \lambda_{i_2} \ldots \partial \lambda_{i_r}} \right|_{\lambda = 0} = A_{i_1i_2 \ldots i_r}^{i_1i_2 \ldots i_r}(C).$$

Lemma A.3. For $C \in \text{Mat}(m, \mathbb{C})$ and $\lambda \in \mathbb{C}^m$ we have

$$G_m(\lambda) = A_{00}^m(C_m(\lambda)) = \det C_m(\lambda) = \det C_m + \sum_{r=1}^{m} \lambda_r A_{r}^r(C_m(\lambda^{[r]})), \quad (28)$$

$$A_k^k(C_m(\lambda)) = A_k^k(C_m) + \sum_{r=1, r \neq k}^{m} \lambda_r A_{r}^k(C_m(\lambda^{[r]})), \quad (29)$$

$$G_m(\lambda) = A_{00}^m(C_m(\lambda)) = \det C_m(\lambda) = \det C_m + \sum_{r=1}^{m} \lambda_r A_{r}^r(C_m(\lambda^{[r]})), \quad (30)$$

$$A_k^k(C_m(\lambda)) = A_k^k(C_m) + \sum_{r=1, r \neq k}^{m} \lambda_r A_{r}^k(C_m(\lambda^{[r]})), \quad (31)$$

where for $\lambda \in \mathbb{C}^m$ and $1 \leq k \leq m$ we have set

$$\lambda^{[k]} = (0, \ldots, 0, \lambda_{k+1}, \ldots, \lambda_m), \quad \lambda^{[k]} = (\lambda_1, \lambda_2, \ldots, \lambda_k, 0, \ldots, 0). \quad (32)$$

Proof. We have for $m = 2$ using (26)

$$G_2(\lambda) = \det C_2 + \lambda_1 A_1^1(C_2) + \lambda_2 A_2^2(C_2) + \lambda_1 \lambda_2 A_{12}^{12}(C_2)$$

$$= \det C_2 + \lambda_1 [A_1^1(C_2) + \lambda_2 A_{12}^{12}(C_2)] + \lambda_2 A_2^2(C_2)$$

$$= \det C_2 + \lambda_1 A_1^1(C_2(\lambda^{[1]})) + \lambda_2 A_2^2(C_2(\lambda^{[2]})),$$
\[ G_2(\lambda) = \det C_2 + \lambda_1 A_1^1(C_2) + \lambda_2 A_2^2(C_2) + \lambda_1 A_1^{12}(C_2) \]
\[ = \det C_2 + \lambda_1 A_1^1(C_2(\lambda^{[1]})) + \lambda_2 A_2^2(C_2(\lambda^{[2]})). \]

For \( m = 3 \) we have
\[ G_3(\lambda) = \det C_3 + \lambda_1 A_1^1(C_3) + \lambda_2 A_2^2(C_3) + \lambda_3 A_3^3(C_3) + \lambda_1 \lambda_2 A_{12}^{12}(C_3) + \lambda_1 \lambda_3 A_{13}^{13}(C_3) \]
\[ + \lambda_2 \lambda_3 A_{23}^{23}(C_3) + \lambda_1 \lambda_2 \lambda_3 A_{123}^{123}(C_3) \]
\[ = \det C_2 + \lambda_1 A_1^{11}(C_3) + \lambda_2 A_2^{12}(C_3) + \lambda_3 A_3^{13}(C_3) + \lambda_1 \lambda_2 A_{12}^{12}(C_3) \]
\[ + \lambda_2 \lambda_3 A_{23}^{23}(C_3) + \lambda_1 \lambda_2 \lambda_3 A_{123}^{123}(C_3) \]
\[ = \det C_3 + \lambda_1 A_1^1(C_3(\lambda^{[1]})) + \lambda_2 A_2^2(C_3(\lambda^{[2]})) + \lambda_3 A_3^3(C_3(\lambda^{[3]})), \]
\[ G_3(\lambda) = \det C_3 + \lambda_1 A_1^1(C_3) + \lambda_2 A_2^2(C_3) + \lambda_1 A_1^{12}(C_3) \]
\[ + \lambda_2 A_2^{23}(C_3) + \lambda_1 \lambda_2 A_{123}^{123}(C_3) \]
\[ = \det C_3 + \lambda_1 A_1^1(C_3(\lambda^{[1]})) + \lambda_2 A_2^2(C_3(\lambda^{[2]})) + \lambda_3 A_3^3(C_3(\lambda^{[3]})). \]

For \( m > 3 \) the proof of (28) and (30) is the same. The identity (29) follows from (28) and (31) follows from (30). □

The proof of Lemma 16 is based on Lemmas A.4, A.6 and A.7 concerning the properties of a positive matrices.

**Lemma A.4.** (Sylvester [10, Chapter II, Section 3]) Let \( C \in \text{Mat}(n, \mathbb{R}) \) and \( 1 \leq p < n \). We consider a matrix \( B = (b_{ik})_{p+1}^n \) defined by \( b_{ik} = M_{12\ldots p}^{12\ldots p_i}(C) \). Then the following Sylvester determinant identity holds:
\[ \det B = [M_{12\ldots p}^{12\ldots p}(C)]^{n-p-1} \det C. \]

**Corollary A.5.** If \( p = n - 2 \) we have in particular
\[ \begin{vmatrix} A_n^p(C) & A_{n-1}^n(C) \\ A_{n-1}^n(C) & A_{n-1}^n(C) \end{vmatrix} = A_{n-1}^{n-1}(C)A_n^q(C). \]

For arbitrary \( 1 \leq p < q \leq n \) we have
\[ \begin{vmatrix} A_p^p(C) & A_q^p(C) \\ A_p^q(C) & A_q^q(C) \end{vmatrix} = A_q^p(C)A_p^q(C) \quad \text{or} \quad \begin{vmatrix} A_p^p(C) & A_q^p(C) \\ A_p^q(C) & A_q^q(C) \end{vmatrix} = A_q^p(C)A_p^q(C). \]

**Lemma A.6.** (Hadamard–Ficher’s inequality [12,13], see also [27]) For any positive definite matrix \( C \in \text{Mat}(m, \mathbb{R}), \ m \in \mathbb{N}, \) and any two subsets \( \alpha \) and \( \beta \) with \( \emptyset \subseteq \alpha, \beta \subseteq \{1, \ldots, m\} \) the following inequality holds:
\[ \begin{vmatrix} M(\alpha) & M(\alpha \cap \beta) \\ M(\alpha \cup \beta) & M(\beta) \end{vmatrix} = \begin{vmatrix} A(\hat{\alpha}) & A(\hat{\alpha} \cup \hat{\beta}) \\ A(\hat{\alpha} \cup \hat{\beta}) & A(\hat{\beta}) \end{vmatrix} \geq 0, \]
where $M(\alpha) = M^\alpha_\alpha(C)$, $A(\alpha) = A^\alpha_\alpha(C)$ and $\hat{\alpha} = \{1, \ldots, m\} \setminus \alpha$.

More precisely, see [12, p. 573]; [13, Chapter 2.5, Problem 36]. See also [27, Corollary 3.2, p. 34].

Let us set as before (see (25)) for $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$ and $C \in \text{Mat}(k, \mathbb{C})$

$$G_k(\lambda) = \det C_k(\lambda), \quad \text{where } C_k(\lambda) = C + \sum_{r=1}^{k} \lambda_r E_{rr}.$$  

In the following lemma we use the notation for $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$:

$$\lambda^{[l]} = (\lambda_1, \ldots, \lambda_{l-1}, 0, \lambda_{l+1}, \ldots, \lambda_k), \quad 1 \leq l \leq k,$$

and $G_l(\lambda) = M_{12 \ldots l}^{12 \ldots l}(C_k(\lambda))$, $1 \leq l \leq k$. For $\alpha$ and $\beta$ such that $\emptyset \subseteq \alpha \subseteq \{1, 2, \ldots, l\}$ and $\emptyset \subseteq \beta \subseteq \{l+1, \ldots, k\}$, with $l < k$, $C \in \text{Mat}(k, \mathbb{C})$ we set

$$(A^\alpha_\alpha(C))^{\beta} := A^{\alpha \cup \beta}_\alpha(C), \quad \text{and } G_l(\lambda)^{\beta} := \sum_{\emptyset \subseteq \alpha \subseteq \{1,2,\ldots,l\}} \lambda_\alpha A^{\alpha \cup \beta}_\alpha(C).$$  

By definition we have

$$G_l(\lambda) = A^{l+1 \ldots k}_l(A_{l+1 \ldots k}(C_k(\lambda))) = (A^\emptyset_\emptyset(C_k(\lambda)))^{l+1 \ldots k}_{l+1 \ldots k} = G_k(\lambda)^{l+1 \ldots k}_{l+1 \ldots k}.$$  

**Lemma A.7.** We have for $1 \leq p \leq l \leq k$ and $C \in \text{Mat}(k, \mathbb{C})$

$$\frac{G_k(\lambda)}{G_l(\lambda)} = \frac{G_k(\lambda)^{\emptyset l}_{\emptyset l} + \lambda_p G_k(\lambda)^{\emptyset l}_{\emptyset l} p}{G_k(\lambda)^{l+1 \ldots k}_{l+1 \ldots k} + \lambda_p G_k(\lambda)^{l+1 \ldots k}_{l+1 \ldots k} p}.$$  

For the positive definite matrix $C$ and $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$ with $\lambda_r \geq 0$, $r = 1, \ldots, k$, we have

$$\left(\frac{G_l(\lambda)}{G_k(\lambda)}\right)^2 \frac{\partial}{\partial \lambda_p} G_k(\lambda) = \left| \begin{array}{cc} G_k(\lambda)^{p l}_{p l} & G_k(\lambda)^{p l}_{l+1 \ldots k} \\ G_k(\lambda)^{l+1 \ldots k}_{p l} & G_k(\lambda)^{l+1 \ldots k}_{l+1 \ldots k} \end{array} \right| \geq 0.$$  

**Proof.** We have for $1 \leq p \leq l \leq k$

$$\frac{\partial G_k(\lambda)}{\partial \lambda_p} = \frac{\partial}{\partial \lambda_p} \det \left( C + \sum_{r=1}^{k} \lambda_r E_{rr} \right) = A^p_p(C(\lambda)) = A^p_p(C(\lambda)) = A^p_p(C(\lambda)), \quad \text{so}$$

$$G_k(\lambda) - \lambda_p G_k(\lambda)^{\emptyset l}_{\emptyset l} p = G_k(\lambda) \big|_{\lambda_p=0} = G_k(\lambda)^{\emptyset l}_{\emptyset l},$$

hence

$$G_k(\lambda) = G_k(\lambda)^{\emptyset l}_{\emptyset l} + \lambda_p G_k(\lambda)^{\emptyset l}_{\emptyset l} p, \quad 1 \leq p \leq k.$$
Similarly, we have
\[ G_l(\lambda) = G_l(\lambda^{p_l}) + \lambda_p G_l(\lambda^{p_l})^{(35)} = G_k(\lambda^{p_l})^{l+1...k} + \lambda_p G_k(\lambda^{p_l})^{pl+1...k}, \quad 1 \leq p \leq l. \]

Finally we get (36). Using the following formula:
\[ \left( \frac{a + bx}{c + dx} \right)' = \frac{bc - ad}{(c + dx)^2} \]
we conclude that (36) implies the identity in (37).

To prove the inequality in (36) we get
\[ \left| \frac{A_k(\lambda^{p_l})}{A_l^{pl+1...k}(\lambda^{p_l})} + \frac{A_{k+1}(\lambda^{p_l})}{A_{l+1}(\lambda^{p_l})} \right| \geq 0, \]}

where \( C = C_k(\lambda^{p_l}), \alpha = \{ p \} \) and \( \beta = \{ l+1, l+2, \ldots, k \} \).

Appendix B. Calculation of the matrix elements \( \phi_p(t) \) for \( t \in \mathbb{R}^p \), their generalizations and \( \Xi_{pq}^n \)

Let us recall (see (10) and (19)) that \( \hat{\lambda}_k = \sum_{r=1}^{k-1} c_{rr}, 2 \leq r \leq m, \hat{\lambda}_1 = 0 \) and
\[ \Xi_{pq}^n = \max_{t \in \mathbb{R}^p} |M\xi_{pq}^n(t)|^2, \quad 1 \leq p \leq q \leq m. \]}

To estimate
\[ \max_{t \in \mathbb{R}^p} |M\xi_{pq}^n(t)|^2 = \max_{t \in \mathbb{R}^p} (|\xi_{pq}^n(t)|^2)^2, \]
where \( \xi_{pq}^n(t) = iyqn \exp(\sum_{r=1}^p t_r \tilde{A}_{rn}) \) we shall find the exact formulas for the matrix elements
\[ \phi_p(t) = \phi_p^{(n)}(t) = \left( T_{R,\mu}^{R,m} \exp(\sum_{r=1}^p t_r E_{rn}) \right) \mathbf{1}, \quad t = (t_r)_{r=1}^p \in \mathbb{R}^p, \quad 1 \leq p \leq m, \]}

of the restriction of the representation \( T_{R,\mu}^{R,m} \) on the commutative subgroup \( \exp(\sum_{r=1}^p t_r E_{rn}) | t \in \mathbb{R}^p) \approx \mathbb{R}^p \) of the group \( B_0^N \) and theirs generalization defined below. We note that \( \exp(\sum_{r=1}^p t_r E_{rn}) = 1 + \sum_{r=1}^p t_r E_{rn}. \)

For \( 1 \leq p \leq q, p, q \in \mathbb{N} \) we get
\[ \xi_{pq}^n(t) = iyqn \exp \left( \sum_{r=1}^p t_r \tilde{A}_{rn} \right) = iyqn \exp \left[ \sum_{r=1}^p t_r \left( \sum_{k=1}^{r-1} x_{kr} y_{kn} + y_{rn} \right) \right], \]}
we have used the expression \( \tilde{A}_{rn} = \sum_{k=1}^{r-1} x_{kr} y_{kn} + y_{rn} = \sum_{k=1}^{r-1} x_{kr} y_{kn} \) (see (9)). We have
\[
\widetilde{\mathcal{R}}^R,\mu^m_B \exp(\sum_{r=1}^p t_r E_{rn}) = \exp \left( \sum_{r=1}^p t_r A_{rn} \right) = \exp \left[ \sum_{r=1}^p \left( \sum_{k=1}^r x_{kr} y_{kn} \right) \right] = \exp \left[ \sum_{k=1}^p \left( \sum_{r=k}^p x_{kr} t_r \right) y_{kn} \right].
\]

To obtain \(\xi^{pp}(t)\) we generalize the function

\[
\widetilde{\mathcal{R}}^R,\mu^m_B \exp(\sum_{r=1}^p t_r E_{rn})
\]

in the following way. We replace in the latter identity the vectors \((t_r, \ldots, t_r) \in \mathbb{R}^{p-k+1}\) by \((t_{rk})_{r=k}^p \in \mathbb{R}^{p-k+1}\) and denote the result by \(\xi^{pp}(t)\):

\[
\xi^{pp}(t) = \xi^{pp} \left( \begin{array}{c}
t_{11} \\
t_{21} \\
t_{22} \\
t_{31} \\
t_{32} \\
t_{33} \\
\vdots \\
t_{p1} \\
t_{p2} \\
t_{pp}
\end{array} \right) := \exp \left[ \sum_{k=1}^p \left( \sum_{r=k+1}^p x_{kr} t_{rk} + t_{kk} \right) y_{kn} \right]. \tag{42}
\]

To obtain \(\xi^{pq}(t)\) we consider the function \(\xi^{pq}(t; t_{qq}) = \xi^{pp}(t) \exp(it_{qq} y_{qn})\). We have

\[
\xi^{pq}(t; t_{qq}) = \xi^{pq} \left( \begin{array}{c}
t_{11} \\
t_{21} \\
t_{22} \\
t_{31} \\
t_{32} \\
t_{33} \\
\vdots \\
t_{p1} \\
t_{p2} \\
t_{pp}
\end{array} \right) := \exp \left[ \sum_{k=1}^p \left( \sum_{r=k+1}^p x_{kr} t_{rk} + t_{kk} \right) y_{kn} + t_{qq} y_{qn} \right].
\]

Finally we have

\[
\frac{\partial \xi^{pp}(t)}{\partial t_{pp}} \bigg|_{t_{kr}=t_k, 1 \leq r \leq k \leq p} \quad \text{and} \quad \frac{\partial \xi^{pq}(t; t_{qq})}{\partial t_{qq}} \bigg|_{t_{qq}=0, t_{kr}=t_k, 1 \leq r \leq k \leq p}.
\]

Let us define \(\phi_p(t) = \int \xi^{pp}(t) d\mu(x, y)\), \(\phi_{pq}(t) = \int \xi^{pq}(t) d\mu(x, y)\), where \(\mu(x, y) = \mu_1(x) \otimes \otimes_{n=m+1}^{\infty} \mu_{C(n)}(y)\) and \(\mu_1(x)\) is the standard Gaussian measure in \(\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}^m\).

Using definition (39) and the previous equalities we have finally

\[
\mathcal{E}^{pp} = \max_{t \in \mathbb{R}_p} \left| \frac{\partial \phi_p(t)}{\partial t_{pp}} \right|_{t_{kr}=t_k, 1 \leq r \leq k \leq p}^2, \quad \mathcal{E}^{pq} = \max_{t \in \mathbb{R}_p} \left| \frac{\partial \phi_{pq}(t; t_{qq})}{\partial t_{qq}} \right|_{t_{qq}=0, t_{kr}=t_k, 1 \leq r \leq k \leq p}^2. \tag{43}
\]

Our aim is to estimate \(\mathcal{E}^{pq}\). We shall use the notation \(C_k := C_{\{1, 2, \ldots, k\}}\) for \(\text{Mat}(m, \mathbb{C})\) and \(1 \leq k \leq m\) (see notation \(C_\alpha\) for \(\emptyset \subseteq \alpha \subseteq \{1, \ldots, m\}\) in Lemma B of Section 3).
Lemma B.1. For $1 \leq p \leq q \leq m$ and $\phi_{pq}(t) = \int \xi_{pq}(t) \, d\mu(x, y)$ we have

$$
\phi_{pq}
\begin{pmatrix}
  t_{11} & t_{21} & t_{31} & \cdots & t_{p1} \\
  t_{21} & t_{22} & t_{32} & \cdots & t_{p2} \\
  t_{31} & t_{32} & t_{33} & \cdots & t_{pp} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  t_{q1} & t_{q2} & t_{q3} & \cdots & t_{qq}
\end{pmatrix}

= \int_{\mathbb{R}^{(p-1)(p-2)/2} + p} \exp \left[ \sum_{k=1}^{p} \left( \sum_{r=k}^{p} x_{kr} t_{rk} \right) y_{kn} + t_{qq} y_{qn} \right] d\mu(x, y)

= \frac{1}{\sqrt{\det C_1(t)}} \exp \left( -\frac{1}{2} \left[ (CT, T) - (C_1(t)^{-1} d, d) \right] \right),
$$

where we set $T = (t_{11}, t_{22}, t_{33}, \ldots, t_{pp}; t_{qq}) \in \mathbb{R}^{p+1}$, $C \in \text{Mat}(p + 1, \mathbb{C})$ is defined by

$$
C := C_{p,q} := C_{[1,2,\ldots,p,q]} :=
\begin{pmatrix}
  c_{11} & c_{12} & c_{13} & \cdots & c_{1p} & c_{1q} \\
  c_{12} & c_{22} & c_{23} & \cdots & c_{2p} & c_{2q} \\
  c_{13} & c_{23} & c_{33} & \cdots & c_{3p} & c_{3q} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  c_{1p} & c_{2p} & c_{3p} & \cdots & c_{pp} & c_{pq} \\
  c_{1q} & c_{2q} & c_{3q} & \cdots & c_{pq} & c_{qq}
\end{pmatrix},
$$

$$
d = (d_{21}(t), d_{31}(t), \ldots, d_{p1}(t); d_{32}(t), d_{42}(t), \ldots, d_{p2}(t); \ldots; d_{pp-1}(t)) \in \mathbb{R}^{(p-1)(p-2)/2},
$$

$$
d_{rs}(t) = t_{rs}e_s(t), \quad 1 \leq s < r < p, \quad e_s(t) = (CT)_s = \sum_{k=1}^{p} c_{sk} t_{kk} + c_{sq} t_{qq}, \quad 1 \leq s \leq p,
$$

the operator

$$
C_1(t) = 1 + C(t) \in \text{Mat} \left( \frac{(p-1)(p-2)}{2}, \mathbb{C} \right)
$$

being defined by

$$
D(t)^{-1} C_1(t) D(t)^{-1}
\begin{pmatrix}
  c_{11} + t_{21}^{-2} & \cdots & c_{11} & c_{12} & \cdots & c_{12} & \cdots & c_{1p-1} \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  c_{11} & \cdots & c_{11} + t_{p1}^{-2} & c_{12} & \cdots & c_{12} & \cdots & c_{1p-1} \\
  c_{12} & \cdots & c_{12} & c_{22} + t_{32}^{-2} & \cdots & c_{22} & \cdots & c_{2p-1} \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  c_{12} & \cdots & c_{12} & c_{22} & \cdots & c_{22} + t_{p2}^{-2} & \cdots & c_{2p-1} \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  c_{1p-1} & \cdots & c_{1p-1} & c_{2p-1} & \cdots & c_{2p-1} & \cdots & c_{p-1p-1} + t_{pp-1}^{-2}
\end{pmatrix},
$$

(45)
where \( D(t) = \text{diag}(t_{21}, \ldots, t_{p1}; t_{32}, \ldots, t_{p2}; t_{43}, \ldots, t_{p3}; \ldots; t_{pp-1}) \). We have

\[
\det C_1(t) = 1 + \sum_{r=1}^{p} \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq p} \alpha_{i_1}^2 \alpha_{i_2}^2 \cdots \alpha_{i_r}^2 M_{i_1i_2\ldots i_r}^{1\ldots r}(C_p) \quad \alpha_k^2 := \sum_{s=k+1}^{p} t_{sk}^2. \tag{46}
\]

**Lemma B.2.** For \( 1 \leq p \leq q \leq m \) we have

\[
\Xi_{pq} \geq \Psi_{pq}, \tag{47}
\]

where

\[
\Psi_{pq} = \frac{(M_{12\ldots p-1q}^{12\ldots p-1q}(C_{p,q}))^2 \exp(-1)}{M_{12\ldots p-1}^{12\ldots p-1}(C_p)M_{12\ldots p}^{12\ldots p}(C_p) + \sum_{k=2}^{p} \hat{\lambda}_k(A_k^p(C_p))^2}. \tag{48}
\]

**List of formulas for** \( \Psi_{pq} \) **for small** \( p \) **and** \( p < q \).

\[
\Psi_{11} = c_{11} \exp(-1), \quad \Psi_{1q} = \frac{c_{1q}^2 \exp(-1)}{c_{11}}, \quad 1 \leq q, \tag{49}
\]

\[
\Psi_{22} = \frac{(M_{12}^{12})^2 \exp(-1)}{c_{11}(M_{12}^{12} + c_{11}^2)}, \quad \Psi_{2q} = \frac{(M_{1q}^{12})^2 \exp(-1)}{c_{11}(M_{12}^{12} + c_{11}^2)}, \quad 2 \leq q, \tag{50}
\]

\[
\Psi_{3q} = \frac{(M_{123}^{123})^2 \exp(-1)}{M_{12}^{12}M_{123}^{123} + c_{11}(M_{13}^{13})^2 + (c_{11} + c_{22})(M_{12}^{12})^2}, \quad 3 \leq q, \tag{51}
\]

\[
\Psi_{4q} = \frac{(M_{1234}^{1234})^2 \exp(-1)}{M_{123}^{123}M_{1234}^{1234} + c_{11}(M_{13}^{13})^2 + (c_{11} + c_{22})(M_{12}^{12})^2 + (c_{11} + c_{22} + c_{33})(M_{13}^{13})^2}. \tag{52}
\]

**Remark B.3.** We have \( \Psi_{pp} > \Psi_{0p}^{pp} \) where

\[
\Psi_{0p}^{pp} := \frac{(M_{12\ldots p-1p}^{12\ldots p-1p}(C_p))^{2e-1}}{A_p^p(C_p)G_p(\hat{\lambda})} = \frac{(A_p^p(C_p))^{2e-1}}{A_p^p(C_p)G_p(\hat{\lambda})} = \frac{(G_p(0))^{2e-1}}{A_p^p(C_p)G_p(\hat{\lambda})}, \tag{53}
\]

and

\[
\Psi_{pq} > \Psi_{0pq}^{pq} := \frac{(M_{12\ldots p-1q}^{12\ldots p-1q}(C_{p,q}))^{2e-1}}{A_p^p(C_p)G_p(\hat{\lambda})}. \tag{54}
\]

**Proof.** For positive definite matrix \( C_p \) we conclude by Sylvester lemma (see Lemma A.4 and (33) of Corollary A.5) that

\[
\begin{vmatrix}
A_k^k(C_p) & A_k^kp(C_p) \\
A_k^p(C_p) & A_p^p(C_p)
\end{vmatrix} = A_k^k(C_p)A_p^p(C_p) - A_k^kp(C_p)A_k^p(C_p) = (A_k^k(C_p))^2,
\]

where

\[
A_k^k(C_p) := \sum_{s=k+1}^{p} t_{sk}^2. \]
hence
\[(A_k^p(Cp))^2 < A_k^k(Cp)A_p^p(Cp), \quad 1 \leq k \leq p.\]

Using the latter inequality we get (see (48))
\[
M_{12\ldots p-1}^{12\ldots p}(Cp)M_{12\ldots p}^{12\ldots p}(Cp) + \sum_{k=2}^{p} \hat{\lambda}_k(A_k^p(Cp))^2
= A_p^p(Cp)A_p^p(Cp) + \sum_{k=2}^{p} \hat{\lambda}_k A_k^p(Cp) + \sum_{k=2}^{p} \hat{\lambda}_k A_k^k(Cp)
\leq A_p^p(Cp)G_p(\hat{\lambda}).\]

**Proof of Lemmas B.1 and B.2.** For a positive definite operator \(C\) in the space \(\mathbb{R}^m\) we have the well-known formulas:
\[
\frac{1}{\sqrt{(2\pi)^m}} \int_{\mathbb{R}^m} \exp\left(-\frac{1}{2}(Cx, x)\right) dx = \frac{1}{\sqrt{\det C}}.
\]  
(55)

Using formula (55) we obtain the following formula for \(d \in \mathbb{R}^m\):
\[
\frac{1}{\sqrt{(2\pi)^m}} \int_{\mathbb{R}^m} \exp\left(-\frac{1}{2}(Cx, x) + (d, x)\right) dx = \frac{1}{\sqrt{\det C}} \exp\left(\frac{(C^{-1}d, d)}{2}\right).
\]  
(56)

and as a particular case for \(m = 1\) we have
\[
\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(-\frac{1}{2}cx^2 + dx\right) dx = \frac{1}{\sqrt{c}} \exp\left(\frac{d^2}{2c}\right).
\]  
(57)

To obtain (56) from (55) we use the following formula:
\[
(Cx, x) - 2(d, x) = (C(x - x_0), (x - x_0)) - (C^{-1}d, d), \quad \text{where} \ x_0 = C^{-1}d.
\]  
(58)

Indeed we find \(x_0 \in \mathbb{R}^m\) and \(d_0 \in \mathbb{R}\) such that
\[
(Cx, x) - 2(d, x) = (C(x - x_0), (x - x_0)) + d_0.
\]

We have
\[
(Cx, x) - 2(d, x) = (C(x - x_0), (x - x_0)) + d_0 = (Cx, x) - 2(Cx_0, x) + (Cx_0, x_0) + d_0,
\]
so \(Cx_0 = d\) hence \(x_0 = C^{-1}d\) and since \((Cx_0, x_0) + d_0 = 0\) we conclude that \(d_0 = -\langle Cx_0, x_0 \rangle = -(CC^{-1}d, C^{-1}d) = -(C^{-1}d, d).\)
Fourier transform for the Gaussian measure $\mu_C$ in the space $\mathbb{R}^m$ is:

$$
\frac{1}{\sqrt{(2\pi)^m}} \int_{\mathbb{R}^m} \exp i(y, x) d\mu_C(x) = \exp \left(-\frac{1}{2} (Cy, y)\right), \quad y \in \mathbb{R}^m.
$$

Let $p = 1$. Using (55)–(57) we have

$$
\phi_1(t_{11}) = \int_{\mathbb{R}} \exp(it_{11}y_1n) d\mu(y) = \exp\left(-\frac{1}{2} c_{11}t_{11}^2\right);
$$

$$
\phi_1(q(t_{11}; t_{qq})) = \int_{\mathbb{R}^2} \exp i(t_{11}y_1n + t_{qq}y_{qn}) d\mu(y) = \exp\left(-\frac{1}{2} (c_{11}t_{11}^2 + 2c_{1q}t_{11}t_{qq} + c_{qq}t_{qq}^2)\right);
$$

$$
M^{1,q}_1(t_{11}) = \int_{\mathbb{R}} iy_{qn} \exp(it_{11}y_1n) d\mu(y) = \left. \frac{\partial \phi_1(q(t_{11}; t_{qq}))}{\partial t_{qq}} \right|_{t_{qq} = 0} = -c_{1q}t_{11} \exp\left(-\frac{1}{2} c_{11}t_{11}^2\right),
$$

$$
|M^{1,q}_1(t_{11})|^2 = c_{1q}^2 t_{11}^2 \exp(-c_{11}t_{11}^2);
$$

$$
\Xi^{1q} = \max_{t_{11} \in \mathbb{R}} |M^{1,q}_1(t_{11})|^2 = \frac{c_{1q}^2 \exp(-1)}{c_{11}} = \Psi^{1q},
$$

we have used the obvious result

$$
\max_{x \in \mathbb{R}} f(x) = f\left(\frac{1}{a}\right) = \frac{1}{ea}, \quad \text{where } f(x) = x \exp(-ax), \quad a > 0.
$$

This proves (47) for $(p, q) = (1, q)$.

To prove (44) in the general case we note that

$$
\sum_{k=1}^{p} \left( \sum_{r=k+1}^{p} x_{kr}t_{rk} + t_{kk} \right) y_{kn} + t_{qq}y_{qn} = (a(x) + T, y)_{\mathbb{R}^{p+1}},
$$

where

$$
y = (y_{1n}, y_{2n}, \ldots, y_{pn}; y_{qn}), \quad T = (t_{11}, t_{22}, \ldots, t_{pp}; t_{qq}) \in \mathbb{R}^{p+1},
$$

$$
a(x) = (a_1(x), a_2(x), \ldots, a_p(x); 0) \in \mathbb{R}^{p+1}, \quad a_k(x) = \sum_{r=k+1}^{p} x_{kr}t_{rk} = (xt)_{kk},
$$

$$
x = \sum_{1 < k < r \leq m} x_{kr}E_{kr}, \quad t = \sum_{1 < r < k \leq m} t_{kr}E_{kr}, \quad 1 \leq k \leq p.
$$

Using the definition of the Fourier transform we have
\[ \phi_{pq}(t; t_{qq}) = \int \int_{\mathbb{R}^{p+1}} \exp \left[ \sum_{k=1}^{p} \left( \sum_{r=k}^{p} x_{kr}t_{rk} \right) \right] \left( y_{kn} + t_{qq}y_{qn} \right) \] 
\[ \times \exp i \left( a(x) + T, y \right) d\mu(x, y) \] 
\[ = \int \exp i \left( a(x) + T, y \right) d\mu(x, y) = \int \exp \left[ -\frac{1}{2} (C(a(x) + T), a(x) + T) \right] d\mu_I(x). \]

Since
\[ (C(a(x) + T), a(x) + T) = (Ca(x), a(x)) + 2(a(x), CT) + (CT, T), \]
we have
\[ \phi_{pq}(t; t_{qq}) = \exp \left[ -\frac{1}{2} (CT, T) \right] \int \exp \left( -\frac{1}{2} \left[ (Ca(x), a(x)) + 2(a(x), CT) \right] \right) d\mu_I(x). \] (60)

To calculate the latter integral we use (56). Let us introduce the notation
\[ X = (x_{12}, x_{13}, x_{23}; \ldots; x_{1p}, \ldots; x_{p-1}p) \in \mathbb{R}^{(p-1)(p-2)/2}. \]

We show that
\[ (Ca(x), a(x)) + 2(a(x), CT) = (C(t)X, X) + 2(d(t), X) \]
for some
\[ d(t) \in \mathbb{R}^{(p-1)(p-2)/2} \text{ and } C(t) \in \text{Mat} \left( \frac{(p-1)(p-2)}{2}, \mathbb{R} \right). \]

We have
\[ (a(x), CT) = \sum_{k=1}^{p} a_k(x)(CT)_k = \sum_{k=1}^{p} \sum_{r=k+1}^{p} x_{kr}t_{rk}e_k(t) = \sum_{1 \leq k < r \leq p} x_{kr}t_{rk}e_k(t) = \sum_{1 \leq k < r \leq p} x_{kr}d_{rk}(t) = (X, d(t)), \]

where
\[ d(t) = (d_{rk}(t))_{1 \leq k < r \leq p} \in \mathbb{R}^{(p-1)(p-2)/2}, \]
\[ d_{rk}(t) = t_{rk}e_k(t) \text{ and } e_k(t) = (CT)_k = \sum_{r=1}^{p} c_{kr}t_{rr} + c_{kq}t_{qq}, 1 \leq k \leq p - 1. \]

Further
\[(Ca(x), a(x)) = \sum_{1 \leq k, n \leq p} c_{kn} a_k(x)a_n(x) = \sum_{1 \leq k, n \leq p} \sum_{r=k+1}^{p} x_{kr}t_{rk} \sum_{s=n+1}^{p} x_{ns}t_{sn}\]

\[= \sum_{1 \leq k < r \leq p} \sum_{1 \leq n < s \leq p} c_{kn}t_{rk}x_{kr}x_{ns} = (C(t)X, X),\]

where the operator \(C(t)\) is defined by its entries:

\[(C(t))_{kr,ns} = c_{kn}t_{rk}t_{sn} \quad \text{for } 1 \leq k < r \leq p \text{ and } 1 \leq n < s \leq p.\] (61)

This prove the representation (45) for the operator \(C_1(t)\). Finally we have

\[(Ca(x), a(x)) = (C(t)X, X) \quad \text{and} \quad (a(x), CT) = (X, d(t)).\]

Putting the latter equalities in (60) we get using (56)

\[
\phi_{pq}(t; t_{qq}) = \exp\left[-\frac{1}{2}(CT, T)\right] \int \exp\left(-\frac{1}{2}\left[(C(t)X, X) + 2(X, d(t))\right]\right) d\mu_I(x)
\]

\[= \frac{1}{\sqrt{\det C_1(t)}} \exp\left(-\frac{1}{2}\left[(CT, T) - (C_1(t)^{-1}d(t), d(t))\right]\right),\]

where \(C_1(t) = I + C(t)\). This proves (44) of Lemma B.1.

We estimate now \(\Sigma^{pq}\). For \((p, q) = (2, 2)\) we get

\[
\phi_2(t) = \frac{1}{\sqrt{\det C_1(t)}} \exp\left(-\frac{1}{2}\left[(CT, T) - (C_1(t)^{-1}d(t), d(t))\right]\right)
\]

\[= \frac{1}{\sqrt{1 + c_{11}t_{21}^2}} \exp\left(-\frac{1}{2}\left[1 + c_{11}t_{21}^2 + 2c_{12}t_{11}t_{22} + c_{22}t_{22}^2 - \frac{(c_{11}t_{11} + c_{12}t_{22})^2t_{21}^2}{1 + c_{11}t_{21}^2}\right]\right),\]

where

\[T = (t_{11}, t_{22}), \quad d(t) = d_{21}(t) = t_{21}e_1(t) = t_{21}(c_{11}t_{11} + c_{12}t_{22}),\]
\[e_1(t) = c_{11}t_{11} + c_{12}t_{22}, \quad e_2(t) = c_{21}t_{11} + c_{22}t_{22},\]
\[C = C_2 = \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix}, \quad C(t) = c_{11}t_{21}^2, \quad C_1(t) = 1 + c_{11}t_{21}^2,\]

\[
\frac{\partial \phi_2(t)}{\partial t_{11}} = \left[-(c_{11}t_{11} + c_{12}t_{22}) + \frac{(c_{11}t_{11} + c_{12}t_{22})c_{11}t_{21}^2}{1 + c_{11}t_{21}^2}\right]
\]

\[\times \frac{\exp\left(-\frac{1}{2}\left[(CT, T) - (C_1(t)^{-1}d(t), d(t))\right]\right)}{\sqrt{\det C_1(t)}}.\]

\[
\frac{\partial \phi_2(t)}{\partial t_{22}} = \left[-(c_{21}t_{11} + c_{22}t_{22}) + \frac{(c_{11}t_{11} + c_{12}t_{22})c_{12}t_{21}^2}{1 + c_{11}t_{21}^2}\right]
\]

\[\times \frac{\exp\left(-\frac{1}{2}\left[(CT, T) - (C_1(t)^{-1}d(t), d(t))\right]\right)}{\sqrt{\det C_1(t)}}.\]
Let \( e_1(t) = c_{11}t_{11} + c_{12}t_{22} = 0 \) so \( t_{11} = -c_{12}t_{22}/c_{11} \). In this case

\[
(CT, T) = c_{11}t_{11}^2 + 2c_{12}t_{11}t_{22} + c_{22}t_{22}^2 = \left(\frac{c_{12}}{c_{11}} - 2\frac{c_{12}}{c_{11}} + c_{22}\right)t_{22}^2 = \frac{M_{12}^2}{c_{11}}t_{22}^2,
\]

\[
c_{12}t_{11} + c_{22}t_{22} = \left( -\frac{c_{12}c_{12}}{c_{11}} + c_{22}\right)t_{22} = \frac{c_{11}c_{22} - c_{12}^2}{c_{11}} = \frac{M_{12}^2}{c_{11}}.
\]

Finally

\[
\left| M_{\xi}^{22}(t) \right|^2 = \left| Mi_{y_{2n}}\exp(it_{11} + it_{22}(x_{12}y_{1n} + y_{2n})) \right|^2 = \left| \frac{\partial \phi_2(t)}{\partial t_{22}} \right|_{e_1(t)=0; t_{21} = t_{22}}^2
\]

\[
= \left( \frac{M_{12}^2}{c_{11}}t_{22} \right)^2 \exp\left(-\frac{M_{12}^2}{c_{11}}t_{22}^2\right) \geq \frac{M_{12}^2}{c_{11}}t_{22}^2 \exp\left[-\left(\frac{M_{12}^2}{c_{11}} + c_{11}\right)t_{22}^2 \right].
\]

We have used the inequality

\[1 + x \leq \exp x, \quad x \in \mathbb{R}.\] (62)

Hence if we denote \( t = (t_{11}, t_{22}) \in \mathbb{R}^2 \) we have using (43)

\[\mathcal{S}^{22} = \max_{t \in \mathbb{R}^2} \left| M_{\xi}^{22}(t) \right|^2 > \Psi^{22} := \frac{(M_{12}^2)^2\exp(-1)}{c_{11}(M_{12}^2 + c_{11}^2)}.
\]

This proves (47) for \((p, q) = (2, 2)\). For \((2, q)\), \(2 < q\), we have

\[
\phi_{2q}(t_{11}^{t_{22}}, t_{qq}) = \int_{\mathbb{R}^{1+3}} \exp i\left[t_{11}y_{1n} + (t_{21}x_{12}y_{1n} + t_{22}y_{2n}) + t_{qq}y_{qn}\right] d\mu(x, y)
\]

\[
= \frac{1}{\sqrt{1 + c_{11}t_{22}^2}} \exp\left(-\frac{1}{2}\left[c_{11}t_{11}^2 + c_{22}t_{22}^2 + c_{qq}t_{qq}^2 + 2c_{12}t_{11}t_{22}
\right.
\]

\[
\left. + 2c_1q t_{11}t_{qq} + 2c_2q t_{22}t_{qq} - \frac{(c_{11}t_{11} + c_{12}t_{22} + c_{1q}t_{qq})^2}{1 + c_{11}t_{22}^2}\right)d\mu(x, y)
\]

\[
= \frac{1}{\sqrt{\text{det} C_1(t)}} \exp\left(-\frac{1}{2}\left[(CT, T) - (C_1(t)^{-1}d(t), d(t))\right]\right),
\]

where

\[
T = (t_{11}, t_{22}; t_{qq}) \in \mathbb{R}^3, \quad d(t) = t_{21}(c_{11}t_{11} + c_{12}t_{22} + c_{1q}t_{qq}) =: t_{21}e_1(t) \in \mathbb{R},
\]

\[
e_1(t) = c_{11}t_{11} + c_{12}t_{22} + c_{1q}t_{qq}, \quad e_2(t) = c_{21}t_{11} + c_{22}t_{22} + c_{2q}t_{qq},
\]

\[
C = C_{2,q} = \begin{pmatrix} c_{11} & c_{12} & c_{1q} \\
 c_{12} & c_{22} & c_{2q} \\
 c_{1q} & c_{2q} & c_{qq} \end{pmatrix}, \quad C_1(t) = \text{det} C_1(t) = 1 + c_{11}t_{21}^2,
\]
By (59) we conclude using (43) that
\[
(CT, T) = c_{11}t_{11}^2 + 2c_{12}t_{11}t_{22} + c_{22}t_{22}^2 = \left(\frac{c_{12}}{c_{11}} - 2\frac{c_{12}^2}{c_{11}^2} + c_{12}\right)t_{22}^2 = \frac{M_{12}^2}{c_{11}}t_{22}^2,
\]
\[c_{1q}t_{11} + c_{2q}t_{22} = \left(-\frac{c_{12}c_{1q}}{c_{11}} + c_{2q}\right)t_{22} = \frac{c_{11}c_{2q} - c_{12}c_{1q}}{c_{11}}t_{22} = \frac{M_{1q}^2}{c_{11}}t_{22}.
\]
Finally, if we denote \( t = (t_{11}, t_{22}) \in \mathbb{R}^2 \), we have
\[
\left|M_{\xi}^{2q}(t)\right|^2 = \left|M_{iy_{12n}} \exp(\imath t_{11} + \imath t_{22}(x_{12}y_{1n} + y_{2n}))\right|^2 = \left|\frac{\partial \phi_2(t; t_{qq})}{\partial t_{qq}}\right|_{t_{qq}=0, c_1(t)=0}^2
\]
\[
= \left(\frac{M_{12}^2}{c_{11}}t_{22}\right)^2 \exp\left(-\frac{M_{12}^2}{c_{11}}t_{22}^2\right) \left(\frac{M_{1q}^2}{c_{11}}t_{22}\right)^2 \exp\left(-\left(\frac{M_{12}^2}{c_{11}} + c_{11}\right)t_{22}^2\right).
\]

By (59) we conclude using (43) that
\[
\mathcal{E}_{2q} = \max_{t_{qq}\in\mathbb{R}^2} \left|M_{\xi}^{2q}(t)\right|^2 \geq \max_{t_{qq}\in\mathbb{R}^2} \left|\frac{\partial \phi_2(t; t_{qq})}{\partial t_{qq}}\right|^2_{t_{qq}=0, e_1(t)=0} \geq \frac{(M_{1q}^2)^2 \exp(-1)}{c_{11}(M_{12}^2 + c_{11}^2)} = \psi^{2q}.
\]
This proves (47) for \((p, q) = (2, q), 2 < q\).

For \( n = 3 \) we have
\[\phi_3 \left(\begin{array}{ccc}
t_{11} & t_{21} & t_{22} \\
t_{31} & t_{32} & t_{33}
\end{array}\right) = \frac{1}{\sqrt{\det C_1(t)}} \exp\left(-\frac{1}{2}[(CT, T) - (C_1(t)^{-1}d, d)]\right),\]

where
\[
T = (t_{11}, t_{22}, t_{33}), \quad d(t) = (d_{21}(t), d_{31}(t), d_{32}(t)),
\]
\[
d_{21}(t) = t_{21}e_1(t), \quad d_{31}(t) = t_{31}e_1(t), \quad d_{32}(t) = t_{32}e_2(t),
\]
\[
e_1(t) = c_{11}t_{11} + c_{12}t_{22} + c_{13}t_{33}, \quad e_2(t) = c_{21}t_{11} + c_{22}t_{22} + c_{23}t_{33},
\]

Finally, if we denote \( t_{qq} = 0 \). We chose \( d(t) = 0 \) so we have \( c_{11}t_{11} + c_{12}t_{22} = 0 \) and \( t_{11} = -\frac{c_{12}t_{22}}{c_{11}} \). In this case
\[
(CT, T) = c_{11}t_{11}^2 + 2c_{12}t_{11}t_{22} + c_{22}t_{22}^2 = \frac{M_{12}^2}{c_{11}}t_{22}^2,
\]
\[
c_{1q}t_{11} + c_{2q}t_{22} = \left(-\frac{c_{12}c_{1q}}{c_{11}} + c_{2q}\right)t_{22} = \frac{c_{11}c_{2q} - c_{12}c_{1q}}{c_{11}}t_{22} = \frac{M_{1q}^2}{c_{11}}t_{22}.
\]
\[
C = C_3 = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{pmatrix}, \quad C(t) = \begin{pmatrix} c_{11}t_{21}^2 & c_{11}t_{21}t_{31} & c_{12}t_{21}t_{32} \\ c_{11}t_{21}t_{31} & c_{11}t_{31}^2 & c_{12}t_{31}t_{32} \\ c_{12}t_{21}t_{32} & c_{12}t_{31}t_{32} & c_{22}t_{32}^2 \end{pmatrix},
\]

hence

\[
C_1(t) = I + C(t) = \begin{pmatrix} 1 + c_{11}t_{21}^2 & c_{11}t_{21}t_{31} & c_{12}t_{21}t_{32} \\ c_{11}t_{21}t_{31} & 1 + c_{11}t_{31}^2 & c_{12}t_{31}t_{32} \\ c_{12}t_{21}t_{32} & c_{12}t_{31}t_{32} & 1 + c_{22}t_{32}^2 \end{pmatrix} = \text{diag}(t_{21}, t_{31}, t_{32})\begin{pmatrix} c_{11} + t_{21}^{-2} & c_{11} & c_{12} \\ c_{11} & c_{11} + t_{31}^{-2} & c_{12} \\ c_{12} & c_{12} & c_{22} + t_{32}^{-2} \end{pmatrix}\text{diag}(t_{21}, t_{31}, t_{32}).
\]

We prove the following inequality for an operator \(C\) of order \(n\) such that \(I + C > 0\):

\[
\det(I + C) \leq \exp \text{tr } C. \tag{63}
\]

Indeed by Hadamard inequality (see [7] or [13, Section 2.5.4]) we have for positive operator \(C\) of order \(n\)

\[
\det C \leq \prod_{i=1}^{n} c_{ii}.
\]

Using the Hadamard inequality and (62) we have for an operator \(C\) such that \(I + C > 0\)

\[
\det(I + C) \leq \prod_{i=1}^{n} (1 + c_{ii}) \leq \prod_{i=1}^{n} \exp c_{ii} = \exp \left( \sum_{i=1}^{n} c_{ii} \right) = \exp(\text{tr } C),
\]

where we denote by \(\text{tr } C\) the trace of an operator \(C\) in the space \(\mathbb{C}^n\). Using (63) and (61) we conclude that

\[
\det(I + C(t)) \leq \text{tr } C(t) = \exp \left[ \sum_{k=1}^{p-1} c_{kk} \left( \sum_{r=k+1}^{p} t_{rk}^2 \right) \right] = \exp \left( \sum_{k=1}^{p-1} c_{kk} \alpha_k^2 \right),
\]

where \(\alpha_k^2 = \sum_{r=k+1}^{p} t_{rk}^2\) since by (61) we have

\[
\text{tr } C(t) = \sum_{1 \leq k < r \leq p} C(t)_{kr,kr} = \sum_{1 \leq k < r \leq p} c_{kk}t_{rk}^2 = \sum_{k=1}^{p-1} c_{kk} \left( \sum_{r=k+1}^{p} t_{rk}^2 \right). \tag{65}
\]

Using (26) we get

\[
\det C_1(t) = t_{21}^2 t_{31}^2 t_{32}^2 (\det B + \lambda_1 A_1^1 + \lambda_1 A_2^2 + \lambda_3 A_3^3 + \lambda_1 \lambda_2 A_{12}^{12} + \lambda_1 \lambda_3 A_{13}^{13} + \lambda_2 \lambda_3 A_{23}^{23} + \lambda_1 \lambda_2 \lambda_3 A_{123}^{123})
\]
\[ \frac{\partial \phi}{\partial t} = \left( \begin{array}{ccc} c_{11} & c_{11} & c_{12} \\ c_{11} & c_{11} & c_{22} \\ c_{12} & c_{12} & c_{22} \end{array} \right) + \left( \begin{array}{ccc} \frac{1}{t_{21}} & \frac{1}{t_{21}} & 0 \\ 0 & \frac{1}{t_{21}} & \frac{1}{t_{21}} \\ 0 & 0 & \frac{1}{t_{21}} \end{array} \right) \left( \begin{array}{ccc} c_{11} & c_{12} \\ c_{12} & c_{22} \end{array} \right) \]
\[ + \frac{1}{t_{21}^2} c_{22} + \left( \begin{array}{ccc} \frac{1}{t_{21}^2} & \frac{1}{t_{21}^2} & 0 \\ 0 & \frac{1}{t_{21}^2} & \frac{1}{t_{21}^2} \\ 0 & 0 & \frac{1}{t_{21}^2} \end{array} \right) \left( \begin{array}{ccc} c_{11} & 0 \\ 0 & c_{22} \end{array} \right) \]
\[ = 1 + c_{11}(t_{21}^2 + t_{31}^2) + c_{22}t_{32}^2 + M_{12}^2(t_{21}^2 + t_{31}^2)t_{32}^2. \]

Finally we have
\[ \det C_1(t) = 1 + c_{11}a_1^2 + c_{22}a_2^2 + M_{12}^2a_1^2a_2^2, \quad \text{where } a_1^2 = t_{21}^2 + t_{31}^2, \ a_2^2 = t_{32}^2. \]

For general \( n \) we have by analogy (it proves thus (46))
\[ \det C_1(t) = 1 + \sum_{r=1}^{n-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq n-1} \alpha_1^{i_1} \alpha_2^{i_2} \cdots \alpha_r^{i_r} M_{i_1i_2\cdots i_r}^r(C_n), \quad \text{where } \alpha_k = \sum_{s=k+1}^{n} t_{sk}. \]

For \( n = 3 \) we have
\[ \frac{\partial \phi_3(t)}{\partial t_{33}} = \left[ -\frac{1}{2} \frac{\partial (CT, T)}{\partial t_{33}} + \frac{\partial (C_1(t) - d(t), d(t))}{\partial t_{33}} \right] \exp\left( -\frac{1}{2} [(CT, T) - (C_1(t) - d(t), d(t))] \right), \]
\[ \frac{\partial \phi_3(t)}{\partial t_{33}} = \left[ -e_3(t) + \frac{\partial (C_1(t) - d(t), d(t))}{\partial t_{33}} \right] \exp\left( -\frac{1}{2} [(CT, T) - (C_1(t) - d(t), d(t))] \right). \]

We calculate \( |\partial \phi_3(t)/\partial t_{33}|^2 \) under the conditions \( e_1(t) = e_2(t) = 0 \) on the variables \( t = (t_{11}, t_{22}, t_{33}) \in \mathbb{R}^3 \). It gives us
\[ \left\{ \begin{array}{l} c_{11}t_{11} + c_{12}t_{22} + c_{13}t_{33} = 0, \\
 c_{21}t_{11} + c_{22}t_{22} + c_{23}t_{33} = 0. \end{array} \right. \]

The solutions are
\[ t_{11} = \frac{M_{23}(C_3)}{M_{12}(C_3)} t_{33} = \frac{A_3^3(C_3)}{A_3^3(C_3)} t_{33}, \quad t_{22} = -\frac{M_{13}(C_3)}{M_{12}(C_3)} t_{33} = \frac{A_3^2(C_3)}{A_3^3(C_3)} t_{33}. \quad (66) \]

In general, for the matrix \( C_n \) conditions \( e_1(t) = e_2(t) = \cdots = e_{n-1}(t) = 0 \) gives us the system
\[ \left\{ \begin{array}{l} c_{11}t_{11} + c_{12}t_{22} + \cdots + c_{1n}t_{nn} = 0, \\
 c_{21}t_{11} + c_{22}t_{22} + \cdots + c_{2n}t_{nn} = 0, \\
 \vdots \\
 c_{n-1}t_{11} + c_{n-2}t_{22} + \cdots + c_{n-1}t_{nn} = 0. \end{array} \right. \quad (67) \]

and the following solutions:
\[ t_{kk} = (-1)^k t_{kk} = \frac{M_{12\cdots k-1k+1\cdots n-1}(C_n)}{M_{12\cdots n-1}(C_n)} t_{nn} = \frac{A_n^k(C_n)}{A_n^k(C_n)} t_{nn}, \quad 1 \leq k \leq n - 1. \quad (68) \]
If we denote \( e_k(t) = \sum_{r=1}^{n} c_{kr} t_{rr} \) we get

\[
(CT, T) = \sum_{1 \leq k, r \leq n} c_{kr} t_{rr} t_{kk} = \sum_{k=1}^{n} e_k(t) t_{kk},
\]

\[
\frac{1}{2} \frac{\partial (CT, T)}{\partial t_{nn}} = e_n(t).
\]

(69)

Under conditions (67) we have

\[
e_n(t) = \sum_{r=1}^{n} c_{nr} A_{rr}^{n}(C_n) / A_{nn}^{n}(C_n) t_{nn} = M_{12 \cdots n}^{12 \cdots n-1}(C_n) t_{nn},
\]

\[
\frac{\partial (C_1(t)^{-1} d(t), d(t))}{\partial t_{nn}} = 0
\]

(70)

and

\[
(CT, T) = \sum_{k=1}^{n} e_k(t) t_{kk} = e_n(t) t_{nn} = M_{12 \cdots n}^{12 \cdots n-1}(C_n) t_{nn}.
\]

(71)

For \( n = 3 \) using (70) and (71) we can calculate

\[
e_3(t) = M_{123}^{123}(C_3) / M_{12}^{12}(C_3) t_{33},
\]

\[
(CT, T) = M_{123}^{123}(C_3) / M_{12}^{12}(C_3) t_{33},
\]

\[
\frac{\partial (C_1(t)^{-1} d(t), d(t))}{\partial t_{33}} = 0.
\]

For \( n = 3 \) we have if \( e_1(t) = e_2(t) = 0 \), using the values for \( t_{22}, e_3(t) \) and \( (CT, T) \)

\[
\left| \frac{\partial \phi_3(t)}{\partial t_{33}} \right|^2 = \frac{e_3^2(t) \exp(- (CT, T))}{\det C_1(t)} \geq e_3^2(t) \exp(- (CT, T) - \text{tr } C(t))
\]

\[
= \left( \frac{M_{123}^{123}(C_3)}{M_{12}^{12}(C_3)} \right)^2 t_{33}^2 \exp \left[ -t_{33}^2 \left( \frac{M_{123}^{123}(C_3)}{M_{12}^{12}(C_3)} + \frac{c_{11} + c_{22}}{c_{11} + c_{22} + c_{11} \left( \frac{M_{13}^{12}(C_3)}{M_{12}^{12}(C_3)} \right)^2} \right) \right].
\]

We get by (59)

\[
\max_{t_{33} \in \mathbb{R}} \left( \frac{M_{123}^{123}(C_3)}{M_{12}^{12}(C_3)} \right)^2 t_{33}^2 \exp \left[ -t_{33}^2 \left( \frac{M_{123}^{123}(C_3)}{M_{12}^{12}(C_3)} + \frac{c_{11} + c_{22}}{c_{11} + c_{22} + c_{11} \left( \frac{M_{13}^{12}(C_3)}{M_{12}^{12}(C_3)} \right)^2} \right) \right]
\]

\[
= \frac{M_{123}^{123}(C_3)}{M_{12}^{12}(C_3)} \exp(-1)
\]

\[
= \frac{M_{123}^{123}(C_3)^2 \exp(-1)}{M_{12}^{12}(C_3) M_{123}^{123}(C_3) + c_{11} (M_{13}^{12}(C_3))^2 + (c_{11} + c_{22})(M_{12}^{12}(C_3))^2} = \Psi_{33}.
\]

Finally we have (see (43))
\[ S^{33} = \max_{t \in \mathbb{R}^2} |M^{33}_{s}(t)|^2 \geq \max_{t_{33} \in \mathbb{R}} \left| \frac{\partial \varphi_3(t)}{\partial t_{33}} \right|^2_{e_1(t) = e_2(t) = 0} \geq \Psi^{33}. \]

This proves (47) for \((p, q) = (3, 3)\).

By analogy we have for general \(n\):

\[ \frac{\partial \varphi_n(t)}{\partial t_{nn}} = \left[ -\frac{1}{2} \frac{\partial (C(T), T)}{\partial t_{nn}} + \frac{\partial (C_1(t)^{-1}d(t), d(t))}{\partial t_{nn}} \right] \exp\left(-\frac{1}{2}[(C(T), T) - (C_1(t)^{-1}d(t), d(t)))]\right), \]

\[ \frac{\partial \varphi_n(t)}{\partial t_{nn}} = \left[ -e_n(t) + \frac{\partial (C_1(t)^{-1}d(t), d(t))}{\partial t_{nn}} \right] \exp\left(-\frac{1}{2}[(C(T), T) - (C_1(t)^{-1}d(t), d(t)))]\right). \]

When \(t_{rk} = t_{rr}, n \geq r \geq k \geq 2\), we have by (65)

\[ \text{tr } C(t) = \sum_{1 \leq k < r \leq n} c_{kk} t_{rk}^2 = \sum_{k=1}^{n-1} c_{kk} \left( \sum_{r=k+1}^{n} t_{rk}^2 \right) = \sum_{k=1}^{n-1} c_{kk} \left( \sum_{r=k+1}^{n} t_{rr}^2 \right). \]

When, in addition, \(e_1(t) = \cdots = e_{n-1}(t) = 0\) we get (see (68) and definition (19) of \(\hat{\lambda}_k\))

\[ \text{tr } C(t) = \sum_{r=1}^{n-1} c_{rr} \left( \sum_{k=r+1}^{n} t_{kk}^2 \right) = \sum_{k=2}^{n} \sum_{r=k+1}^{n} c_{rr} t_{kk}^2 = \sum_{k=2}^{n} \hat{\lambda}_k t_{kk}^2 = \sum_{k=2}^{n} \hat{\lambda}_k \left( \frac{A_k^n(C_n)}{A_k^n(C_n)} \right)^2 t_{nn}. \]

Finally for general \(n\) we have if \(e_1(t) = \cdots = e_{n-1}(t) = 0\)

\[ \left| \frac{\partial \varphi_n(t)}{\partial t_{nn}} \right|^2 = \frac{\text{tr } C(t)}{\det C_1(t)} \geq \frac{e_n^2(t) \exp\left(-CT, T\right) \exp\left(-(CT, T) - \text{tr } C(t)\right)}{\left( \frac{M_{12 \cdots n}^n(C_n)}{M_{12 \cdots n-1}^n(C_n)} \right)^2 t_{nn} \exp\left(-t_{nn} \left( \frac{M_{12 \cdots n}^n(C_n)}{M_{12 \cdots n-1}^n(C_n)} + \sum_{k=2}^{n} \hat{\lambda}_k \left(A^n_k(C_n)\right)^2 \right)\right)}). \]

Using (59) we get

\[ \left| \frac{\partial \varphi_{nq}(t)}{\partial t_{qq}} \right|^2 = \frac{e_q^2(t) \exp\left(-CT, T\right) \exp\left(-(CT, T) - \text{tr } C(t)\right)}{\det C_1(t)} \geq \frac{e_q^2(t) \exp\left(-(CT, T) - \text{tr } C(t)\right)}{\left( \frac{M_{12 \cdots n}^n(C_n)}{M_{12 \cdots n-1}^n(C_n)} \right)^2 \exp\left(-1\right)} \geq \frac{\left(M_{12 \cdots n}^n(C_n)\right)^2 \exp\left(-1\right)}{\left(M_{12 \cdots n-1}^n(C_n) + \sum_{k=2}^{n} \hat{\lambda}_k \left(A^n_k(C_n)\right)^2 \right)} = \Psi^{nn}. \]

Finally for general \((n, q), n \leq q\), we have if \(e_1(t) = \cdots = e_{n-1}(t) = 0, t_{qq} = 0\),

\[ \left| \frac{\partial \varphi_{nq}(t)}{\partial t_{qq}} \right|^2 = \frac{e_q^2(t) \exp\left(-CT, T\right) \exp\left(-(CT, T) - \text{tr } C(t)\right)}{\det C_1(t)} \geq \frac{e_q^2(t) \exp\left(-(CT, T) - \text{tr } C(t)\right)}{\left( \frac{M_{12 \cdots n}^n(C_n)}{M_{12 \cdots n-1}^n(C_n)} \right)^2 \exp\left(-1\right)} \geq \frac{\left(M_{12 \cdots n}^n(C_n)\right)^2 \exp\left(-1\right)}{\left(M_{12 \cdots n-1}^n(C_n)M_{12 \cdots n}^n(C_n) + \sum_{k=2}^{n} \hat{\lambda}_k \left(A^n_k(C_n)\right)^2 \right)} = \Psi^{nn}. \]
where $C = C_{n,q}$ and $T$ are defined in Lemma B.1. Moreover, the above conditions gives us the same solutions (68) as before, hence using the decomposition of the minor $M_{12\ldots n-1}^{12\ldots n-1}(C_{n,q})$ we have

$$e_q(t) = (C_{n,q}T)_q = \sum_{r=1}^{n} c_{qr} t_{rr} = \sum_{r=1}^{n} c_{qr} A^n(C_n) t_{nn} = \frac{M_{12\ldots n-1}^{12\ldots n-1}(C_{n,q}) t_{nn}}{A^n(C_n)}.$$ 

Finally we get if $e_1(t) = \cdots = e_{n-1}(t) = 0$ and $t_{qq} = 0$

$$Z^{nq} \geq \max_{t_{nn} \in \mathbb{R}} \frac{\partial^{2} \phi_{nq}(t; t_{qq})}{\partial t_{qq}^{2}} \geq \max_{t_{nn} \in \mathbb{R}} e_{q}^{2}(t) \exp\left(-(C_T, T) - \text{tr} C(t)\right)$$

$$= \max_{t_{nn} \in \mathbb{R}} \left(\frac{M_{12\ldots n-1}^{12\ldots n-1}(C_{n,q})}{M_{12\ldots n-1}^{12\ldots n-1}(C_n)}\right)^2 t_{nn}^{2} \exp\left(-t_{nn}^{2} \left(\frac{M_{12\ldots n-1}^{12\ldots n-1}(C_{n,q})}{M_{12\ldots n-1}^{12\ldots n-1}(C_n)} + \sum_{k=2}^{n} \hat{\lambda}_k(\hat{A}_k^n(C_n))^2\right)\right)$$

$$= \frac{(M_{12\ldots n-1}^{12\ldots n-1}(C_{n,q}))^2 \exp(-1)}{M_{12\ldots n-1}^{12\ldots n-1}(C_n)M_{12\ldots n-1}^{12\ldots n-1}(C_n) + \sum_{k=2}^{n} \hat{\lambda}_k(\hat{A}_k^n(C_n))^2} = \psi^{nq}.$$ 

Appendix C. Proof of Lemma 16

**Proof.** Firstly, we prove by induction the inequalities $I_k^k \geq 0$ for $k \geq 2$. Secondly, we show that inequality $I_k^k \geq 0$ and Lemma A.6 imply the inequality $I_m^m \geq 0$ for $m \geq k$ where (see (24)):

$$I_m^k := f_k A_k^k(C_m(\hat{\lambda})) - \hat{\lambda}_k A_k^k(C_m(\hat{\lambda}^{|k|})) \geq 0, \quad 2 \leq k \leq m.$$ 

We shall show also that $I_m^2 = 0$. In the case $m = 2$ we have

$$I_2^2 = f_2 A_2^2(C_2(\hat{\lambda})) - \hat{\lambda}_2 A_2^2(C_2(\hat{\lambda}^{|2|})) = 0$$

since $f_2 = \hat{\lambda}_2 = c_{11}$ by (19), (20) and (49), and

$$A_2^2(C_2(\hat{\lambda})) = A_2^2(C_2(\hat{\lambda}^{|2|})) = A_2^2(C_2) = c_{11},$$

where

$$C_2(\hat{\lambda}) = \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{11} + c_{22} \end{pmatrix}, \quad C_2(\hat{\lambda}^{|2|}) = C_2 = \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix}.$$ 

In the case $m = 3$ we prove the following inequalities:

$$I_3^2 := f_2 A_2^2(C_3(\hat{\lambda})) - \hat{\lambda}_2 A_2^2(C_3(\hat{\lambda}^{|2|})) \geq 0, \quad (72)$$

$$I_3^3 := f_3 A_3^3(C_3(\hat{\lambda})) - \hat{\lambda}_3 A_3^3(C_3(\hat{\lambda}^{|3|})) \geq 0. \quad (73)$$

Since (see (21))
Since we use here the definition of $f_q$ and $C_I$, we have

$$C_3(\hat{\lambda}) = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{11} + c_{22} & c_{13} \\ c_{13} & c_{23} & c_{11} + c_{22} + c_{33} \end{pmatrix}, \quad C_3(\hat{\lambda}^{[2]}) = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{13} \\ c_{13} & c_{23} & c_{11} + c_{22} + c_{33} \end{pmatrix},$$

and $C_3(\hat{\lambda}^{[3]}) = C_3$ we have by (26)

$$A_2^2(C_3(\hat{\lambda})) = A_2^2(C_3(\hat{\lambda}^{[2]})) = A_2^2(C_3) + \hat{\lambda}_3 A_{23}^{23}(C_3), \quad A_3^3(C_3(\hat{\lambda}^{[3]})) = A_3^3(C_3).$$

The latter equalities give us $I_3^2 = 0$. This proves (72). Indeed we have

$$I_3^2 = \hat{\lambda}_2 (A_2^2(C_3) + \hat{\lambda}_3 A_{23}^{23}(C_3)) - \hat{\lambda}_2 (A_2^2(C_3) + \hat{\lambda}_3 A_{23}^{23}(C_3)) = 0.$$

Since $f_2 = \hat{\lambda}_2 = c_{11}$ and $\hat{\lambda}_1 = 0$ we have $A_2^2(C_m(\hat{\lambda})) = A_2^2(C_m(\hat{\lambda}^{[2]}))$ hence

$$I_m^2 := f_2 A_2^2(C_m(\hat{\lambda})) - \hat{\lambda}_2 A_2^2(C_m(\hat{\lambda}^{[2]})) = 0, \quad 2 \leq m.$$

**Remark C.1.** In what follows we take $\lambda = (\lambda, \hat{\lambda}) = (0, c_{11})$.

To prove (73) for $k = 3$ we use the identity for $\lambda = (0, \lambda_2) \in \mathbb{R}^2$, $\hat{\lambda} = (0, c_{11})$,

$$A_3^3(C_3(\hat{\lambda})) = M_{12}^{12}(C_3(\hat{\lambda})) = M_{12}^{12}(C_3) + c_{11}^2 = M_{12}^{12}(C_3) + \hat{\lambda}_2 c_{11},$$

$$A_3^3(C_3(\lambda)) = M_{12}^{12}(C_3(\lambda)) = M_{12}^{12}(C_3) + \lambda_2 c_{11}, \quad \frac{\partial M_{12}^{12}(C_3(\lambda))}{\partial \lambda_2} = c_{11}.$$

We have

$$I_3^3 := f_3 A_3^3(C_3(\hat{\lambda})) - \hat{\lambda}_3 A_3^3(C_3(\hat{\lambda}^{[3]}))$$

$$= \left( c_{11} + \frac{c_{12}^2}{c_{11}} + \frac{(M_{12}^{12}(C_3))^2}{c_{11}(M_{12}^{12}(C_3) + c_{11}^2)} \right) \left( M_{12}^{12}(C_3) + c_{11}^2 \right) - (c_{11} + c_{22}) M_{12}^{12}(C_3)$$

$$= \left( c_{11} + \frac{c_{12}^2}{c_{11}} + \frac{(M_{12}^{12}(C_3))^2}{c_{11} M_{12}^{12}(C_3(\hat{\lambda}))} \right) M_{12}^{12}(C_3(\hat{\lambda})) - (c_{11} + c_{22}) M_{12}^{12}(C_3),$$

we use here the definition of $f_q = e^{\sum_{1 \leq r, p < q} \Psi^{rp}}$ and $\Psi^{pq}$ (see (20), (48)–(50)),

$$f_3 = e^{(\Psi^{11} + \Psi^{12} + \Psi^{22})} = c_{11} + \frac{c_{12}^2}{c_{11}} + \frac{(M_{12}^{12}(C_3))^2}{c_{11}(M_{12}^{12}(C_3) + c_{11}^2)}.$$

We define the function $I_3^3(\lambda)$ for $\lambda = (0, \lambda_2)$ by

$$I_3^3(\lambda) := \left( c_{11} + \frac{c_{12}^2}{c_{11}} + \frac{(M_{12}^{12}(C_3))^2}{c_{11} M_{12}^{12}(C_3(\lambda))} \right) M_{12}^{12}(C_3(\lambda)) - (c_{11} + c_{22}) M_{12}^{12}(C_3)$$

$$= \left( c_{11} + \frac{c_{12}^2}{c_{11}} \right) \left( M_{12}^{12}(C_3) + \lambda_2 c_{11} \right) + \frac{(M_{12}^{12}(C_3))^2}{c_{11}} - (c_{11} + c_{22}) M_{12}^{12}(C_3).$$
Since $I_3^3 = I_3^3(\hat{\lambda})$ it is sufficient to prove that $I_3^3(\lambda) > 0$ for $\lambda_2 > 0$. We show that

$$I_3^3(0) = 0 \quad \text{and} \quad \frac{\partial I_3^3(\lambda)}{\partial \lambda_2} > 0.$$ 

Indeed we have $M_{12}^1(C_3(0)) = M_{12}^1(C_3)$ hence

$$I_3^3(0) = \left( c_{11} + \frac{c_{12}^2 + M_{12}^1(C_3)}{c_{11}} \right) M_{12}^1(C_3) - (c_{11} + c_{22}) M_{12}^1(C_3)$$

$$= M_{12}^1(C_3) \left( \frac{c_{2}^2 + M_{12}^1(C_3)}{c_{11}} - c_{22} \right) = 0 \quad \text{and} \quad \frac{\partial I_3^3(\lambda)}{\partial \lambda_2} = \left( c_{11} + \frac{c_{2}^2}{c_{11}} \right) c_{11} > 0.$$

Finally $I_3^3(\lambda) > 0$ for $\lambda_2 > 0$ so $I_3^3 = I_3^3(\hat{\lambda}) = I_3^3(0, c_{11}) > 0$ and (73) is proved. To prove that $I_k^k \geq 0$ let us denote $f^q = e \sum_{1 \leq r \leq p < q} \Psi^{rp}$. Using (20) we have

$$f_q = e \sum_{1 \leq r \leq p < q} \Psi^{rp} = e \sum_{1 \leq r \leq p < q-1} \Psi^{rp} + e \sum_{r=1}^{q-1} \Psi^{rp} = f_{q-1} + f^q, \quad f_1 := 0, \quad (75)$$

for $2 \leq q \leq m$. We prove by induction that

$$I_k^k = f_k A_k^k(C_k(\hat{\lambda})) - \hat{\lambda}_k A_k^k(C_k) \geq 0, \quad 2 \leq k. \quad (76)$$

For $k = 2$ and $k = 3$ it is proved. Let us suppose that it holds for $k$. To find the general formula for $I_k^k(\hat{\lambda})$ with $I_k^k \geq I_k^k(\hat{\lambda})$ we consider the cases $m = 4$.

$$I_4^4 = f_4 A_4^4(C_4(\hat{\lambda})) - \hat{\lambda}_4 A_4^4(C_4) = (f_3 + f^4) A_4^4(C_4(\hat{\lambda})) - \hat{\lambda}_4 A_4^4(C_4) \bigg|_{\lambda = \hat{\lambda}} \quad (73)$$

$$\geq \left( \frac{\hat{\lambda}_3 A_3^{34}(C_4)}{A_3^{34}(C_4(\lambda))} + f^4 \right) A_4^4(C_4(\lambda)) - \hat{\lambda}_4 A_4^4(C_4) \bigg|_{\lambda = \hat{\lambda}} \quad (49)-(51)$$

$$= \left( \frac{(c_{11} + c_{22}) M_{12}^{1}(C_4)}{M_{12}^{1}(C_4(\lambda))} + \frac{c_{13}^2}{c_{11}} + \frac{(M_{13}^{12}(C_4))^2}{c_{11} M_{12}^{1}(C_4(\lambda))} \right)$$

$$\times \left( M_{12}^{1}(C_4(\lambda)) - (c_{11} + c_{22} + c_{33}) M_{123}^{1}(C_4) \bigg|_{\lambda = \hat{\lambda}} \right)$$

$$\times \left( M_{123}^{1}(C_4(\lambda)) - (c_{11} + c_{22} + c_{33}) M_{123}^{1}(C_4) \bigg|_{\lambda = \hat{\lambda}} \right). \quad (54)$$

So we have $I_4^4 > I_4^4(\lambda) \bigg|_{\lambda = \hat{\lambda}}$, where $I_4^4(\lambda)$ is defined by the formula
Let us consider the function

\[ I_4^4(\lambda) := \left( \frac{(c_{11} + c_{22})M_{12}^{12}(C_4)}{M_{12}^{12}(C_4(\lambda))} + \frac{a_2}{c_{11}} \frac{(M_{13}^{12}(C_4))^2}{c_{11}M_{12}^{12}(C_4(\lambda))} + \frac{(M_{12}^{12}(C_4))^2}{M_{12}^{12}(C_4)M_{12}^{123}(C_4(\lambda))} \right) \times M_{123}^{123}(C_4(\lambda)) - (c_{11} + c_{22} + c_{33})M_{123}^{123}(C_4) \]

where

\[ a_1 = \frac{c_{3}^2}{c_{11}} > 0, \quad a_2 = (c_{11} + c_{22})M_{12}^{12}(C_4) + \frac{(M_{13}^{12}(C_4))^2}{c_{11}} > 0, \]

\[ b_1 = \frac{(M_{12}^{12}(C_4))^2}{M_{12}^{12}(C_4)} - (c_{11} + c_{22} + c_{33})M_{123}^{123}(C_4). \]

We prove that \( I_4^4(\lambda) \geq 0 \) for \( \lambda = (0, \lambda_2, \lambda_3) \), when \( \lambda_2 \geq 0, \lambda_3 \geq 0 \). It then gives us \( I_4^4 \geq \hat{I}_4^4 \geq 0 \). We have (see below the proof of \( I_k^k(0) = 0, k \geq 3 \))

\[ I_4^4(0) = \left( \frac{c_{3}^3}{c_{11}} + \frac{(M_{13}^{12}(C_4))^2}{c_{11}M_{12}^{12}(C_4)} + \frac{M_{12}^{123}(C_4)}{M_{12}^{12}(C_4)} - c_{33} \right)M_{123}^{123}(C_4) = 0. \]

Moreover, by inequality (37) of Lemma A.7 we have for \( \lambda_2 \geq 0, \lambda_3 \geq 0 \)

\[ \frac{\partial I_4^4(\lambda)}{\partial \lambda_2} = a_1 \frac{\partial M_{123}^{123}(C_4(\lambda))}{\partial \lambda_2} + a_2 \frac{\partial M_{123}^{123}(C_4(\lambda))}{\partial \lambda_2} \geq 0, \]

\[ \frac{\partial I_4^4(\lambda)}{\partial \lambda_3} = \left( a_1 + \frac{a_2}{M_{12}^{12}(C_4(\lambda))} \right) \frac{\partial M_{123}^{123}(C_4(\lambda))}{\partial \lambda_3} \geq 0. \]

Let us consider the function

\[ i_4^4(t) = I_4^4(t \hat{\lambda}) = I_4^4(0, t \hat{\lambda}_2, t \hat{\lambda}_3), \quad t \in \mathbb{R}. \]

We have

\[ i_4^4(0) = I_4^4(0) = 0 \quad \text{and} \quad \frac{di_4^4(t)}{dt} \geq \frac{\partial I_4^4(\lambda)}{\partial \lambda_2} \hat{\lambda}_2 + \frac{\partial I_4^4(\lambda)}{\partial \lambda_3} \hat{\lambda}_3 \geq 0 \]

hence \( i_4^4(t) \geq 0 \) by the previous inequalities for \( t > 0 \). So

\[ I_4^4 > I_4^4(0, \hat{\lambda}_2, \hat{\lambda}_3) = i_4^4(t) \big|_{t=1} \geq 0. \]

To prove that \( I_k^k(\hat{\lambda}) \geq 0 \) we show that

\[ I_k^k(0) = 0, \quad 2 \leq k \quad \text{and} \quad \frac{\partial I_k^k(\lambda)}{\partial \lambda_p} \geq 0, \quad 2 \leq p < k. \]
To define the function $I_{k+1}^{k+1}(\lambda)$ with $I_{k+1}^{k+1} \geq I_{k+1}^{k+1}(\hat{\lambda})$ we have

$$
I_{k+1}^{k+1} = f_{k+1} A_{k+1}^{k+1}(C_{k+1}(\hat{\lambda})) - \hat{\lambda}_{k+1} A_{k+1}^{k+1}(C_{k+1})
$$

$$
(75) \quad (f_k + f_k^{k+1}) A_{k+1}^{k+1}(C_{k+1}(\lambda)) - \hat{\lambda}_{k+1} A_{k+1}^{k+1}(C_{k+1})|_{\lambda=\hat{\lambda}}
$$

$$
(76) \quad \left( \hat{\lambda}_{k} A_{kk}^{kk+1}(C_{k+1}) + e \sum_{r=1}^{k} \Psi_r k \right) A_{k+1}^{k+1}(C_{k+1}(\lambda)) - \hat{\lambda}_{k+1} A_{k+1}^{k+1}(C_{k+1})|_{\lambda=\hat{\lambda}}
$$

$$
(54) \quad \left( \hat{\lambda}_{k} A_{kk}^{kk+1}(C_{k+1}) + e \sum_{r=1}^{k} \Psi_0 r \right) A_{k+1}^{k+1}(C_{k+1}(\lambda)) - \hat{\lambda}_{k+1} A_{k+1}^{k+1}(C_{k+1})|_{\lambda=\hat{\lambda}}
$$

$$
:= I_{k+1}^{k+1}(\hat{\lambda}),
$$

where the function $I_{k+1}^{k+1}(\lambda)$ is defined by (see definition (54) of $\Psi_0^{pq}$):

$$
I_{k+1}^{k+1}(\lambda) = \left( \frac{\hat{\lambda}_{k} M_{12...k}^{12...k-1}(C_{k+1})}{M_{12...k-1}^{12...k}(C_{k+1}(\lambda))} + C_{1k}^{1k} + \sum_{r=2}^{k} \frac{(M_{12...r-1}^{12...r-1}(C_{k+1}))^2}{M_{12...r}^{12...r}(C_{k+1}(\lambda))} \right)
$$

$$
\times M_{12...k}^{12...k}(C_{k+1}(\lambda)) - \hat{\lambda}_{k+1} M_{12...k}^{12...k}(C_{k+1})
$$

$$
= \left( \frac{\hat{\lambda}_{k} M_{12...k-1}^{12...k-1}(C_{k+1})}{M_{12...k-1}^{12...k}(C_{k+1}(\lambda))} + C_{1k}^{1k} + \sum_{r=2}^{k-1} \frac{(M_{12...r-1}^{12...r-1}(C_{k+1}))^2}{M_{12...r}^{12...r}(C_{k+1}(\lambda))} \right)
$$

$$
\times M_{12...k}^{12...k}(C_{k+1}(\lambda)) + \frac{(M_{12...k}^{12...k}(C_{k+1}))^2}{M_{12...k-1}^{12...k}(C_{k+1})} - \hat{\lambda}_{k+1} M_{12...k}^{12...k}(C_{k+1}).
$$

Finally we have the following expression for $I_{k+1}^{k+1}(\lambda)$ with corresponding positive constants $a_r$, $2 \leq r \leq k-1$ (depending on $k$) and $b_1 \in \mathbb{R}$:

$$
I_{k+1}^{k+1}(\lambda) = \left( a_1 + \sum_{r=2}^{k-1} \frac{a_r M_{12...r}^{12...r}(C_{k+1}(\lambda))}{G_r(\lambda)} \right) M_{12...k}^{12...k}(C_{k+1}(\lambda)) + b_1
$$

$$
= \left( a_1 + \sum_{r=2}^{k-1} \frac{a_r}{G_r(\lambda)} \right) G_k(\lambda) + b_1.
$$

By (37) of Lemma A.7 we conclude that for $\lambda_r > 0$, $2 \leq r \leq k$, holds

$$
\frac{\partial I_{k+1}^{k+1}(\lambda)}{\partial \lambda_k} = \left( a_1 + \sum_{r=2}^{k-1} \frac{a_r}{G_r(\lambda)} \right) \frac{\partial G_k(\lambda)}{\partial \lambda_k} \geq 0,
$$

$$
\frac{\partial I_{k+1}^{k+1}(\lambda)}{\partial \lambda_p} = a_1 \frac{\partial G_k(\lambda)}{\partial \lambda_p} + \sum_{r=2}^{k-1} a_r \frac{\partial G_k(\lambda)}{\partial \lambda_p G_r(\lambda)} \geq 0, \quad 2 \leq p \leq k.
$$

(78)
Remark C.2. In fact $\partial I_{k+1}^k(\lambda)/\partial \lambda_p > 0$, $2 \leq p \leq k$, for $\lambda = (\lambda_r)_{r=1}^{k+1} \in \mathbb{R}^{k+1}$, $\lambda_r \geq 0$, $1 \leq r \leq k + 1$, since by (38) we have $\partial G_k(\lambda)/\partial \lambda_p = A_p^\lambda (C(\lambda)^p)) > 0$.

Let us suppose that $I_{k+1}^k(0) = 0$, i.e.

$$0 = I_{k+1}^k(0) = M_{12...k-1}^{12...k}(c_{ik-1}^2/c_{11} + (M_{1k}^{12})^2/c_{11}M_{12}^{12} + \cdots + (M_{12...k-3k-1}^{12...k})^2/c_{11}M_{12...k-3}^{12...k} + \cdots)(M_{12...k-3k-1}^{12...k-2}c_{k-1k-1}).$$

For $k = 3, k = 4$ and $k = 5$ we have

$$I_3^3(0) = M_{12}^{12}(c_{12}^2/c_{11} + M_{12}^{12}c_{22}) = 0,$$
$$I_4^4(0) = M_{123}^{123}(c_{13}^2/c_{11} + (M_{13}^{12})^2/c_{11}M_{12}^{12} + \cdots) + (M_{12...k-3k-1}^{12...k})^2/c_{11}M_{12...k-3}^{12...k} + \cdots)(M_{12...k-3k-1}^{12...k-2} - c_{k-1k-1}),$$
$$I_5^5(0) = M_{1234}^{1234}(c_{14}^2/c_{11} + (M_{14}^{12})^2/c_{11}M_{12}^{12} + \cdots) + (M_{12...k-3k-1}^{12...k})^2/c_{11}M_{12...k-3}^{12...k} + \cdots)(M_{12...k-3k-1}^{12...k-2} - c_{k-1k-1}).$$

We prove that $I_{k+1}^k(0) = 0$. Indeed, we get

$$I_{k+1}^k(0) = M_{12...k}^{12...k}(c_{ik}^2/c_{11} + (M_{1k}^{12})^2/c_{11}M_{12}^{12} + \cdots + (M_{12...k-2k-1}^{12...k})^2/c_{11}M_{12...k-2}^{12...k} + \cdots)(M_{12...k-2k-1}^{12...k-2} - c_{kk}).$$

Since by Corollary A.5 we have

$$\left| A_{k-1}^{k-1}(C_k) \begin{array}{c}
A_{k-1}^{k-1}(C_k) \\
A_{k-1}^{k}(C_k)
\end{array} \right| = A_{k-1}^{k}(C_k)A_{k-1}^{k-1}(C_k) \quad \text{or} \quad \left| A_{k-1}^{k-1}(C_k) \begin{array}{c}
A_{k-1}^{k-1}(C_k) \\
A_{k}^{k}(C_k)
\end{array} \right| = (A_{k-1}^{k}(C_k))^2,$$

we conclude that

$$\left| \begin{array}{cc}
M_{12...k-1}^{12...k-1}(C_k) & M_{12...k-2}^{12...k-2}(C_k) \\
M_{12...k}^{12...k}(C_k) & M_{12...k-2}^{12...k-2}(C_k)
\end{array} \right| = (M_{12...k-2k-1}^{12...k}(C_k))^2.$$

Hence

$$\frac{(M_{12...k-2k-1}^{12...k}(C_k))^2}{M_{12...k-2}^{12...k}(C_k)M_{12...k-1}^{12...k}(C_k)} + \frac{M_{12...k}^{12...k}(C_k)}{M_{12...k-1}^{12...k}(C_k)} = \frac{M_{12...k-2}^{12...k}(C_k)}{M_{12...k-2}^{12...k}(C_k)},$$
and
\[ I_{k+1}^{k+1}(0) = M_{12...k}^{12...k} \left( \frac{c_{1k}^2}{c_{11}^2} + \frac{(M_{1k}^{12})^2}{c_{11}^2 M_{12}^{12}} + \frac{(M_{12k}^{123})^2}{M_{12}^{12} M_{123}^{123}} + \cdots \right. \]
\[ + \left. \frac{(M_{12...k-3k-2k})^2}{M_{12...k-3}^{12} M_{12...k-2}^{12}} + \frac{M_{12...k-2k}}{M_{12...k-2}^{12}} - c_{kk} \right). \]

If we change \( k \) with \( k - 1 \) in the last expression we obtain the right-hand part (up to a positive factor) of the expression for \( I_k^k(0) \).

Finally we have proved (77) for \( I_{k+1}^{k+1}(\lambda) \). Let us consider the function
\[ i_{k+1}^{k+1}(t) = I_{k+1}^{k+1}(\hat{\lambda}), \quad t \in \mathbb{R}. \]

We have
\[ i_{k+1}^{k+1}(0) = I_{k+1}^{k+1}(0) = 0 \quad \text{and} \quad \frac{di_{k+1}^{k+1}(t)}{dt} = \sum_{p=2}^{k} \frac{\partial I_{k+1}^{k+1}(\hat{\lambda})}{\partial \lambda_p} \hat{\lambda}_p > 0 \]

by (78) and Remark C.2. So
\[ I_k^k > I_k^k(\hat{\lambda}) = i_k^k(t) \bigg|_{t=1} \geq 0. \]

We recall (see (32)) that for \( \lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m \) and \( 1 \leq k \leq m \) we denote
\[ \lambda^{[k]} = (0, \ldots, 0, \lambda_{k+1}, \ldots, \lambda_m), \quad \lambda^{[k]} = (\lambda_1, \ldots, \lambda_k, 0, \ldots, 0). \]

Using (27)
\[ G_m(\lambda) = A_{\emptyset}^{\emptyset}(C_m(\lambda)) = \sum_{\emptyset \subseteq \delta \subseteq \{1,2,\ldots,m\}} \lambda_{\delta} A_{\delta}^{\delta}(C), \]
we get
\[ A_k^k(C_m(\lambda)) = \sum_{\emptyset \subseteq \delta \subseteq \{1,2,\ldots,k-1,k+1,\ldots,m\}} \lambda_{\delta} A_{k\cup \delta}^{k\cup \delta}(C_m). \quad (79) \]

If we put \( C_m(\lambda^{[k]}) = C_m + \sum_{r=k+1}^{m} \lambda_r E_{rr} \) in (79) we get
\[ A_k^k(C_m(\lambda^{[k]})) = \sum_{\emptyset \subseteq \delta \subseteq \{k+1,k+2,\ldots,m\}} \lambda_{\delta} A_{k\cup \delta}^{k\cup \delta}(C_m). \quad (80) \]

Similarly, if we put \( C_m(\lambda) = C_m(\lambda^{[k]}) + \sum_{r=k+1}^{m} \lambda_r E_{rr} \) we get
\[ A_k^k(C_m(\lambda)) = \sum_{\emptyset \subseteq \delta \subseteq \{k+1,k+2,\ldots,m\}} \lambda_{\delta} A_{k\cup \delta}^{k\cup \delta}(C_m(\lambda^{[k]})). \quad (81) \]
Using (76) we have

\[ f_k \geq \hat{\lambda}_k A_k^k(C_k)(A_k^k(C_k(\hat{\lambda})))^{-1} = \hat{\lambda}_k A_{kk+1\ldots m}^{kk+1\ldots m}(C_m)(A_{kk+1\ldots m}^{kk+1\ldots m}(C_m(\hat{\lambda})))^{-1} \]

hence \( I_m^k = f_k A_k^k(C_m(\hat{\lambda})) - \hat{\lambda}_k A_k^k(C_m(\hat{\lambda}^{[k]})) \geq I_m^k(\hat{\lambda}) \), where the function \( I_m^k(\hat{\lambda}) \) is defined by

\[
I_m^k(\hat{\lambda}) := \hat{\lambda}_k(A_{kk+1\ldots m}^{kk+1\ldots m}(C_m(\hat{\lambda})))^{-1} \begin{vmatrix} A_{kk+1\ldots m}^{kk+1\ldots m}(C_m) & A_k^k(C_m(\hat{\lambda}^{[k]})) \\ A_{kk+1\ldots m}^{kk+1\ldots m}(C_m(\hat{\lambda})) & A_k^k(C_m(\hat{\lambda})) \end{vmatrix}
\]

\[
\geq \hat{\lambda}_k(A_{kk+1\ldots m}^{kk+1\ldots m}(C_m(\hat{\lambda})))^{-1} \sum_{\emptyset \subseteq \delta \subseteq \{k+1, k+2, \ldots, m\}} \hat{\lambda}_\delta \begin{vmatrix} A_{kk+1\ldots m}^{kk+1\ldots m}(C_m) & A_k^k(C_m) \\ A_{kk+1\ldots m}^{kk+1\ldots m}(C_m(\hat{\lambda})) & A_k^k(C_m(\hat{\lambda}^{[k]})) \end{vmatrix}.
\]

Using (26) or (27) we conclude for \( \lambda = (0, \lambda_2, \ldots, \lambda_m) \in C^m \)

\[
A_{kk+1\ldots m}^{kk+1\ldots m}(C_m(\lambda)) = \sum_{\emptyset \subseteq \gamma \subseteq \{2, 3, \ldots, k-1\}} \lambda_\gamma \gamma \cup \{k, k+1, \ldots, m\}(C_m),
\]

\[
A_k^k(C_m(\lambda^{[k]})) = \sum_{\emptyset \subseteq \gamma \subseteq \{2, 3, \ldots, k-1\}} \lambda_\gamma \gamma \cup \{k\}(C_m).
\]

Finally we obtain

\[
I_m^k(\hat{\lambda}) = \hat{\lambda}_k(A_{kk+1\ldots m}^{kk+1\ldots m}(C_m(\hat{\lambda})))^{-1} \sum_{\emptyset \subseteq \delta \subseteq \{k+1, k+2, \ldots, m\}} \hat{\lambda}_\delta \sum_{\emptyset \subseteq \gamma \subseteq \{2, 3, \ldots, k-1\}} \hat{\lambda}_\gamma \begin{vmatrix} A_{kk+1\ldots m}^{kk+1\ldots m}(C_m) & A_{\gamma \cup \{k\}}^\gamma(C_m) \\ A_{kk+1\ldots m}^{kk+1\ldots m}(C_m(\hat{\lambda})) & A_{\gamma \cup \{k\}}^\gamma(C_m(\hat{\lambda}^{[k]})) \end{vmatrix} \geq 0
\]

due to the Hadamard–Fisher’s inequality (Lemma A.6), for \( \alpha = \{k, k+1, \ldots, m\} \) and \( \beta = \gamma \cup \{k\} \cup \emptyset \). This completes the proof of Lemma 16. \( \square \)

References