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Rank-One Singular Perturbations with a Dual Pair of Eigenvalues

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Abstract. We discuss the eigen-values problem for rank one singular perturbations $\tilde{A} = A \tilde{+} \alpha\langle\cdot,\omega\rangle\omega$ of a self-adjoint unbounded operator A with a gap in its spectrum. We give a the constructive description of operators $\tilde{A}$ which possess at least two new eigenvalues, one in the resolvent set and other in the spectrum of A.

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1. Introduction

Many recent publications (see, e.g., [1–21]) have been devoted to the spectral theory of rank-one perturbations of self-adjoint operators,

$$\tilde{A} = A \tilde{+} \alpha\langle\cdot,\omega\rangle\omega, \quad \alpha \in \mathbf{R} \cup \infty,\ \omega \in \mathcal{H}_{-2},$$

where $\mathcal{H}_k$ denotes the usual A-scale of spaces. In fact, this spectral theory is rather rich and instructive even though rank-one perturbations are, in a sense, the simplest kind of perturbations. In this Letter, we expose a new phenomenon which can be described in this theory: a rank-one singular perturbation with a special relation between the coupling constant and the element ω characterizing the perturbation may produce the appearance of a dual pair of eigenvalues.

We investigate the inverse eigenvalues problem in the setting developed in [18] and [9]. We give an explicit construction of the operator $\tilde{A} = A \tilde{+} \alpha\langle\cdot,\omega\rangle\omega$ which solves the eigenvalue problem with a pair of dual eigenvalues,

$$\tilde{A}\varphi = \mu\varphi, \qquad \tilde{A}\psi = \lambda\psi, \quad \mu \in \rho(A),\ \lambda \in \sigma(A), \quad (\lambda-\mu)^{-1} = (\psi, (A-\mu)^{-1}\psi).$$

Let $A = A^*$ be a self-adjoint unbounded operator defined on dom $A = \mathcal{D}(A)$ in the separable Hilbert space $\mathcal{H}$ with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$. $\sigma(A)$,

$\sigma_p(A)$, and $\rho(A)$ denote the spectrum, the point spectrum, and, resp., the regular points set of A.

Another self-adjoint operator $\tilde{A}$ in $\mathcal{H}$ is called a (pure) singular perturbation of A (notation $\tilde{A} \in \mathcal{P}_s(A)$) ([3, 17]) if the set

$$\mathcal{D} := \{f \in \mathcal{D}(A) \cap \mathcal{D}(\tilde{A}) | Af = \tilde{A}f\}$$

is dense in $\mathcal{H}$. It is clear that for each $\tilde{A} \in \mathcal{P}_s(A)$, there exists a densely defined symmetric operator $\mathring{A} := A \upharpoonright \mathcal{D}$ with nontrivial deficiency indices $\mathbf{n}^{\pm}(\mathring{A}) = \dim \ker(\mathring{A} \mp z)^* \neq 0$. In this Letter we discuss only the case of rank-one singular perturbations, $\tilde{A} \in \mathcal{P}_s^1(A)$, i.e., we assume that $\mathbf{n}^{\pm}(\mathring{A}) = 1$.

Let $\{\mathcal{H}_k(A)\}_{k \in \mathbf{R}^1}$ denote the associated A-scale of Hilbert spaces where $\mathcal{H}_k \equiv \mathcal{H}_k(A) = \mathcal{D}(|A|^{k/2})$, $k = 1, 2$, in the norm $||\varphi||_k := ||(|A| + I)^{k/2}\varphi||$ (I stands for the identity) and $\mathcal{H}_{-k} \equiv \mathcal{H}_{-k}(A)$ is the dual space ($\mathcal{H}_{-k}$ is the completion of $\mathcal{H}$ in the norm $||f||_{-k} := ||(|A| + I)^{-k/2}f||$). Let $\langle \cdot, \cdot \rangle$ denote the dual inner product between $\mathcal{H}_k$ and $\mathcal{H}_{-k}$. Obviously, A is bounded as a map from $\mathcal{H}_1$ to $\mathcal{H}_{-1}$, and from $\mathcal{H}$ to $\mathcal{H}_{-2}$ and, therefore, the expression $\langle \varphi, \omega \rangle$, $\omega = \mathbf{A}\psi$ has a sense for any $\varphi, \psi \in \mathcal{H}_1$, where $\mathbf{A}$ denotes the closure of A: $\mathcal{H}_1 \to \mathcal{H}_{-1}$. Moreover, $R_\lambda = (\mathbf{A} - \lambda)^{-1}$ is densely defined in $\mathcal{H}_{-2}$ if $\lambda \notin \sigma_p(A)$.

Each $\tilde{A} \in \mathcal{P}_s^1(A)$ admits the representation $\tilde{A} = A \tilde{+} \alpha\langle \cdot, \omega \rangle \omega$, where $0 \neq \alpha \in \mathbf{R} \cup \infty$ ($\infty^{-1} := 0$), $\omega \in \mathcal{H}_{-2}$, and $\tilde{+}$ stands for the generalized sum (see [12, 20]). The resolvent of $\tilde{A}$ may be written by Krein's formula (see [5, 6, 10, 14]) as

$$\tilde{R}_z = (A - z)^{-1} + b_z^{-1}(\cdot, \eta_{\bar{z}})\eta_z, \quad \operatorname{Im} z \neq 0, \tag{1}$$

where the scalar function b_z satisfies the equation

$$b_\xi = b_z + (z - \xi)(\eta_z, \eta_{\bar{\xi}}), \qquad \bar{b}_z = b_{\bar{z}}, \quad \operatorname{Im} z, \ \operatorname{Im} \xi \neq 0 \tag{2}$$

and where the vector function η_z belongs to $\mathcal{H} \setminus \mathcal{D}(A)$ and one has

$$\eta_z = (A - \xi)R_z\eta_\xi. \tag{3}$$

In the case where $\omega \in \mathcal{H}_{-1}$, we have

$$b_z = -\alpha^{-1} - \langle \omega, \eta_{\bar{z}} \rangle, \quad \eta_z = (\mathbf{A} - z)^{-1}\omega.$$

Vice-versa, the operator function (1) uniquely defines the resolvent of some operator $\tilde{A} \in \mathcal{P}_s^1(A)$ if (2), (3) are fulfilled (see Theorem 2 below).

We are able to formulate our main result.

THEOREM 1. *Let A be a self-adjoint unbounded operator with a nonempty connected spectral gap (i.e., the set $\rho(A) \cap \mathbf{R} \neq \emptyset$ is connected). Then for any vector $\psi \in \mathcal{H} \setminus \mathcal{D}(A)$, $||\psi|| = 1$ and any $\mu \in \rho(A)$, there exists a rank-one singular perturbation $\tilde{A} \in \mathcal{P}_s^1(A)$ uniquely defined by* (1) *with*

$$\eta_z := (A - \lambda)R_z\psi, \qquad b_z := (\lambda - z)(\psi, \eta_{\bar{z}}), \tag{4}$$

which solves the eigenvalue problems with a dual pair of values:

$$\tilde{A}\psi = \lambda\psi, \qquad \tilde{A}\varphi = \mu\varphi, \quad \mu \in \rho(A),\ \lambda \in \sigma(A), \tag{5}$$

where

$$\lambda = \mu + \frac{1}{(\psi, R_\mu\psi)}, \qquad \varphi = (A - \lambda)R_\mu\psi.$$

If $\psi \in \mathcal{H}_1 \setminus \mathcal{D}(A)$, *then* $\tilde{A}$ *admits the representation,* $\tilde{A} = A \tilde{+} \alpha\langle\cdot, \omega\rangle\omega$, *with*

$$\omega = (A - \mu)\psi - \frac{1}{(\psi, R_\mu\psi)}\psi, \quad \alpha = -\frac{1}{\langle\psi, \omega\rangle}. \tag{6}$$

For the proof, see Section 4.

2. Preliminaries

Let $\psi \in \mathcal{D}(A)$ and $\lambda \in \rho(A)$ be fixed. Consider a rank-one (regular) perturbation $\tilde{A} = A + \alpha(\cdot, \omega)\omega$ with $\omega = (A - \lambda)\psi$ and $\alpha = -(1/(\psi, \omega))$. Then obviously $\tilde{A}$ solves the eigenvalue problem $\tilde{A}\psi = \lambda\psi$.

One can repeat this construction for $\tilde{A} = A \tilde{+} \alpha\langle\cdot, \omega\rangle\omega$ in the case where $\psi \in \mathcal{H}_1 \setminus \mathcal{D}(A)$. Then $\tilde{A} \in \mathcal{P}_s^1(A)$ and $\tilde{A}\psi = \lambda\psi$ is fulfilled if $\alpha = -(1/\langle\psi, \omega\rangle)$ with $\omega = (\mathbf{A} - \lambda)\psi$. The resolvent of $\tilde{A}$ has the form

$$\tilde{R}_z := R_z - \frac{1}{\frac{1}{\alpha} + \langle\omega, \eta_{\bar{z}}\rangle}(\cdot, \eta_{\bar{z}})\eta_z,$$

where $\eta_z := (A - \lambda)R_z\psi \equiv R_z\omega$.

Moreover, we assert that one can take any $\psi \in \mathcal{H} \setminus \mathcal{D}(A)$ and any $\lambda \in \mathbf{R}$.

THEOREM 2. *Let A be a self-adjoint unbounded operator. Given* $\lambda \in \mathbf{R}$ *and a vector* $\psi \in \mathcal{H} \setminus \mathcal{D}(A)$, $||\psi|| = 1$, *there exists a rank-one singular perturbation* $\tilde{A} \in \mathcal{P}_s^1(A)$ *uniquely defined by* (1) *with* η_z *and* b_z *given by* (4). $\tilde{A}$ *solves the eigenvalue problem* $\tilde{A}\psi = \lambda\psi$. *If* $\psi \in \mathcal{H}_1 \setminus \mathcal{D}(A)$, *the operator* $\tilde{A}$ *admits the representation in a form,* $\tilde{A} = A \tilde{+} \alpha\langle\cdot, \omega\rangle\omega$, *with* $\omega = (A - \lambda)\psi$, $\alpha^{-1} = -\langle\psi, \omega\rangle$.

For the proof see Appendix and ([9]).

3. Rank-One Singular Perturbations with Two New Eigenvalues

Let A be as in Theorem 1. Let the vector $\psi \in \mathcal{H}_1 \setminus \mathcal{D}(A)$, $||\psi|| = 1$, and the number $\mu \in \rho(A)$ be fixed. Consider the operator $\tilde{A}_0 = A \tilde{+} \alpha_0\langle\cdot, \omega_0\rangle\omega_0 \in \mathcal{P}_s^1(A)$ with

$$\omega_0 = (\mathbf{A} - \mu)\psi \in \mathcal{H}_{-1} \quad \text{and} \quad \alpha_0 = -\frac{1}{\langle\psi, \omega_0\rangle}.$$

From the above considerations, this operator solves the eigenvalue problem $\tilde{A}_0\psi = \mu\psi$.

Now we will construct another operator $\tilde{A} \in \mathcal{P}_s^1(A)$ which solves the eigenvalue problems with a pair of values: the same $\mu \in \rho(A)$ and an additional one, $\lambda \in \sigma(A)$. We define $\tilde{A}$ by Krein's formula (1) with $b_z = (\lambda - z)(\psi, \eta_{\bar{z}})$:

$$\tilde{R}_z := R_z - \frac{1}{(\lambda - z)(\psi, \eta_{\bar{z}})}(\cdot, \eta_{\bar{z}})\eta_z,$$

where

$$\eta_z := (A - \lambda)R_z\psi \quad \text{and} \quad \lambda := \mu + (\psi, R_\mu\psi)^{-1}.$$

$\tilde{A}$ solves the eigenvalue problem $\tilde{A}\eta_\lambda = \lambda\eta_\lambda$ with $\eta_\lambda = \psi$, since obviously $b_\lambda = 0$ (see Theorem 2 above and Proposition 3 in [3]). The operator $\tilde{A}$ also solves the eigenvalue problem $\tilde{A}\eta_\mu = \mu\eta_\mu$ with $\eta_\mu = (A - \lambda)R_\mu\psi$, since $b_\mu = 0$. Indeed, $b_\mu = (\lambda - \mu)(\psi, \eta_\mu) = 0$ because

$$(\psi, \eta_\mu) = (\psi, (A - \lambda)R_\mu\psi) = 1 + (\mu - \lambda)(\psi, R_\mu\psi) = 0,$$

due to the above connection between λ and μ. We note that $\lambda \in \sigma(A)$ since a rank-one perturbation may produce only one new eigenvalue in each spectral gap of the starting operator. Thus, we described the construction of a rank-one singular perturbation $\tilde{A} \in \mathcal{P}_s^1(A)$ which solves the eigenvalues problems with two new values, one lying in the gap of the spectrum $\sigma(A)$ of the original operator A. Since $\omega_0 = (\mathbf{A} - \mu)\psi \in \mathcal{H}_{-1}$, one can present $\tilde{A}$ in the form $\tilde{A} = A \tilde{+} \alpha\langle\cdot, \omega\rangle\omega$ with

$$\alpha = -\langle\psi, \omega\rangle^{-1} \quad \text{and} \quad \omega = (\mathbf{A} - \lambda)\psi = \omega_0 + (\psi, R_\mu\psi)^{-1}\psi.$$

We remark that the same operator appears in another (dual) way. Namely, using $\lambda = \mu + (\psi, R_\mu\psi)^{-1}$ and putting $\varphi := (A - \lambda)R_\mu\psi$, we can define the resolvent of $\tilde{A}$ in the form

$$\tilde{R}_z = R_z - \frac{1}{(\mu - z)(\varphi, \eta_{\bar{z}})}(\cdot, \eta_{\bar{z}})\eta_z,$$

where

$$\eta_z = (A - \mu)R_z\varphi = (A - \lambda)R_z\psi = R_z\omega$$

with $\omega = (\mathbf{A} - \lambda)\psi = (\mathbf{A} - \mu)\varphi$, and where $b_z = (\mu - z)(\varphi, \eta_{\bar{z}})$ coincides with $b_z = (\lambda - z)(\psi, \eta_{\bar{z}})$. The latter is true due to (11) (see below) and since, by the Hilbert identity, one has

$$(\lambda - z)(\psi, \eta_{\bar{z}}) = \langle R_\lambda\omega, \omega\rangle - \langle R_z\omega, \omega\rangle, (\mu - z)(\varphi, \eta_{\bar{z}}) = \langle R_\mu\omega, \omega\rangle - \langle R_z\omega, \omega\rangle.$$

Thus, one can to calculate the coupling constant α in the representation $\tilde{A} = A \tilde{+} \alpha\langle\cdot, \omega\rangle\omega$ by two formulas.

$$\alpha = -\langle\psi, \omega\rangle^{-1} \quad \text{and} \quad \alpha = -\langle\varphi, \omega\rangle^{-1}.$$

Obviously, α is negative for positive A, since $\langle\varphi, \omega\rangle = \langle\varphi, (\mathbf{A} - \mu)\varphi\rangle > 0$ for all $\mu < 0$.

EXAMPLE. Let

$$\mathcal{H} = L_2(\mathbf{R}, \mathrm{d}x) \quad \text{and} \quad A = -\Delta = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}.$$

First consider the perturbed operator $-\tilde{\Delta}_{\alpha_0,y} = -\Delta \tilde{+} \alpha_0 \langle \cdot, \delta_y \rangle \delta_y$, where the coupling constant is real and δ_y is the Dirac distribution concentrated at the point $y \in \mathbf{R}$. For each $\alpha_0 < 0$, the operator $-\tilde{\Delta}_{\alpha_0,y}$ has a single eigenvalue $\mu = -\alpha_0^2/4 < 0$ with the corresponding eigenfunction $\psi(x) = \mathrm{e}^{\alpha_0|x-y|/2}$ (for more details, see ([1])).

Now we will construct the new rank-one singular perturbation of the Laplace operator which has a pair of dual eigenvalues.

Fix $\mu = -1$ and $\psi = \mathrm{e}^{-|x|}$ and define

$$\xi = ((-\Delta + 1)^{-1}\psi, \psi) = ||\psi||_{-1}^2 < 1$$

and

$$\lambda \equiv \mu + \xi^{-1} = -1 + \xi^{-1} > 0.$$

Put

$$\varphi \equiv \psi - \xi^{-1}(-\Delta + 1)^{-1}\psi$$

and

$$\omega \equiv (-\Delta - \lambda)\psi = (-\Delta + 1)\varphi = 2\delta - \xi^{-1}\psi,$$

where we used $(-\Delta + 1)\psi = 2\delta$, (with $\delta = \delta_0$). Introduce the operator

$$-\tilde{\Delta} = -\Delta \tilde{+} \alpha\langle \cdot, \omega \rangle \omega$$

where

$$\alpha = -1/\langle \psi, \omega \rangle \equiv -1/\langle \varphi, \omega \rangle = -(2 - \xi^{-1})^{-1}.$$

If we put $\alpha = -1/\langle \varphi, \omega \rangle$, then by direct calculation, we find that

$$\begin{aligned}(-\tilde{\Delta} + 1)\varphi &= (-\Delta + 1)\varphi + \alpha\langle \varphi, \omega \rangle \omega = (-\Delta + 1)\psi - \xi^{-1}\psi + \alpha\langle \varphi, \omega \rangle \omega \\ &= 2\delta - \xi^{-1}\psi - \omega = 2\delta - \xi^{-1}\psi - 2\delta + \xi^{-1}\psi = 0,\end{aligned}$$

i.e., $-\tilde{\Delta}\varphi = -\varphi$. Moreover, if we put $\alpha = -1/\langle \psi, \omega \rangle$, then

$$-\tilde{\Delta}\psi = -\Delta\psi + \alpha\langle \psi, \omega \rangle \omega = 2\delta - \psi - \omega = 2\delta - \psi - 2\delta + \xi^{-1}\psi = (-1 + \xi^{-1})\psi = \lambda\psi.$$

Of course, we can verify that

$$(\psi, \varphi) = (\psi, \psi - \xi^{-1}(-\Delta + 1)^{-1}\psi) = 1 - \xi^{-1}((-\Delta + 1)^{-1}\psi, \psi) = 1 - \xi^{-1}\xi = 0.$$

Simple calculations show that the above terms and expressions have the following explicit values: $\xi = 3/4,\ \lambda = 1/3,\ \alpha = -3/2,$

$$\varphi(x) = \mathrm{e}^{-|x|} - \tfrac{2}{3}(1 + |x|)\mathrm{e}^{-|x|}, \qquad \omega(x) = 2\delta(x) - \tfrac{4}{3}\mathrm{e}^{-|x|}.$$

Thus, $-\tilde{\Delta}$ possesses the two eigenvalues, $\mu = -1 < 0$ and $\lambda = \frac{1}{3} > 0$.

4. Dual Eigenvalues Pairs

For a fixed vector $\psi \in \mathcal{H} \setminus \mathcal{D}(A)$, $||\psi|| = 1$, a point $\lambda \in \sigma(A)$ will be said to be dual with respect to a given $\mu \in \rho(A)$ if $(\lambda - \mu)^{-1} = (\psi, R_\mu \psi)$.

Let us consider a positive operator, $A \geqslant 0$. If $\sigma(A) = [0, \infty)$, then for any $\psi \in \mathcal{H} \setminus \mathcal{D}(A)$ and any point $\mu < 0$, there exists a dual point λ which is uniquely defined by

$$\lambda = \mu + \frac{1}{(\psi, R_\mu \psi)}. \tag{7}$$

We note that $\lambda > 0$, since for $A \geqslant 0$

$$0 < (\psi, R_\mu \psi) < -\frac{1}{\mu}, \quad \mu < 0.$$

Our main result in this case reads as follows:

THEOREM 3. *Let $A = A^* \geqslant 0$ and $\sigma(A) = [0, \infty)$. Then for any vector $\psi \in \mathcal{H} \setminus \mathcal{D}(A)$, $|\psi| = 1$, and any $\mu < 0$, there exists a uniquely defined rank-one singular perturbation $\tilde{A} \in \mathcal{P}_s^1(A)$ which solves the eigenvalue problems with a dual pair of values*

$$\tilde{A}\psi = \lambda\psi, \quad \tilde{A}\varphi = \mu\varphi, \tag{8}$$

where $\varphi = (A - \lambda) R_\mu \psi$ and $\lambda > 0$ is given by (7)*. If $\psi \in \mathcal{H}_1 \setminus \mathcal{D}(A)$, then the operator $\tilde{A}$, which solves* (8)*, admits the representation*

$$\tilde{A} = A \tilde{+} \alpha\langle \cdot, \omega \rangle \omega, \quad \alpha = -\frac{1}{\langle \psi, \omega \rangle}, \quad \omega = (\mathbf{A} - \lambda)\psi.$$

Proof. Given $\psi \in \mathcal{H} \setminus \mathcal{D}(A)$ and $\mu < 0$, let us consider λ to be connected with μ by (7). Define the operator $\tilde{A}$ by Krein's resolvent formula

$$\tilde{R}_z = R_z + b_z^{-1}(\cdot, \eta_{\bar{z}})\eta_z = R_z + \frac{1}{(\lambda - z)(\psi, \eta_{\bar{z}})}(\cdot, \eta_{\bar{z}})\eta_z,$$

where $\eta_z := (A - \lambda) R_z \psi$. By Theorem 2, the operator $\tilde{A}$ solves the problem $\tilde{A}\psi = \lambda\psi$. Let us directly show that $\tilde{A}$ also solves the second problem in (8). To this aim, we will show that

$$\tilde{R}_z \varphi = \frac{1}{\mu - z}\varphi,$$

i.e., that

$$R_z \varphi + b_z^{-1}(\varphi, \eta_{\bar{z}})\eta_z = \frac{1}{\mu - z}\varphi.$$

So in the notation $\varphi = (A-\lambda)R_\mu\psi \equiv \eta_\mu$, we have to prove that

$$R_z\eta_\mu + \frac{1}{(\lambda - z)(\psi, \eta_{\bar z})}(\eta_\mu, \eta_{\bar z})\eta_z = \frac{1}{\mu - z}\eta_\mu, \tag{9}$$

which is equivalent to $\tilde{A}\varphi = \mu\varphi$. We observe that

$$\eta_z = (A-\mu)R_z\eta_\mu = \eta_\mu + (z-\mu)R_z\eta_\mu.$$

Therefore, (9) will be true only if

$$\frac{(\eta_\mu, \eta_{\bar z})}{(\lambda - z)(\psi, \eta_{\bar z})} = \frac{1}{\mu - z}. \tag{10}$$

Let us prove (10). By the resolvent identity, we have with $\omega = (\mathbf{A} - \lambda)\psi = (\mathbf{A} - \mu)\varphi, \ \varphi \equiv \eta_\mu$,

$$(\eta_\mu, \eta_{\bar z}) = (R_\mu\omega, R_{\bar z}\omega) = \frac{1}{\mu - z}[\langle R_\mu\omega, \omega\rangle - \langle R_z\omega, \omega\rangle].$$

Besides

$$(\lambda - z)(\psi, \eta_{\bar z}) = (\lambda - z)(R_\lambda\omega, R_{\bar z}\omega) = \langle R_\lambda\omega, \omega\rangle - \langle R_z\omega, \omega\rangle.$$

So we have only to prove that

$$\langle R_\mu\omega, \omega\rangle = \langle R_\lambda\omega, \omega\rangle. \tag{11}$$

It follows from $\psi \equiv \eta_\lambda \perp \lambda_\mu \equiv \varphi$, which is true since

$$(\psi, \eta_\mu) = (\psi, (A-\lambda)R_\mu\psi) = 1 + (\mu - \lambda)(\psi, R_\mu\psi) = 0,$$

by virtue of (7). Now we have

$$0 = (\eta_\lambda, \eta_\mu) = (R_\lambda\omega, R_\mu\omega) = \frac{1}{\lambda - \mu}[\langle R_\lambda\omega, \omega\rangle - \langle R_\mu\omega, \omega\rangle].$$

That proves (11) and therefore (10) too. The uniqueness of $\tilde{A}$ was proved in Theorem 2. □

Proof of Theorem 1. In the general case where the nonempty set $\rho(A)$ is connected, the proof is the same as for a positive operator. We only have to be sure that $\lambda \in \sigma(A)$. This follows from the fact that any singular rank-one perturbation (as a self-adjoint extension of the symmetric operator $\mathring{A}$) may produce only a single new eigenvalue in each spectral gap. Since $\mu \in \rho(A)$, this implies that λ, which is an eigenvalue by construction, belongs to $\sigma(A)$ by necessity. □

Appendix

Proof of Theorem 2. Consider the operator function defined by (1)

$$\tilde{R}_z := R_z + b_z^{-1}(\cdot, \eta_{\bar z})\eta_z,$$

where the vector function η_z and the scalar function b_z have the form

$$\eta_z := (A - \lambda)R_z\psi, \qquad b_z := (\lambda - z)(\psi, \eta_{\bar{z}}), \quad \lambda \in \mathbf{R}, \ \psi \in \mathcal{H} \setminus \mathcal{D}(A).$$

First, we will prove that $\tilde{R}_z$ is the resolvent of a some self-adjoint operator $\tilde{A} \in \mathcal{P}_s^1(A)$. With this aim, we check by direct calculations, using the Hilbert identity for the resolvent $R_z = (A - z)^{-1}$, that η_z and b_z satisfiy (2). Then again by direct calculations, we check that the operator function $\tilde{R}_z$ is a pseudo-resolvent (see [13]), i.e., that $\tilde{R}_z$ satisfies the Hilbert identity. To be sure that $\tilde{R}_z$ is the resolvent of a some closed operator, we have to show that $\mathrm{Ker}\tilde{R}_z = \{0\}$.

This is a consequence of the condition $\psi \in \mathcal{H} \setminus \mathcal{D}(A)$. Indeed

$$\tilde{R}_z h = R_z h + b_z^{-1}(h, \eta_{\bar{z}})\eta_z = 0$$

implies that $h = 0$ because

$$R_z h \in \mathcal{D}(A) \quad \text{and} \quad \eta_z = \psi + (z - \lambda)R_z\psi \notin \mathcal{D}(A)$$

due to $\psi \notin \mathcal{D}(A)$. In fact, $\tilde{R}_z$ is the resolvent a self-adjoint operator. Denote it by $\tilde{A}$, since

$$(\tilde{R}_z)^* = R_{\bar{z}} + \bar{b}_z^{-1}(\cdot, \eta_z)\eta_{\bar{z}} = \tilde{R}_{\bar{z}},$$

where we used, $\bar{b}_z^{-1} = b_{\bar{z}}^{-1}$. Further, $\eta_z \notin \mathcal{D}(A)$, $\mathrm{Im}\, z > 0$ implies that the set

$$\mathcal{D} = \{f \in \mathcal{H} : f = \tilde{R}_z h = R_z h\} \equiv \{f \in \mathcal{D}(A) : \langle f, \omega \rangle = 0, \ \omega = (\mathbf{A} - \lambda)\psi\}$$

does not depend on z and is dense in $\mathcal{H}$. Thus, $\tilde{A}$ is a self-adjoint extension of the symmetric operator $\mathring{A} = A \restriction \mathcal{D}$. The deficiency indices of $\mathring{A}$ are (1,1) because the deficiency subspaces $\mathfrak{N}_z := \mathrm{Ker}(\mathring{A} - z)^*$ are one-dimensional (they are spanned by η_z). Therefore, $\tilde{A} \in \mathcal{P}_s^1(A)$).

The operator $\tilde{A}$ solves the problem $\tilde{A}\psi = \lambda\psi$, since due to $\eta_z = \psi + (z - \lambda)R_z\psi$, we have

$$\tilde{R}_z\psi = R_z\psi + \frac{1}{(\lambda - z)(\psi, \eta_{\bar{z}})}(\psi, \eta_{\bar{z}})\eta_z = \frac{1}{\lambda - z}\psi.$$

Finally, let us prove the uniqueness. Suppose that there exists another operator $\hat{A} \in \mathcal{P}_s^1(A)$, which also solves the problem $\hat{A}\psi = \lambda\psi$. From the above considerations, its resolvent admits the representation

$$\hat{R}_z := R_z + \hat{b}_z^{-1}(\cdot, \hat{\eta}_{\bar{z}})\hat{\eta}_z, \quad \mathrm{Im}\, z \neq 0,$$

and, moreover, for the above λ and ψ we have

$$\hat{R}_z\psi = \tilde{R}_z\psi = \frac{1}{\lambda - z}\psi = R_z\psi + \hat{b}_z^{-1}(\cdot, \hat{\eta}_{\bar{z}})\hat{\eta}_z = R_z\psi + b_z^{-1}(\cdot, \eta_{\bar{z}})\eta_z.$$

Now it is clear that $\hat{b}_z^{-1} = b_z^{-1}$, and $\hat{\eta}_z = \eta_z$ up to a constant of modulus one. Therefore $\hat{A} = \tilde{A}$. □

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