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SPECTRAL PROPERTIES OF IMAGE PROBABILITY MEASURES AFTER CONFLICT INTERACTIONS

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Abstract. We introduce a composition of the conflict interaction between a pair of image probability measures and study the associated dynamical system. We establish the existence of invariant limiting measures and find conditions for these measures to be a pure point, absolutely continuous, or singular continuous. Besides we investigate the fractal properties of their supports, in particular we find condition when the limiting measure has the Cantor, Salem, or Pratsiovytyi type of arbitrary Hausdorff dimension.

Introduction

We deal with a pair of so-called image probability measures μ, ν on the segment $[0,1]$, which are uniquely associated with infinite products: $\mu^* = \prod_{k=1}^{\infty} \mu_k,\ \nu^* = \prod_{k=1}^{\infty} \nu_k$, where $\mu_k,\ \nu_k$ are discrete probability measures defined on some space of $n < \infty$ points $\Omega_k = \{\omega_0, \omega_1, ...\omega_n\} \equiv \{1, 2, ..., n\}$. We consider Ω_k as a set of conflicting positions for each pair $\mu_k,\ \nu_k$ in the following interpretation. A position ω_i may be occupied by μ_k or ν_k with probabilities $\mu_k(\omega_i) \geq 0$ or $\nu_k(\omega_i) \geq 0$ resp. The non-linear and non-commutative conflict composition (see (7) below) between μ_k, ν_k is defined in such a way that on the Nth step of the conflict, $N = 1, 2, ...$, we get a pair of new probability measures: $\mu_k^N(\omega_i) = p_{ik}^{(N)} \geq 0,\ \nu_k^N(\omega_i) = r_{ik}^{(N)} \geq 0,\ i = 0, ..., n$ on Ω_k. The infinite iteration of the conflict composition generates a certain dynamical system. We show the existence of the limiting points for their trajectories, i.e., the existence of limiting values $\mu_k^{\infty}(\omega_i) = p_{ik}^{(\infty)} \geq 0$ and $\nu_k^{\infty}(\omega_i) = r_{ik}^{(\infty)} \geq 0$. Thus we get a pair of the limiting product measures $\mu^{*,\infty} = \prod_{k=1}^{\infty} \mu_k^{\infty}, \nu^{*,\infty} = \prod_{k=1}^{\infty} \nu_k^{\infty}$, and therefore a pair of image probability measures $\mu^{\infty}, \nu^{\infty}$ on $[0,1]$.

We assert that measures $\mu^{\infty}, \nu^{\infty}$ posses rather rich metric and topological structures. We find conditions (see Theorems 4-6 below) for μ^{∞} to be a pure point, pure absolutely continuous, or pure singular continuous as well as to have any topological type (Theorem 7). Moreover we show that its support may meet any Hausdorff dimension (Theorem 8).

We note that in the case of $n = 2$ the similar results was obtained in [1].

A sub-class of image measures on $[0,1]$

We start with a sequence $\{\mathbf{q}_k\}_{k=1}^{\infty}$ of stochastic vectors in $\mathbf{R}^n$, $n > 1$,

$$\mathbf{q}_k = (q_{1k}, q_{2k}, ..., q_{nk}),\ q_{1k}, ..., q_{nk} > 0,\ q_{1k} + \cdots + q_{nk} = 1.$$

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By Q we denote the infinite matrix of the form

$$Q = \{\mathbf{q}_k\}_{k=1}^{\infty} = \begin{pmatrix} q_{11} & q_{12} & \cdot & \cdot & \cdot & q_{1k} & \cdot & \cdot & \cdot \\ q_{21} & q_{22} & \cdot & \cdot & \cdot & q_{2k} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q_{n1} & q_{n2} & \cdot & \cdot & \cdot & q_{nk} & \cdot & \cdot & \cdot \end{pmatrix} \equiv \{q_{ik}\}_{i=1,k=1}^{n,\infty} \qquad (1)$$

Given Q consider a family of intervals:

$$\Delta_{i_1},\ \Delta_{i_1 i_2}, ...,\ \Delta_{i_1 i_2 \cdots i_k}, ... \subset [0,1] \quad \text{(each index } i_1, i_2, ...i_k, ... \text{ runs } 1,2,...,n),$$

such that

$$[0,1] = \bigcup_{i_1=1}^{n} \Delta_{i_1},\ \Delta_{i_1} = \bigcup_{i_2=1}^{n} \Delta_{i_1 i_2}, \quad \Delta_{i_1 i_2 \ldots i_{k-1}} = \bigcup_{i_k=1}^{n} \Delta_{i_1 i_2 \ldots i_k}, \ \cdots$$

and

$$q_{i_1 1} = \lambda(\Delta_{i_1}),\ q_{i_2 2} = \frac{\lambda(\Delta_{i_1 i_2})}{\lambda(\Delta_{i_1})}, \ \cdots, q_{i_k k} = \frac{\lambda(\Delta_{i_1 i_2 \ldots i_k})}{\lambda(\Delta_{i_1 i_2 \ldots i_{k-1}})} \ \cdots,$$

where $\lambda(A)$ stands for Lebesgue measure of a set A. Obviously for any $k = 1, 2, ...$

$$\lambda(\Delta_{i_1 i_2 \ldots i_k} \bigcap \Delta_{i_1 i_2 \ldots j_k}) = 0, i_k \neq j_k.$$

Assume

$$\prod_{k=1}^{\infty} \max_i \{q_{ik}\} = 0. \qquad (2)$$

Then it is easily seen that the σ-algebra generated by the family $\{\Delta_{i_1 i_2 \ldots i_k}\}_{k=1}^{\infty}$ coincides with the Borel σ-algebra $\mathcal{B}$ on $[0,1]$.

Fixed Q, we associate a sub-class of probability measures on the segment $[0,1]$, notation $\mathcal{M}(Q)$, to a family of matrices

$$P \equiv \{\mathbf{p}_k\}_{k=1}^{\infty} = \begin{pmatrix} p_{11} & p_{12} & \cdot & \cdot & \cdot & p_{1k} & \cdot & \cdot & \cdot \\ p_{21} & p_{22} & \cdot & \cdot & \cdot & p_{12} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{n1} & p_{n2} & \cdot & \cdot & \cdot & p_{nk} & \cdot & \cdot & \cdot \end{pmatrix} \equiv \{p_{ik}\}_{i=1,k=1}^{n,\infty} \qquad (3)$$

Namely, a matrix P is associated to a Borel measure $\mu_P \equiv \mu \in \mathcal{M}(Q)$ if

$$\mu([0,1]) = 1,\ \mu(\Delta_{i_1}) = p_{i_1 1},\ \mu(\Delta_{i_1 i_2}) = p_{i_1 1} \cdot p_{i_2 2},$$

and so on, for any $i_1, i_2, ..., i_k,\ k = 1, 2, ...$

$$\mu(\Delta_{i_1 i_2 \cdots i_k}) = p_{i_1 1} \cdot p_{i_2 2} \cdots p_{i_k k}. \qquad (4)$$

Obviously $\mu_{P^1} \neq \mu_{P^2}$ if matrices P^1, P^2 are different. Thus we have a one-to-one correspondence between measures $\mu \in \mathcal{M}(Q)$ and matrices P of the form (3).

We note that Lebesgue measure λ on $[0,1]$ belongs to $\mathcal{M}(Q)$ and corresponds to the matrix $P \equiv Q$.

We will show now that each $\mu \in \mathcal{M}(Q)$ in really is the image probability measure in the sense [6,7].

Fixed Q and given P let us consider a sequence of discrete probability spaces $(\Omega_k, \mathcal{A}_k, \mu_k)$, $k = 1, 2, ...$, where the finite space $\Omega_k = \{\omega_{ik}\} \equiv \{1, 2, ..., n\}$ is the same for all k, and $\mu_k(\omega_{ik}) = p_{ik}$. Let $(\Omega, \mathcal{A}, \mu^*) = \prod_{k=1}^{\infty} (\Omega_k, A_k, \mu_k)$ denote the infinite product of the above probability spaces. We observe that μ^* is also uniquely associated with a matrix P. Indeed, on the cylindrical sets $\Omega_{i_1 \cdots i_k} \subset \Omega$,

$$\mu^*(\Omega_{i_1 \cdots i_k}) = \prod_{l=1}^{k} p_{i_l l}. \qquad (5)$$

Its extension on any set from $\mathcal{A}$ is defined by the standard way. From (4) and (5) it follows that measures μ^* and $\mu \equiv \mu_P$ are equivalent since $\mu^*(\Omega_{i_1 \cdots i_k}) = \mu(\Delta_{i_1 \cdots i_k})$. For more details one can consider a measurable mapping π from Ω into $[0,1]$ defined as follows,

$$\pi : \Omega \ni \omega = \{\omega_{i1}, \omega_{i2}, \cdots \omega_{ik}, \cdots\} \to x \in [0,1], \qquad (6)$$

where a point $x := \bigcap_{k=1}^{\infty} \Delta_{i_1 i_2 \dots i_k}$ is uniquely defined due to (2). So for cylindrical sets we have, $\pi(\Omega_{i_1 \cdots i_k}) = \Delta_{i_1 \cdots i_k}$ and therefore $\mu^*(\Omega_{i_1 \cdots i_k}) = \mu(\pi(\Omega_{i_1 \cdots i_k}))$. Conversely, for any Borel set E from $[0,1]$, $\mu(E) = \mu^*(\pi^{-1}(E))$, where $\pi^{-1}(E) := \{\omega : \pi(\omega) \in E\}$. This means that π and π^{-1} are measure-preserving mappings and therefore measures $\mu \equiv \mu_P$ and μ^* are equivalent. By this reason we refer on a measure $\mu \in \mathcal{M}(Q)$ as the image probability measure with respect to the mapping (6).

The following result (see Theorem 1 below) on image measures is well known (see e.g. [2-7]). For the formulation of it we need in the following notations. We write, $\mu \in \mathcal{M}_{\rm pp}, \mathcal{M}_{\rm ac}, \mathcal{M}_{\rm sc}$ if a measure μ is a pure point, pure absolutely continuous, or pure singular continuous, resp. Further, for above matrices Q and P we define

$$P_{max}(\mu) := \prod_{k=1}^{\infty} \max_i \{p_{ik}\}$$

and

$$\rho(\mu, \lambda) := \prod_{k=1}^{\infty} (\sqrt{\mathbf{p}_k}, \sqrt{\mathbf{q}_k}), \text{ where } \quad \sqrt{\mathbf{p}_k} := (p_{1k}^{1/2}, ..., p_{nk}^{1/2}), \quad \sqrt{\mathbf{q}_k} := (q_{1k}^{1/2}, ..., q_{nk}^{1/2}).$$

Theorem 1. *Each image probability measure $\mu \in \mathcal{M}(Q)$ has a pure spectral type:*

(*a*) $\mu \in \mathcal{M}_{\rm pp}$ *iff* $P_{max}(\mu) > 0$,

(*b*) $\mu \in \mathcal{M}_{\rm ac}$ *iff* $\rho(\mu, \lambda) > 0$,

(*c*) $\mu \in \mathcal{M}_{\rm sc}$ *iff* $P_{max}(\mu) = 0$ *and* $\rho(\mu, \lambda) = 0$.

Conflict interaction between image measures

At first we define the non-commutative non-linear conflict composition, notation $\divideontimes$, between a pair of stochastic vectors $\mathbf{p}, \mathbf{r} \in \mathbf{R}^n$, associated to abstract discrete measures, as follows (for more details see [8,9]):

$$\mathbf{p}^1 := \mathbf{p} \divideontimes \mathbf{r}, \ \mathbf{r}^1 := \mathbf{r} \divideontimes \mathbf{p},$$

where coordinates of new vectors $\mathbf{p}^1, \mathbf{r}^1$ are given by formulae:

$$p_i^{(1)} := \frac{p_i(1-r_i)}{1-(\mathbf{p},\mathbf{r})}, \quad r_i^{(1)} := \frac{r_i(1-p_i)}{1-(\mathbf{p},\mathbf{r})}, \ i = 1, ..., n, \tag{7}$$

where $(\mathbf{p}, \mathbf{r})$ stands for the inner product in $\mathbf{R}^n$. Obviously we have to exclude the blow-up case $(\mathbf{p}, \mathbf{r}) = 1$.

The iteration of the composition $\divideontimes$ generates a dynamical system in the space $\mathbf{R}^n \times \mathbf{R}^n$ with mapping,

$$f^{\divideontimes} : \begin{pmatrix} \mathbf{p}^{N-1} \\ \mathbf{r}^{N-1} \end{pmatrix} \to \begin{pmatrix} \mathbf{p}^{N} \\ \mathbf{r}^{N} \end{pmatrix}, N \geq 1, \quad \mathbf{p}^0 \equiv \mathbf{p}, \mathbf{r}^0 \equiv \mathbf{p}, \tag{8}$$

where coordinates of the vectors $\mathbf{p}^N, \mathbf{r}^N$ are defined by induction,

$$p_i^{(N)} := \frac{p_i^{(N-1)}(1-r_i^{(N-1)})}{z^{N-1}}, \quad r_i^{(N)} := \frac{r_i^{(N-1)}(1-p_i^{(N-1)})}{z^{N-1}}, \ i = 1, ..., n, \tag{9}$$

with $z^{N-1} = 1 - (\mathbf{p}^{N-1}, \mathbf{r}^{N-1}) > 0$.

Theorem 2. *([8,9]) For each pair of stochastic vectors $\mathbf{p}^0 \equiv \mathbf{p}, \mathbf{r}^0 \equiv \mathbf{r} \in \mathbf{R}^n$, $(\mathbf{p}, \mathbf{r}) \neq 1$, there exist the invariant with respect to $\divideontimes$ limits,*

$$\mathbf{p}^\infty = \lim_{N\to\infty} \mathbf{p}^N, \quad \mathbf{r}^\infty = \lim_{N\to\infty} \mathbf{r}^N,$$

such that $\mathbf{p}^\infty \perp \mathbf{r}^\infty$, if $\mathbf{p} \neq \mathbf{r}$, and $\mathbf{p}^\infty = \mathbf{r}^\infty = (1/n, ..., 1/n)$, if $\mathbf{p} = \mathbf{r}$ and all starting coordinates p_i, r_i are strongly positive.

Remark 1. We note that if $p_i > r_i$, then $p_i^\infty > 0, r_i^\infty = 0$, in particular if $p_i > r_i$ only for one fixed i, then $\mathbf{p}^\infty = \mathbf{1}_i$, where $\mathbf{1}_i = (\underbrace{0, ...0}_{i-1}, 1, 0, ...0)$.

Remark 2. If $\mathbf{p} \neq \mathbf{r}$, but $p_i = r_i$ for some i, then $p_i^{(\infty)} = r_i^{(\infty)} = 0$.

We will now extend the above conflict composition for any pair of image measures $\mu, \nu \in \mathcal{M}(Q)$ and then to study the spectral properties of the limiting measures μ^∞, ν^∞.

Let μ and ν be a couple of image measures associated to a pair of matrices $P = \{\mathbf{p}_k\}_{k=1}^\infty$ and $R = \{\mathbf{r}_k\}_{k=1}^\infty$. We introduce the composition of conflict interaction between μ and ν, notation $\mu^1 := \mu \divideontimes \nu$, $\nu^1 := \nu \divideontimes \mu$, using the above defined conflict compositions for stochastic vectors in $\mathbf{R}^n$. Namely, a new couple of measures $\mu^1, \nu^1 \in \mathcal{M}(Q)$ is associated to matrices $P^1 = \{\mathbf{p}_k^1\}_{k=1}^\infty$ and $R^1 = \{\mathbf{r}_k^1\}_{k=1}^\infty$, where coordinates of vectors $\mathbf{p}_k^1, \mathbf{r}_k^1$ are defined according to formulae (7), i.e.,

$$p_{ik}^{(1)} := \frac{p_{ik}(1-r_{ik})}{1-(\mathbf{p}_k,\mathbf{r}_k)}, \quad r_{ik}^{(1)} := \frac{r_{ik}(1-p_{ik})}{1-(\mathbf{p}_k,\mathbf{r}_k)}, \ i = 1, ..., n, \ k = 1, 2, ... \tag{10}$$

Of course we have to assume

$$(\mathbf{p}_k, \mathbf{r}_k) \neq 1, \ k = 1, 2, ... \qquad (11)$$

By induction we introduce matrices $P^N = \{\mathbf{p}_k^N\}_{k=1}^{\infty}$ and $R^N = \{\mathbf{r}_k^N\}_{k=1}^{\infty}$ for any $N = 1, 2, ...$ with stochastic vectors $\mathbf{p}_k^N = \mathbf{p}_k^{N-1} \circledast \mathbf{r}_k^{N-1}$, $\mathbf{r}_k^N = \mathbf{r}_k^{N-1} \circledast \mathbf{p}_k^{N-1}$ defined as N-times iteration of the vector composition $\circledast$, i.e., coordinates of $\mathbf{p}_k^N, \mathbf{r}_k^N$ are calculated by formulae of view (9) with $p_{ik} \equiv p_{ik}^{(0)}$, $r_{ik} \equiv r_{ik}^{(0)}$ and $\mathbf{r}_k \equiv \mathbf{r}_k^0$, $\mathbf{r}_k \equiv \mathbf{r}_k^0$.

Further, with each pair P^N, R^N we associate the couple of image measures $\mu^N \equiv \mu_{P^N}$ and $\nu^N \equiv \nu_{R^N}$ from $\mathcal{M}(Q)$. Therefore the mapping $f^{\circledast}$ (see (8)) extended on measures generates the dynamical system in the space $\mathcal{M}(Q) \times \mathcal{M}(Q)$. The following theorem entirely based on Theorem 2 establishes the existence and characterizes the structure of the limiting points of this dynamical system.

Theorem 3. *For each couple of image measures $\mu, \nu \in \mathcal{M}(Q)$ satisfying condition (11), there exist the invariant with respect to $\circledast$ limiting measures,*

$$\mu^{\infty} = \lim_{N \to \infty} \mu^N, \nu^{\infty} = \lim_{N \to \infty} \nu^N.$$

The measures $\mu^{\infty}, \nu^{\infty}$ are mutually singular, if $\mu \neq \nu$, and $\mu^{\infty}, \nu^{\infty}$ are identical, if $\mu = \nu$.

Metric properties of the limiting measures

Starting from a couple of image measures $\mu, \nu \in \mathcal{M}(Q)$ associated to the above matrices $P = \{\mathbf{p}_k\}_{k=1}^{\infty}$ and $R = \{\mathbf{r}_k\}_{k=1}^{\infty}$ let us introduce the following notations:

$$\mathbf{N}_= := \{k : \mathbf{p}_k = \mathbf{r}_k\}, \quad \mathbf{N}_{\neq} := \mathbf{N} \setminus \mathbf{N}_= \equiv \{k : \mathbf{p}_k \neq \mathbf{r}_k\};$$

$$V_k(\mu) := \{i : p_{ik} > r_{ik}\}, \quad V_k(\nu) := \{i : p_{ik} < r_{ik}\}, \quad E_k := \{i : p_{ik} = r_{ik}\};$$

$$S_m(\mu) := \{k : |V_k(\mu)| = m\}, \quad S_m(\nu) := \{k : |V_k(\nu)| = m\}, \ m = 1, ..., n-1,$$

where $|A|$ stands for cardinality of a set A. Obviously $V_k(\mu) \bigcup V_k(\nu) \bigcup E_k = \{1, 2, ..., n\}$, $|V_k(\mu)| + |V_k(\nu)| + |E_k| = n$ for each k, and $\mathbf{N}_{\neq} = \bigcup_{m=1}^{n-1} S_m(\mu) = \bigcup_{m=1}^{n-1} S_m(\nu)$. Let us denote else $n_k := n - |E_k^0|$, where $E_k^0 := \{i : p_{ik} = r_{ik} = 0\}$.

Theorem 4. *Assume that one of the following conditions is fulfilled:*

$$(a) \ |E_k| = n - 2, \ k \in \mathbf{N}_{\neq}; \quad (b) \ \sum_{m=2}^{n-1} |S_m(\mu)| < \infty.$$

Then μ^{∞} (as well as ν^{∞}) belongs to $\mathcal{M}_{pp}$ iff $|\mathbf{N}_=| < \infty$.

Proof. Excepting a finite many $k \in S_m(\mu)$ each of conditions (a), (b) implies that coordinates p_{ik}^{∞} are equal to 0 or 1 if $k \in \mathbf{N}_{\neq}$. Therefore by Theorem 1, $P_{max}(\mu^{\infty}) > 0$ if and only if $|\mathbf{N}_=| < \infty$. $\square$

It is easy to see that under conditions of Theorem 4 suppμ^{∞} consists of exactly $2^{|\mathbf{N}_=|} \cdot \prod_{m=2}^{n-1} m^{|S_m(\mu)|}$ points.

We remark that in general the condition $|\mathbf{N}_=| < \infty$ is only the necessary one for $\mu^{\infty} \in \mathcal{M}_{pp}$. In particular we have $P_{max}(\mu^{\infty}) > 0$ and hence $\mu^{\infty} \in \mathcal{M}_{pp}$ if the sequence $\mathbf{p}_k^{\infty}$, $k \in \mathbf{N}_{\neq}$ converges to a some vector $\mathbf{1}_i$ as $k \to \infty$.

Now let $|\mathbf{N}_=| = \infty$ and therefore μ^{∞}, ν^{∞} belong to $\mathcal{M}_{ac} \bigcup \mathcal{M}_{sc}$. By Theorem 1 $\mu^{\infty} \in \mathcal{M}_{ac}$ iff $\rho(\mu^{\infty}, \lambda) > 0$.

Theorem 5. *(i) Let $|\mathbf{N}_{\neq}| < \infty$. Then μ^∞ belong to $\mathcal{M}_{\rm ac}$ if*

$$q_{ik} = \frac{1}{n_k},\ k \in \mathbf{N}_{=}.$$

(ii) Let $|\mathbf{N}_{\neq}| = \infty$. Assume $|E_k^0| = n-2,\ k \in \mathbf{N}_{=}$. Then $\mu^\infty \in \mathcal{M}_{ac}$ iff $|\mathbf{N}_{=}| = \infty$ and the following inequalities are fulfilled

$$\sum_{k\in\mathbf{N}_{=},\ i\in V_k(\mu),\ j\in V_k(\nu)} (1-4q_{ik}q_{jk}) < \infty, \qquad \sum_{k\in\mathbf{N}_{\neq},\ j\in V_k(\mu)} q_{jk} < \infty. \quad (12)$$

(iii) Let $|\mathbf{N}_{=}| = |\mathbf{N}_{\neq}| = \infty$ and $|\mathbf{N}_{\neq} \setminus S_1(\mu)| < \infty$. Then $\mu^\infty \in \mathcal{M}_{\rm ac}$, if

$$\sum_{k\in S_1(\mu)} \left(\sum_{i\in E_k \bigcup V_k(\nu)} q_{ik} \right) < \infty, \qquad (13)$$

and for all $k \in \mathbf{N}_{=}$ the condition $q_{ik} = \frac{1}{n_k}$ is fulfilled.

Proof. (i) Put

$$c_{\neq} = \prod_{k\in\mathbf{N}_{\neq}} \left(\sum_{i=1}^{n} \sqrt{p_{ik}^{(\infty)} \cdot q_{ik}} \right)$$

Then

$$\rho(\mu^\infty, \lambda) = c_{\neq} \prod_{k\in\mathbf{N}_{=}} \left(\sum_{i=1}^{n} \sqrt{p_{ik}^{(\infty)} \cdot q_{ik}} \right) = c_{\neq} > 0,$$

since $p_{ik}^\infty = 1/n_k$ and $(\sqrt{\mathbf{p}_k^\infty}, \sqrt{\mathbf{q}_k}) = 1$ for all $k \in \mathbf{N}_{=}$.

(ii) Proof easy follows from Theorem 3(b) in [1].

(iii) Since $|\mathbf{N}_{=}| = \infty$, $\mu^\infty \in \mathcal{M}_{ac} \bigcup \mathcal{M}_{sc}$. By Theorem 1 $\mu^\infty \in \mathcal{M}_{ac}$ iff $\rho(\mu^\infty, \lambda) > 0$.

Taking into account that $p_{ik}^{(\infty)} = 1/n_k$ for all $i = 1, 2, ...n$ and $k \in \mathbf{N}_{=}$, and the fact that $p_{ik}^{(\infty)} = 0$, if $k \in \mathbf{N}_{\neq},\ i \in E_k \bigcup V_k(\nu)$, $\quad p_{ik}^{(\infty)} = 1$, if $k \in S_1(\mu),\ i \in V_k(\mu)$, and $p_{ik}^{(\infty)} = a,\ 0 < a < 1$, if $k \in \mathbf{N}_{\neq} \setminus S_1(\mu), i \in V_k(\mu)$, we have

$$\rho(\mu^\infty, \lambda) = \prod_{k\in\mathbf{N}_{=}} \left(\frac{1}{\sqrt{n}_k} \sum_{i=1}^{n} \sqrt{q_{ik}} \right) \cdot \prod_{k\in S_1(\mu)} \left(\sum_{i\in V_k(\mu)} \sqrt{q_{ik}} \right) \times$$
$$\times \prod_{k\in N_{\neq}\setminus S_1(\mu)} \left(\sum_{i\in V_k(\mu)} \sqrt{p_{ik}^{(\infty)} q_{ik}} \right).$$

Obviously the first term

$$\prod_{k\in\mathbf{N}_{=}} \left(\frac{1}{\sqrt{n}_k} \sum_{i=1}^{n} \sqrt{q_{ik}} \right) = 1$$

if $q_{ik} = 1/n_k,\ k \in \mathbf{N}_{=}$, and the latter term is positive due to $|\mathbf{N}_{\neq} \setminus S_1(\mu)| < \infty$. Therefore $\rho(\mu^\infty, \lambda) > 0$ if (13) is fulfilled. Indeed

$$\prod_{k\in S_1(\mu)} \left(\sum_{i\in V_k(\mu)} \sqrt{q_{ik}} \right) > 0 \Leftrightarrow \prod_{k\in S_1(\mu)} \left(\sqrt{1 - \sum_{i\in E_k \bigcup V_k(\nu)} q_{ik}} \right) > 0 \Leftrightarrow$$
$$\Leftrightarrow \prod_{k\in S_1(\mu)} \left(1 - \sum_{i\in E_k \bigcup V_k(\nu)} q_{ik} \right) > 0 \Leftrightarrow \sum_{k\in S_1(\mu)} \left(\sum_{i\in E_k \bigcup V_k(\nu)} q_{ik} \right) < \infty. \ \square$$

Theorem 6. *(i) Assume* $|E_k| = n-2,\ k \in \mathbf{N}_{\neq}$ *and* $|E_k^0| = n-2,\ k \in \mathbf{N}_{=}$. *Then* $\mu^\infty \in \mathcal{M}_{sc}$ *iff* $|\mathbf{N}_=| = \infty$ *and at least one of the conditions (12) is fulfilled.*

(ii) Let $|\mathbf{N}_=| = \infty$ *and* $|\mathbf{N}_{\neq}| = \infty$. *Then* $\mu^\infty \in \mathcal{M}_{sc}$ *if at least for one* $m = 1, 2, ...n-1$ *such that* $|S_m(\mu)| = \infty$, *we have*

$$\sum_{k \in S_m(\mu)} \left(\sum_{i \in E_k \bigcup V_k(\nu)} q_{ik} \right) = \infty.$$

Proof. (i) If one of conditions (12) is fulfilled and $|\mathbf{N}_=| = \infty$, then $\mu^\infty \in \mathcal{M}_{ac} \bigcup \mathcal{M}_{sc}$ and under the theorem assumptions we have $\mu^\infty \perp \lambda$ due to Theorem 5 (ii). Therefore $\mu^\infty \in \mathcal{M}_{sc}$.

Conversely, if we assume $|E_k| = n-2$, $k \in \mathbf{N}_{\neq}$, $|E_k^0| = n-2$, $k \in \mathbf{N}_=$ and $\mu^\infty \in \mathcal{M}_{sc}$, then $|\mathbf{N}_=| = \infty$, and $\rho(\mu^\infty, \lambda) = 0$ since $\mu^\infty \perp \lambda$. Therefore one of the conditions (12) is fulfilled. $\square$

(ii) We recall that by Theorem 1 $\mu^\infty \in \mathcal{M}_{sc}$ iff $P_{max}(\mu^\infty) = 0$ and $\rho(\mu^\infty, \lambda) = 0$ Let $|\mathbf{N}_=| = \infty$, then $P_{max}(\mu^\infty) = 0$, and therefore $\mu^\infty \in \mathcal{M}_{ac} \bigcup \mathcal{M}_{sc}$. We can write

$$\rho(\mu^\infty, \lambda) = \prod_{k \in \mathbf{N}_=} \left(\sum_{i=1}^{n} \sqrt{p_{ik}^{(\infty)} q_{ik}} \right) \cdot \prod_{k \in S_1(\mu)} \left(\sum_{i \in V_k(\mu)} \sqrt{p_{ik}^{(\infty)} q_{ik}} \right) \cdot ... \times$$

$$\times \prod_{k \in S_{n-1}(\mu)} \left(\sum_{i \in V_k(\mu)} \sqrt{p_{ik}^{(\infty)} q_{ik}} \right).$$

Let us consider some $m_0 \in \{1, 2, ...n-1\}$ such that $|S_{m_0}| = \infty$ and

$$\sum_{k \in S_{m_0}(\mu)} \left(\sum_{i \in E_k \bigcup V_k(\nu)} q_{ik} \right) = \infty. \qquad (14)$$

Condition (14) is equivalent to

$$\prod_{k \in S_{m_0}(\mu)} \left(1 - \sum_{i \in E_k \bigcup V_k(\nu)} q_{ik} \right) = 0, \text{or} \prod_{k \in S_{m_0}(\mu)} \sqrt{\sum_{i \in V_k(\mu)} q_{ik}} = 0.$$

So we get

$$\prod_{k \in S_{m_0}(\mu)} \left(\sum_{i \in V_k(\mu)} p_{ik}^{(\infty)} q_{ik} \right)^{1/2} = 0$$

since

$$q_{ik} \geq p_{ik}^{(\infty)} q_{ik}.$$

Therefore $\rho(\mu^\infty, \lambda) = 0$ and $\mu^\infty \in \mathcal{M}_{sc}$. $\square$

Topological properties of suppμ^∞

We recall here some definition for details see [2]. A Borel measure μ on $\mathbf{R}$ has the S type if its support, supp$\mu \equiv S_\mu$, is a regularly closed set, i.e., $S_\mu = (\mathrm{int} S_\mu)^{\mathrm{cl}}$, where int$A$ denotes the interior part of a set A and cl stands for the closure. A measure μ has the C type if S_μ is a set of zero Lebesgue measure. A measure μ has the P type if S_μ is a nowhere dense set with property: $\forall x \in S_\mu,\ \forall \varepsilon > 0 :\ \lambda(B(x,\varepsilon)\bigcap S_\mu) > 0$.

We write $\mu \in \mathcal{M}^{\mathrm{S}}, \mathcal{M}^{\mathrm{C}}$, or $\mathcal{M}^{\mathrm{P}}$, if μ has S, C, or P type, resp.

Let us introduce a set $\mathbf{N}_{=,0} := \{k \in \mathbf{N}_= : \exists i, p_{ik} = r_{ik} = 0\}$ and put

$$W(\mu) := \sum_{k\in\mathbf{N}_{\neq}} \left(\sum_{i\in E_k \bigcup V_k(\nu)} q_{ik} \right) + \sum_{k\in\mathbf{N}_{=,0}} \left(\sum_{i:p_{ik}=0} q_{ik} \right)$$

Theorem 7. *The infinite conflict interaction between image measures $\mu, \nu \in \mathcal{M}(Q)$ produce the limiting invariant measures μ^∞, ν^∞ (see Theorem 2) of a pure topological type. Namely:*

(a) $\mu^\infty \in \mathcal{M}^{\mathrm{S}}$, *iff* $|\mathbf{N}_{\neq} \bigcup \mathbf{N}_{=,0}| < \infty$,

(b) $\mu^\infty \in \mathcal{M}^{\mathrm{C}}$, *iff* $|\mathbf{N}_{\neq} \bigcup \mathbf{N}_{=,0}| = \infty$ *and* $W(\mu) = \infty$,

(c) $\mu^\infty \in \mathcal{M}^{\mathrm{P}}$, *iff* $|\mathbf{N}_{\neq} \bigcup \mathbf{N}_{=,0}| = \infty$ *and* $W(\mu) < \infty$.

Proof. (a) By Theorem 8 in [3] the measure μ^∞ has S-type iff the matrix P^∞ contains only a finite number of zero elements. It occurs iff $|\mathbf{N}_{\neq} \bigcup \mathbf{N}_{=,0}| < \infty$.

(b) The measure μ^∞ has C type (see Theorem 8 in [3]) iff the matrix P^∞ contains infinitely many columns which contain zero elements, and besides, $\sum_{k=1}^{\infty}(\sum_{i:p_{ik}=0} q_{ik}) = \infty$. This is just equivalent to $|\mathbf{N}_{\neq} \bigcup \mathbf{N}_{=,0}| = \infty$ and $W(\mu) = \infty$.

(c) Finally the measure μ^∞ has P type (see again Theorem 8 in [3]) iff the matrix P^∞ contains infinitely many columns with some zero elements p_{ik}, and besides, $\sum_{k=1}^{\infty}(\sum_{i:p_{ik}=0} q_{ik}) < \infty$, i.e., iff $|\mathbf{N}_{\neq} \bigcup \mathbf{N}_{=,0}| = \infty$ and $W(\mu) < \infty$. $\square$

Remark 1. The assertions of Theorem 7 are also true for the measure ν^∞ if one changes $W(\mu)$ by $W(\nu)$.

Remark 2. Measures μ^∞, ν^∞ could not have the P type simultaneously. So if one of them has the P type, then other has necessarily the C type.

The measures μ^∞ and ν^∞ in general have rather complicated local structures and their supports might posses arbitrary Hausdorff dimensions.

Let us denote $\dim_{\mathrm{H}}(E)$ the Hausdorff dimension of a set $E \subset \mathbf{R}$.

Assume for simplicity that matrix Q contains only elements of a form $q_{ik} = 1/n$.

Theorem 8. *Given a measure $\mu \in \mathcal{M}(Q)$, and a number $c_0 \in [0,1]$ there exists the measure $\nu \in \mathcal{M}(Q)$ such that*

$$dim_{\mathrm{H}}(supp\ \mu^\infty) = c_0.$$

Proof. According to results of [10]

$$\dim_{\mathrm{H}}(\mathrm{supp}\ \mu^\infty) = \frac{1}{\ln n} \lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} \left(\sum_{i=1}^{n} p_{ij}^{(\infty)} \ln p_{ij}^{(\infty)} \right),$$

where we put $0 \cdot \ln 0 = 0$.

Let us consider a set $\mathcal{M}' \subset \mathcal{M}(Q)$ of probability measures such that for $\nu \in \mathcal{M}'$ $\mathbf{N}_{\neq} = S_1(\mu)$ and for any $k \in \mathbf{N}_{=}$ the kth vector of the corresponding matrix has no zero elements. Then for $\nu \in \mathcal{M}'$ we get that the limiting measure μ^{∞} corresponds to the matrix P with vectors

$$\mathbf{p}_k^{\infty} = \begin{cases} \mathbf{1}_i, \text{ if } k \in \mathbf{N}_{\neq}, \\ (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}), \text{ if } k \in \mathbf{N}_{=}. \end{cases}$$

Further for vectors $\mathbf{p}_k^{\infty} = \mathbf{1}_i$,

$$\sum_{i=1}^{n} p_{ij}^{(\infty)} \ln p_{ij}^{(\infty)} = \sum_{i=1}^{n-1} 0 \cdot \ln 0 + 1 \cdot \ln 1 = 0,$$

and for vectors $\mathbf{p}_k^{\infty} = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})$,

$$\sum_{i=1}^{n} p_{ij}^{(\infty)} \ln p_{ij}^{(\infty)} = \sum_{i=1}^{n} \frac{1}{n} \cdot \ln \frac{1}{n} = \ln \frac{1}{n}.$$

Thus, with notation $\mathbf{N}_{=,k} := \{s \in \mathbf{N}_{=} : s \leq k\}$, we get

$$dim_H(\text{supp } \mu^{\infty}) = \lim_{k \to \infty} \frac{1}{k} |\mathbf{N}_{=,k}|.$$

So, for any number $c_0 \in [0, 1]$ and any probability image measure μ one can always find a probability image measure $\nu \in \mathcal{M}'$ such that

$$\dim_{\mathrm{H}}(\text{supp } \mu^{\infty}) = c_0. \quad \square$$

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