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The conflict interaction between two complex systems. Cyclic migration

Abstract

We construct and study a discrete time model describing the conflict interaction between two complex systems with non-trivial internal structures. The external conflict interaction is based on the model of alternative interaction between a pair of non-annihilating opponents. The internal conflict dynamics is similar to the one of a predator-prey model. We show that the typical trajectory of the complex system converges to an asymptotic attractive cycle. We propose an interpretation of our model in terms of migration processes.

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1 Introduction

Since the beginning of 20-th century the Lotka-Volterra model of prey-predator interaction is one of the main models for the simulation of many processes in population dynamics and economics. As a rule, continuous models where Lotka-Volterra equations have ratio-dependent parameters are studied (see, for example [3, 5, 6, 11, 12, 13, 16, 17, 18]). Logistic and Ricker's type models are also studied in some works, for example [5]. In the majority of works the prey-predator

interaction is treated only inside of a single region, and no migration from one region to other regions is considered.

In some works [6, 5] models with a random process of migration are studied.

There are also only few works (see [5] and references wherein), in which discrete models are considered, though in reality such processes are more natural, since they take better into account seasonal phenomena (reproduction, migration, etc.).

The main aim of the majority of works is the discussion of questions like the determination of stable points or bifurcation points, the study of the asymptotic behavior, and the analysis of model's dependence on the coefficients of the equations describing the dynamics.

In [3] the synchronization of population dynamics with natural phenomena (like change of seasons and floods) is studied. In the work [18] the dependence of the population dynamics on population density and migration is studied. In the work [6] migration is not assumed to be random, but aims at the maximization of some function of the population. At last, in [3] the influence of stochastic terms in a Lotka-Volterra model is described, and interesting figures are presented.

In the present work we construct a model that joins two most rarely studied variants of models of the Lotka-Volterra type, i.e., a discrete model with migration. Here individuals migrate not randomly, but according to strategies which we interpret in section 5.

Our models contains a conflict interaction between a pair of complex systems A and B. The whole system is described by a finite set of positive numbers: $\mathbf{P} = (P_1, \ldots, P_N)$ for A and $\mathbf{R} = (R_1, \ldots, R_N)$ for B, where $N \geq 2$ means the quantity of parameters that characterize the system. We study dynamics in the discrete time denoted by $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. The evolution of every subsystem is described by a sequence of vectors with non-negative coordinates $\mathbf{P}^n = (P_1^{(n)}, \ldots, P_N^{(n)})$ for A, and $\mathbf{R}^n = (R_1^{(n)}, \ldots, R_N^{(n)})$ for B, $n \in \mathbb{N}_0$. The vectors $\mathbf{P}$ and $\mathbf{R}$ correspond to the initial time $n = 0$. The model is such that each subsystem A or B tries to reach optimal values of its coordinates. Due to the conflict interaction, every coordinate changes however in a complicated way. The evolution of the system is determined by a double dependence: by the conflict interaction between subsystems (which we describe in section 3), and by a prey-predator type interaction inside every subsystem. We suppose that every subsystem is complex in the sense that its elements may be treated as one of the types: dominant (predators) or dependent (preys). So, every coordinate $P_i^{(n)}$, resp. $R_i^{(n)}$ may be regarded as the quantity (population) of dominant, resp. dependent species at the position i at time n.

The law of evolution inside of each (independent) system is described in section

2. We suppose this law is identical in every system and is based on the discrete Lotka-Volterra equation.

In section 3 we shortly recall the main results on conflict interaction between non-annihilating opponents.

In section 4, that includes the main results of the work, we construct a dynamical system describing the conflict interactions both inside any subsystem A or B and between the subsystems.

In our model two operations happen at any fixed moment of time: redistribution of probabilities to occupy some controversial positions by the systems in opposition, and quantitative changes (namely population) of all species inside both systems.

A computer modeling of such a complex interaction shows the appearance of some interesting phenomena. In this work we limit however ourselves to present only the observation that under a certain appropriate choice of parameters and initial data, the complex system oscillates. Indeed we find a rather wide range of initial data for which the population trajectory in phase-space becomes cyclic. Moreover, we observe the stability of the limit cycle, which is thus an attractor.

2   Traditional models of population dynamics

Let us recall here shortly some traditional models of population dynamics, see e.g. [16] for more details.

Malthus proposed in 1798 an idealized evolution equation for a population:

$$\frac{dP}{dt} = (b - d)P, \tag{2.1}$$

where P is the cumulative number of individuals (species), and b, d are the natural birth and death rates. The solution is

$$P(t) = P(0)e^{(b-d)t}.$$

In practice, however, one expects exponential rise, if $b - d > 0$, resp. decrease, if $b - d < 0$ at most in a local period of time.

Verhulst introduced in 1838 a more realistic equation with saturation terms:

$$\frac{dP}{dt} = (b - d)P - cP^2, \tag{2.2}$$

where the coefficient $c > 0$ represents the competition activity of individuals for living resources. The presence of the square power takes care of an alternative law of access to the living resource.
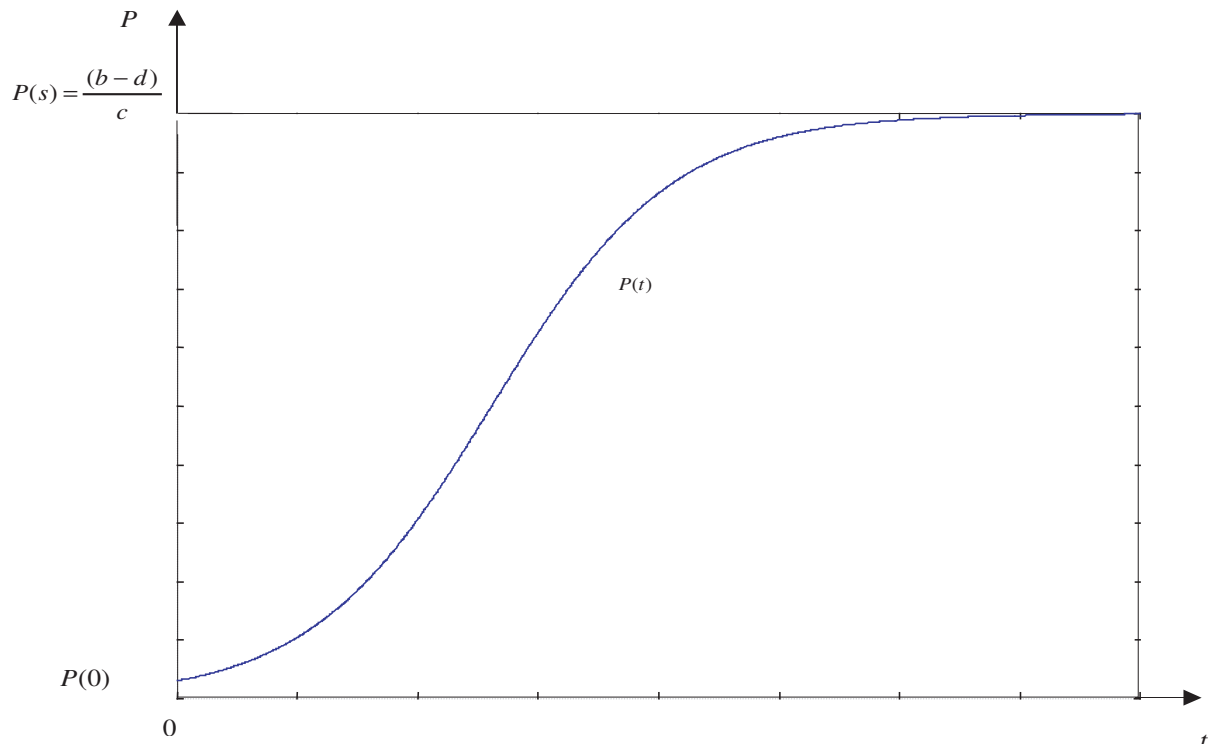
Figure 1: *A typical shape of the logistical curve $\frac{dP}{dt} = (b-d)P - cP^2$.*

The solution of (2.2) describes the S-shaped logistical curve (see Figure 1) and corresponds better to the actual behavior of many population processes.

The curve starts with a small value $P(0)$, increases exponentially , and then saturate at the capacity $P(s) = \frac{b-d}{c}, b - d > 0$.

In the economic context, equation (2.2) can be written as follows

$$\frac{dM}{dt} = (g - l)M - fM^2, \tag{2.3}$$

where M is the capital (money), g and l are the average gain and loss percentages on the capital, and f stands for the coefficient of confrontation between individuals. If $g - l < 0$, the capital decays to 0 exponentially; if $g - l > 0$, at the beginning the capital increases exponentially quick, but then the growth slows down, so that it never reaches the asymptotic value of saturation $M(s) = \frac{g-l}{f}$.

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Lotka (1907) and Volterra (1901) extended the Verhulst logistical equation to the well-known Lotka-Volterra equations intended for the description of the amount of changes in populations of two species in interaction. These equations are also known under the name of predator-prey model. We will refer to the Lotka-Volterra equations in the following form:

$$\begin{aligned}\frac{dP}{dt} &= aP - bPR - cP^2 \\ \frac{dR}{dt} &= -dR + ePR - fR^2,\end{aligned} \tag{2.4}$$

where all coefficients $a, b, ..., f$ are nonnegative.

The population of prey is described by the first equation. Without presence of any predators it grows exponentially at the beginning and then converges to the fixed capacity $P(s) = a/c$. The predators, without any prey to feed on, die out. When both species are present, the growth of the prey is limited by the predators, due to the term $-bPR$, and the predators grow if the amount ePR of prey available, is large enough. See for example [16] and references wherein for more information or the corresponding analysis of Lotka-Volterra equations.

The models with discrete time are also studied. In this case, equations (2.4) have the following form:

$$\begin{aligned}P_1^{(n)} &= P_1^{(n-1)} + P_1^{(n-1)}(a - bP_2^{(n-1)} - cP_1^{(n-1)}), \\ P_2^{(n)} &= P_2^{(n-1)} + P_2^{(n-1)}(-d + eP_1^{(n-1)} - fP_2^{(n-1)}).\end{aligned} \tag{2.5}$$

The typical behaviour of discrete Lotka-Volterra model is shown in Figure 2.

3 Conflict interaction between non-annihilating opponents

In this section we shortly recall an alternative approach to describe the redistribution of conflicting positions between two opponents, say A and B, concerning an area of common interests.

We consider the simplest case where the existence space of common interests is a finite set of positions $\Omega = \{\omega_1, \ldots, \omega_N\}, N \geq 2$. Each of the opponents A and B tries to occupy a position $\omega_i, i = 1, \ldots, N$ with a probability $P_A(\omega_i) = p_i \geq 0$ resp. $P_B(\omega_i) = r_i \geq 0$. The starting distributions of A and B along Ω are arbitrary and normalized: $\sum_{i=1}^N p_i = 1 = \sum_{i=1}^N r_i$. A and B can not be present simultaneously in the same position ω_i. The interaction between A and B is considered in discrete time $t \in \mathbb{N}_0$. We introduce the noncommutative conflict composition between stochastic vectors $\mathbf{p}^0 = (p_1, \ldots, p_N), \mathbf{r}^0 = (r_1, \ldots, r_N) \in \mathbb{R}_+^N$:

$$\mathbf{p}^1 := \mathbf{p}^0 * \mathbf{r}^0, \mathbf{r}^1 = \mathbf{r}^0 * \mathbf{p}^0, \mathbf{p}^0 \equiv \mathbf{p}, \mathbf{r}^0 \equiv \mathbf{r},$$
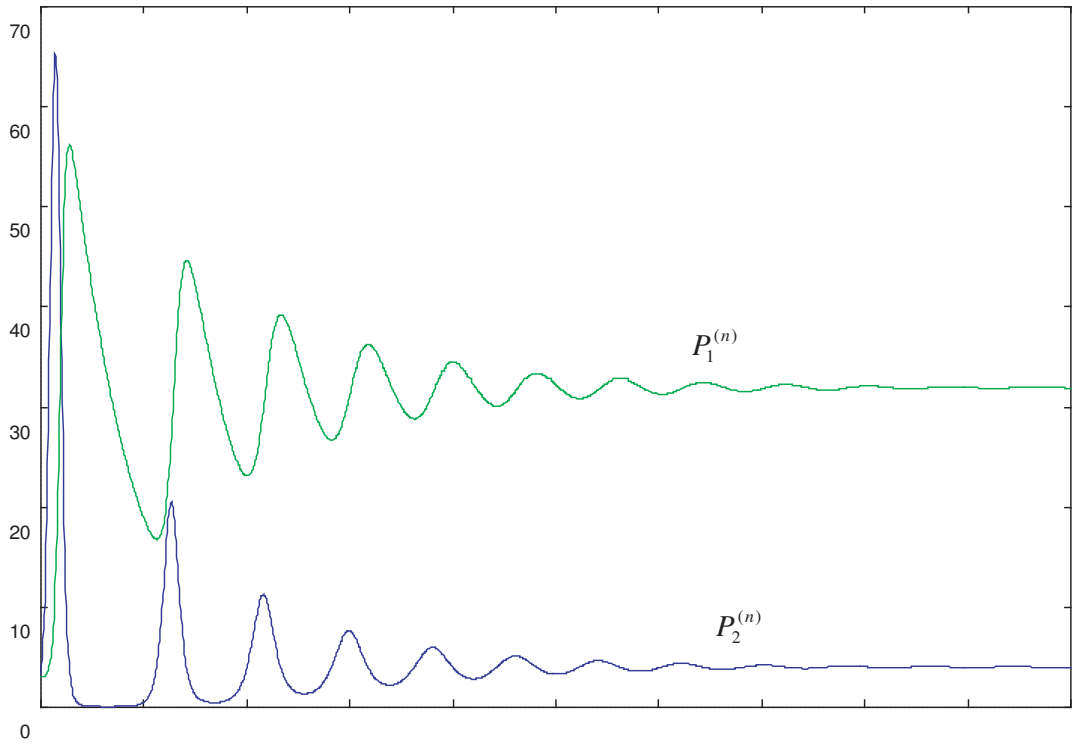
Figure 2: *Lotka-Volterra model with discrete time*

$$P_1^{(n)} = P_1^{(n-1)}(a - bP_1^{(n-1)} - cP_1^{(n-1)})$$

$$P_2^{(n)} = P_2^{(n-1)}(-d + eP_2^{(n-1)} - fP_2^{(n-1)})$$

$$a = 0.2, b = 0.006, c = 0.002, d = 0.008, e = 0.002, f = 0,$$

$$P_1^{(0)} = 3, P_2^{(0)} = 5.$$

where the coordinates of $\mathbf{p}^1, \mathbf{r}^1$ are defined as follows

$$p_i^{(1)} = \frac{p_i^{(0)}(1 - \alpha r_i^{(0)})}{1 - \alpha \sum_{i=1}^N p_i^{(0)} r_i^{(0)}}, r_i^{(1)} = \frac{r_i^{(0)}(1 - \alpha p_i^{(0)})}{1 - \alpha \sum_{i=1}^N p_i^{(0)} r_i^{(0)}}, \tag{3.1}$$

and where the coefficient $-1 \le \alpha \le 1$, $\alpha \ne 0$ standing for the activity interaction. At the nth step of the conflict dynamics we get two vectors

$$\mathbf{p}^n = \mathbf{p}^{n-1} * \mathbf{r}^{n-1} \equiv \mathbf{p}^0 *^n \mathbf{r}^0, \mathbf{r}^n = \mathbf{r}^{n-1} * \mathbf{p}^{n-1} \equiv \mathbf{r}^0 *^n \mathbf{p}^0$$

with coordinates

$$p_i^{(n)} = \frac{p_i^{(n-1)}(1 - \alpha r_i^{(n-1)})}{z_n}, r_i^{(n)} = \frac{r_i^{(n-1)}(1 - \alpha p_i^{(n-1)})}{z_n},$$

with z_n a normalization coefficient given by

$$z_n = 1 - \alpha(\mathbf{p}^{n-1}, \mathbf{r}^{n-1}),$$

$(\cdot, \cdot)$ being the inner product in $\mathbb{R}^N$.

The behavior of the state $\{\mathbf{p}^n, \mathbf{r}^n\}$ at time $t = n$, for $n \to \infty$, has been investigated in [1, 4, 7, 8, 9, 10]. We shortly describe the results (for proofs see, e.g. [8, 1]).

Theorem.1. *For any pair of non-orthogonal stochastic vectors $\mathbf{p}, \mathbf{r} \in \mathbb{R}_+^N$, $(\mathbf{p}, \mathbf{r}) > 0$, and fixed interaction intensity parameter $\alpha \ne 0$, $-1 \le \alpha \le 1$, with condition $\alpha \ne \frac{1}{(\mathbf{p},\mathbf{r})}$, the sequence of states $\{\mathbf{p}^n, \mathbf{r}^n\}$ tends to the limit state $\{\mathbf{p}^\infty, \mathbf{r}^\infty\}$*

$$\mathbf{p}^\infty = \lim_{n\to\infty} \mathbf{p}^n, \quad \mathbf{r}^\infty = \lim_{n\to\infty} \mathbf{r}^n.$$

This limit state is invariant with respect to the conflict interaction:

$$\mathbf{p}^\infty = \mathbf{p}^\infty \divideontimes \mathbf{r}^\infty, \quad \mathbf{r}^\infty = \mathbf{r}^\infty \divideontimes \mathbf{p}^\infty.$$

Moreover,

$$\begin{cases} \mathbf{p}^\infty \perp \mathbf{r}^\infty, & \text{if } \mathbf{p} \ne \mathbf{r} \quad \text{and} \quad 0 < \alpha \le 1 \\ \mathbf{p}^\infty = \mathbf{r}^\infty, & \text{in all other cases.} \end{cases}$$

We emphasize that in the case of a purely repulsive interaction, $0 < \alpha \le 1$, if the starting distributions are different, then the limiting vectors are orthogonal. Therefore each of the vectors $\mathbf{p}^\infty, \mathbf{r}^\infty$ contains by necessity some amount of zero
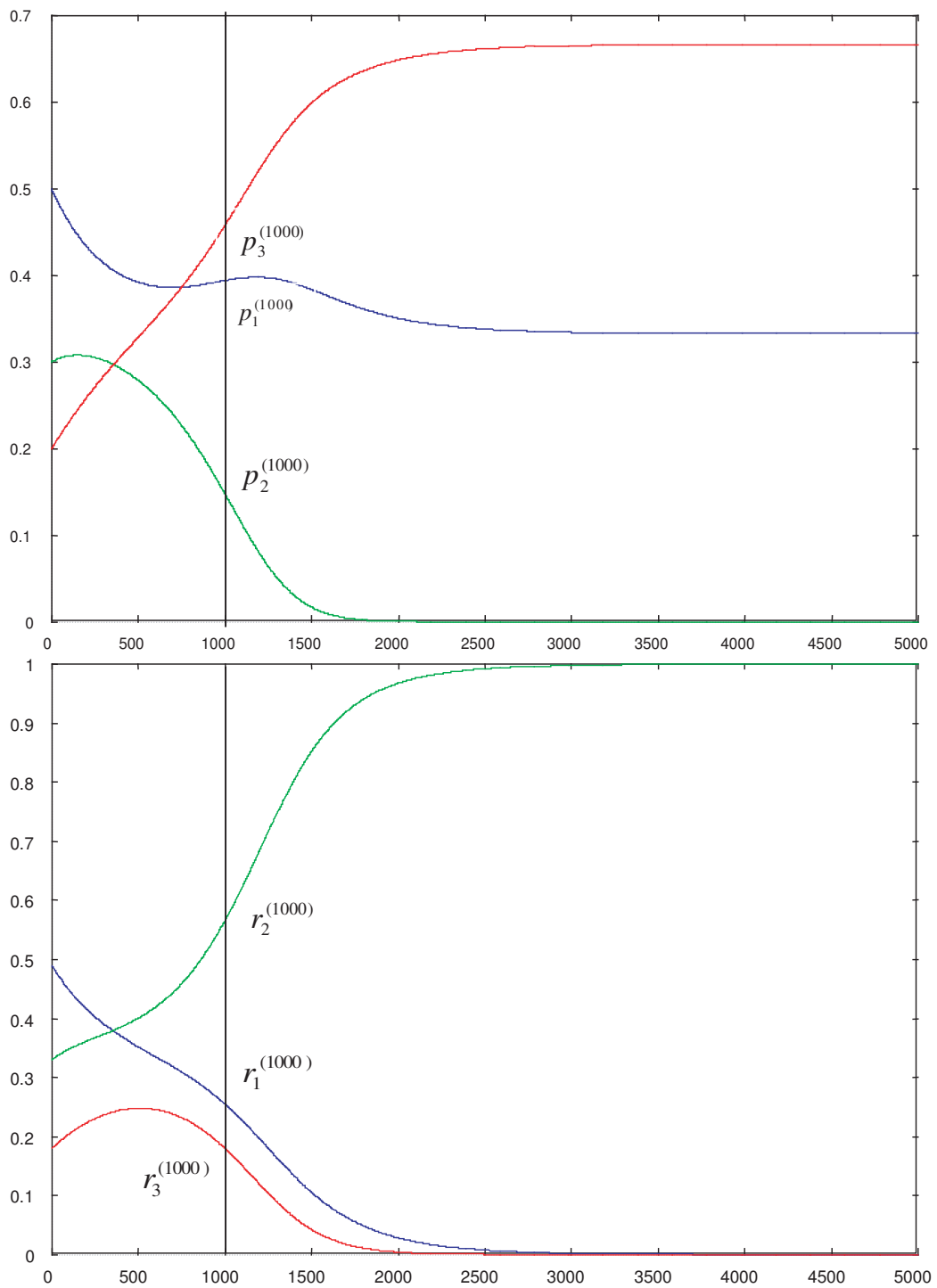
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Figure 3: $\alpha = 1, \mathbf{p}^0 = (0.5; 0.3; 0.2), \mathbf{p}^0 = (0.48; 0.34; 0.18)$
$\mathbf{p}^\infty = (0.33; 0; 0.67), \mathbf{r}^\infty = (0; 1; 0).$

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coordinates. For example the typical limit picture for $\mathbf{p}^n, \mathbf{r}^n \in \mathbb{R}_+^3$ is presented in Figure 3.

If we start with a pair of identical vectors, $\mathbf{p} = \mathbf{r}$, then $\mathbf{p}^\infty = \mathbf{r}^\infty$ too. That is, all non-zero coordinates of the limit vectors are equal.

In the general case, $\mathbf{p}, \mathbf{r} \in \mathbb{R}_+^N$, the coordinates $p_i^{(n)}, r_i^{(n)}$ have at most several oscillations and then reach monotonically their positive or zero limits. The limiting values p_i^∞, r_i^∞ may be described in terms of starting states.

Given a couple of stochastic vectors $\mathbf{p}, \mathbf{r} \in \mathbb{R}_+^n$, $\mathbf{p} \neq \mathbf{r}$, $(\mathbf{p}, \mathbf{r}) > 0$, define

$$D_+ := \sum_{i \in \mathbb{N}_+} d_i, \ D_- := \sum_{i \in \mathbb{N}_-} d_i,$$

where

$$d_i = p_i - r_i, \ \ \mathbb{N}_+ := \{i : d_i > 0\}, \ \mathbb{N}_- := \{i : d_i < 0\}.$$

Obviously

$$0 < D_+ = -D_- < 1,$$

since $\mathbf{p} \neq \mathbf{r}$, and $\sum_i p_i - \sum_i r_i = 0 = D_+ + D_-$.

Theorem 2. *Let $\mathbf{p} \neq \mathbf{r}$, $(\mathbf{p}, \mathbf{r}) > 0$. In the purely repulsive case, $\alpha = 1$, the coordinates of the limit vectors $\mathbf{p}^\infty, \mathbf{r}^\infty$ have the following explicit distributions:*

$$p_i^\infty = \begin{cases} d_i/D, & i \in \mathbb{N}_+ \\ 0, & otherwise \end{cases}, \ r_i^\infty = \begin{cases} -d_i/D, & i \in \mathbb{N}_- \\ 0, & otherwise, \end{cases} \quad (3.2)$$

where $D := D_+ = -D_-$.

Remark. From (3.2) it follows that any transformation $\mathbf{p}, \mathbf{r} \to \mathbf{p}', \mathbf{r}'$, which does not change the values d_i and D, preserves the same limit distribution as for the vectors $\mathbf{p}^\infty, \mathbf{r}^\infty$. A class of such transformations may be presented by a shift transformation of coordinates, $p_i \to p_i' = p_i + a_i$, $r_i \to r_i' = r_i + a_i$ with appropriated $a_i's$.

In the case $-1 \leq \alpha < 0$ of the pure attractive interaction we have another limit distribution.

Define the set $\mathbb{S}_0 := \{k | p_k^\infty = r_k^\infty = 0\}$ and set

$$\mathbb{S}^\infty := \{1, \ldots, N\} \backslash \mathbb{S}_0.$$

Theorem 3. *In the purely attractive case, $\alpha = -1$, the limit vectors $\mathbf{p}^\infty, \mathbf{r}^\infty$ are equal and their coordinates have the following distributions:*

$$p_i^\infty = r_i^\infty = \begin{cases} 1/m, & i \in \mathbb{S}^\infty \\ 0, & otherwise, \end{cases} \quad (3.3)$$

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where $m = |\mathbb{S}^\infty|$ denotes the cardinality of the set $\mathbb{S}^\infty$.

In general, it is an open question to give a complete characterization of $\mathbb{S}_0$. Below we present several sufficient conditions for k to belong to the set $\mathbb{S}_0$. Simultaneously these conditions give some characterization for the points to be in $\mathbb{S}^\infty$.

We will use the following notations:

$$\sigma_i := p_i + r_i, \ \ \rho_i := p_i r_i, \ \ \sigma_i^1 := p_i^1 + r_i^1 \ \ \rho_i^1 := p_i^1 r_i^1. \tag{3.4}$$

Proposition 1. *If*

$$\sigma_i \geq \sigma_k, \ \ \rho_i > \rho_k, \ \ \text{or} \ \ \sigma_i > \sigma_k, \ \ \rho_i \geq \rho_k, \tag{3.5}$$

then

$$p_k^\infty = r_k^\infty = 0,$$

and therefore $k \in \mathbb{S}_0$.

Proof. By (3.4) we have

$$\sigma_k^1 = p_k^1 + r_k^1 - 1/z(p_k + r_k + 2p_k r_k) = 1/z(\sigma_k + 2\rho_k)$$

where we recall that $z = 1 + (\mathbf{p}, \mathbf{r})$. Therefore each of the conditions (3.5) implies that $\sigma_i^1 > \sigma_k^1$. Further, since

$$\rho_k^1 = 1/z^2(\rho_k + (\rho_k)^2 + \rho_k \sigma_k), \tag{3.6}$$

again from (3.5) it also follows that $\rho_i^1 > \rho_k^1$. Thus, by induction, $\sigma_i^N > \sigma_k^N$ and $\rho_i^N > \rho_k^N$ for all $N \geq 1$.

Or, in other words,

$$1 < \frac{p_i}{p_k} < \frac{p_i^1}{p_k^1} < \ldots < \frac{p_i^N}{p_k^N} \ldots,$$

$$1 < \frac{p_i}{p_k} < \frac{r_i^1}{r_k^1} < \ldots < \frac{r_i^N}{r_k^N} \ldots, N = 1, 2, \ldots \tag{3.7}$$

Thus, the sequences of the ratios

$$\frac{p_i^N}{p_k^N}, \frac{r_i^N}{r_k^N}$$

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are monotone increasing as $N \to \infty$. Assume for a moment that there exists a finite limit,

$$1 < \lim_{N \to \infty} \frac{p_i^N}{p_k^N} = \frac{p_i^\infty}{p_k^\infty} \equiv \frac{p_i^\infty}{p_k^\infty} \cdot \frac{1 + r_i^\infty}{1 + r_k^\infty} = M < \infty.$$

This is only possible if $r_i^\infty = r_k^\infty$, which contradicts (3.7). Thus, $M = \infty$ and therefore $p_k^\infty = 0$, as well as $r_k^\infty = 0$. $\square$

Let us consider now the critical situation where for a fixed pair of indices, say i and k, the values $\sigma_k - \sigma_i$, $\rho_k - \rho_i$ have opposite signs, for example, $\sigma_k - \sigma_i > 0$, $\rho_k - \rho_i < 0$. In such a case it is not clear what behavior the coordinates p_i^N, r_i^N and p_k^N, r_k^N will have when $N \to \infty$. We will show that the limits depend on which of the two values, $2\rho_i + \sigma_i$ or $2\rho_k + \sigma_k$, is larger. Moreover we will show that even if p_k is the largest coordinate, it may happen that $p_k^\infty = 0$. Let for example, $p_k = \max_j \{p_j, r_j\}$ and $\sigma_k = p_k + r_k > p_i + r_i = \sigma_i$, however the value of r_k is such that $\rho_k = p_k r_k < p_i r_i = \rho_i$. Then under some additional condition it is possible to have $p_k^\infty = 0$. In fact we have:

Proposition 2. *Let for the coordinates p_i, r_i, p_k, r_k, $i \neq k$, the following conditions be fulfilled:*

$$\sigma_k > \sigma_i \tag{3.8}$$

but

$$\rho_k < \rho_i. \tag{3.9}$$

Assume

$$2\rho_k + \sigma_k \leq 2\rho_i + \sigma_i. \tag{3.10}$$

Then

$$p_k^\infty = r_k^\infty = 0, \tag{3.11}$$

i.e., $k \in \mathbb{S}_0$

Proof. We will show that (3.8), (3.9), and (3.10) imply

$$p_k^1 + r_k^1 = \sigma_k^1 \leq \sigma_i^1 = p_i^1 + r_i^1 \tag{3.12}$$

and

$$p_k^1 r_k^1 = \rho_k^1 < \rho_i^1 = p_i^1 r_i^1. \tag{3.13}$$

Then (3.11) follows from Proposition 1. In reality (3.12) follows from (3.10) directly, without condition (3.9). So, we have only to prove (3.13).

With this aim we find the representation of ρ_i^1 in terms σ_i and σ_i^1. Since $\sigma_i^1 = 1/z(\sigma_i + 2\rho_i)$ we have

$$\rho_i = 1/2(z\sigma_i^1 - \sigma_i). \tag{3.14}$$

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By (3.6) and (3.14) we get

$$\rho_i^1 = 1/z^2(\rho_i + \rho_i^2 + \rho_i\sigma_i) = \frac{1}{2z^2}(z\sigma_i^1 - \sigma_i)[1 + 1/2(z\sigma_i^1 - \sigma_i) + \sigma_i]$$

$$= \frac{1}{4z^2}(z\sigma_i^1 - \sigma_i)(2 + z\sigma_i^1 + \sigma_i) = \frac{1}{4z^2}[2z\sigma_i^1 + z^2(\sigma_i^1)^2 + z\sigma_i^1\sigma_i - 2\sigma_i - z\sigma_i^1\sigma_i - \sigma_i^1]$$

$$= \frac{1}{4z^2}[2z\sigma_i^1 + z^2(\sigma_i^1)^2 - \sigma_i^2 - 2\sigma_i].$$

Therefore

$$\rho_k^1 - \rho_i^1 = 1/z^2[\rho_k(1 + \rho_k + \sigma_k) - \rho_i(1 + \rho_i + \sigma_i)].$$

Thus, we have

$$\rho_k^1 - \rho_i^1 = 1/4z^2[2z(\sigma_k^1 - \sigma_i^1) + z^2((\sigma_k^1)^2 - (\sigma_i^1)^2) + ((\sigma_i)^2 - (\sigma_k)^2) + 2(\sigma_i - \sigma_k)] < 0$$

due to starting condition (3.9), and (3.12). Thus $\rho_k^1 < \rho_i^1$, i.e., (3.13) holds. □

We stress that (3.11) is true in spite of $\sigma_k > \sigma_i$. Of course, if $\sigma_k < \sigma_i$ and $\rho_k < \rho_i$, then (3.11) holds without any additional condition of the form (3.10).

4   Model of conflict interaction between complex systems

In this section we construct a dynamical model of conflict interaction between a pair of complex systems. Each of the systems is subject to the inner conflict between their elements. For simplicity, we assume both systems to be similar and described by discrete prey-predator models of the form (2.5). We introduce the conflict interaction between these systems using an approach developed in [1, 2, 4, 7, 8, 9, 10]. With such a rather complex interaction we obtain a wide spectrum of evolution. In this work we study qualitative characteristics of the behavior of corresponding dynamical systems for some choice of parameters a, b, c, d, e, f, α (see (2.5), (3.1)) and values of the initial populations of species P_i, R_i.

The coefficient α, giving the intensity of the interaction between subsystems, has an important effect. Increase α from zero to unit causes the appearance of a series of bifurcation. For $\alpha = 0$ we have two copies of independent Lotka-Volterra subsystems. For small values of α both subsystems behave like pure Lotka-Volterra systems, converging then to a stable state (see Figure 4).
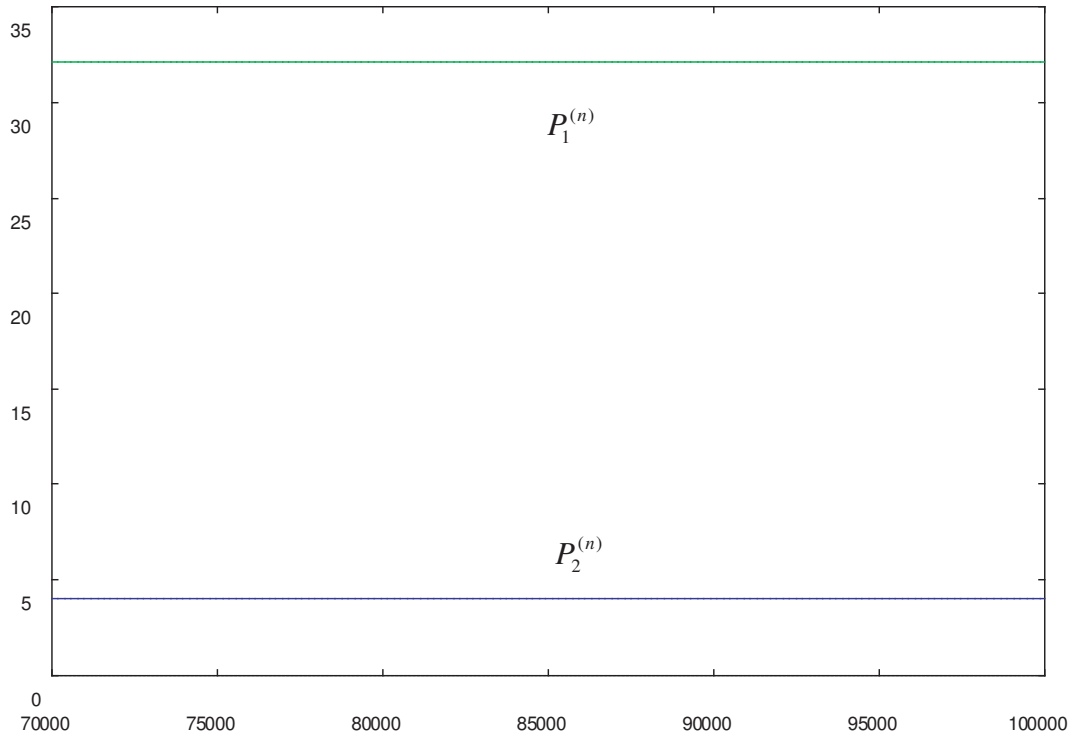
Figure 4: *Trajectory plot of Lotka-Volterra model with discrete time and a small conflict interaction after 70000 steps of iteration,*
$a = 0.2, b = 0.006, c = 0.002, d = 0.008, e = 0.002, f = 0,$

$$\alpha = 0.005,$$

$$P_1^{(0)} = 3, P_2^{(0)} = 10,$$
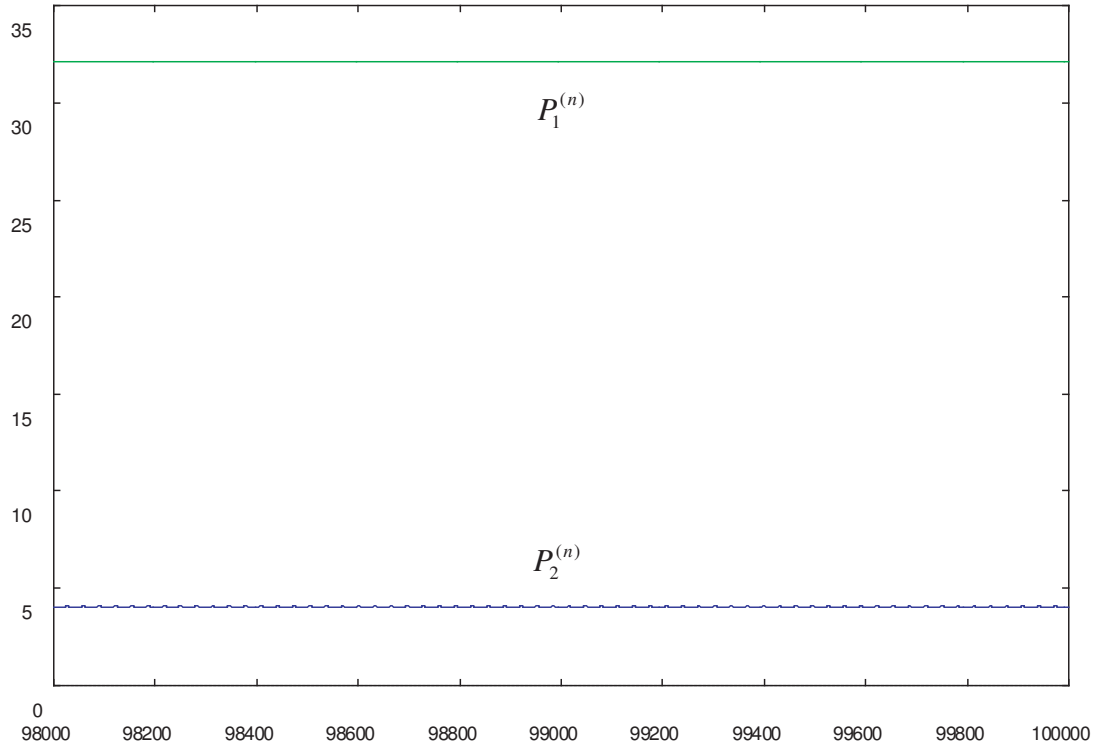$$R_1^{(0)} = 5, R_2^{(0)} = 20.$$

Figure 5: *The existence of the bifurcation point involves small oscillations. Lotka-Volterra model with discrete time and the conflict interaction after 98000 steps of iteration,*

$$a = 0.2, b = 0.006, c = 0.002, d = 0.008, e = 0.002, f = 0,$$

$$\alpha = 0.007,$$

$$P_1^{(0)} = 3, P_2^{(0)} = 10,$$

$$R_1^{(0)} = 5, R_2^{(0)} = 20.$$

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Under fixed parameters and the starting coordinates (see Figure 5) we have the first bifurcation point at $\alpha \approx 0.0056781739$. The coordinates $P_i^{(n)}(R_i^{(n)})$ oscillate and a cycle of a small period appears.

The following increase of α shows the appearance of new bifurcation points that are characterized by an increasing value of the cyclic period. For the value $\alpha = 0.4815545975$ a cycle of infinite period appears. This means that all coordinates rapidly reach the stable state. In this case some species may disappear, even if they had some stable positive values in a pure ($\alpha = 0$) Lotka-Volterra model.

The role of the coefficients a, b, c, d, e, f and the initial quantity of the species P_i, R_i in a pure Lotka-Volterra model is well-known and described (e.g. in [16, 12]). The coefficients a, d govern the increase of the prey population when predators are absent and the predator population decreases when the preys are absent. In turn, the coefficients b, e are responsible respectively for the prey quantity decreasing with an increasing number of predators, as well as for the increase of the predator population with an increase of the number of preys. The last coefficients in each of the equations give the limitation on the increase of both populations. In other words, each population "makes pressure" on itself, it does not permit an infinite reproducibility.

Questions about stable points, orbits, asymptotic behavior of orbits are well described for the classical Lotka-Volterra model. We shall recall that usually there are at least three equilibrium points. They are mentioned in the literature as follows (see, e.g., [16]):

(1) trivial (0,0);

(2) axial (a/b,0);

(3) inner positive

$$\left(\frac{a}{b} - \frac{b}{c}\frac{ae-cd}{be+cf}, \frac{ae-cd}{be+cf} \right). \tag{4.1}$$

An equilibrium point is called a stable point if after a sudden change of population it comes back to an equilibrium point some time later. This may happen monotonically, or with some oscillations.

We should note that under the existence of stable points the behavior of the system is well defined by the coefficients a, b, c, d, e, f. But under the absence of stable points, the behavior of the system is defined by the initial data P_i, R_i. Depending on how close the initial data are situated with respect to the equilibrium point, the system may evolve in a different way.

The role of all these coefficients is preserved in the case of our model. But now their influence is much more complex. We present here only first steps in this direction. We shall discuss not only stability zones, as it was pointed out

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above, but also the values of the coefficients for which the system oscillates along some closed cycles.

The state of our dynamical system is fixed by a pair of vectors $\mathbf{P}^n = (P_1^{(n)}, \ldots, P_N^{(n)})$, $\mathbf{R}^n = (R_1^{(n)}, \ldots, R_N^{(n)})$ with non-negative coefficients, where $n \in \mathbb{N}_0$ denotes the discrete time, $N \geq 2$ stands for the number of conflict positions. Here we study the most simple situation, when every system consists of only two agents: prey and predator, i.e., $N = 2$. The complex conflict transformation is denoted by the mapping

$$\begin{pmatrix} \mathbf{P}^n \\ \mathbf{R}^n \end{pmatrix} \xrightarrow{F} \begin{pmatrix} \mathbf{P}^{n+1} \\ \mathbf{R}^{n+1} \end{pmatrix},$$

where F is the composition of four operations, $\mathcal{N}^{-1}, *, \mathcal{N}$, and U.

Let us describe these operations in an explicit form for the first step of the complex conflict transformation.

The first operation U describes the interaction between elements inside every system separately according to the prey-predator model. The corresponding mathematical transformation of vectors (the interaction composition) $\{\mathbf{P}^0, \mathbf{R}^0\} \xrightarrow{U} \{\widetilde{\mathbf{P}}^0, \widetilde{\mathbf{R}}^0\}$ is described by the system of equations of the form (2.5):

$$\begin{aligned} \widetilde{P}_1^{(0)} &= P_1^{(0)} + P_1^{(0)}(a - bP_2^{(0)} - cP_1^{(0)}), \\ \widetilde{P}_2^{(0)} &= P_2^{(0)} + P_2^{(0)}(-d + eP_1^{(0)} - fP_2^{(0)}), \end{aligned}$$

and

$$\begin{aligned} \widetilde{R}_1^{(0)} &= R_1^{(0)} + R_1^{(0)}(a - bR_2^{(0)} - cR_1^{(0)}), \\ \widetilde{R}_2^{(0)} &= R_2^{(0)} + R_2^{(0)}(-d + eR_1^{(0)} - fR_2^{(0)}), \end{aligned}$$

where the passage to new values of the coordinates is marked by a tilde, but not by a change of upper index, like in (2.5).

The following operation involves the interaction $*$ (see (3.1)) between the previous subsystems according to the theory of the alternative conflict for non-annihilating opponents (see, e.g. [1, 2, 4, 7, 8, 9, 10]). To describe this operation we at first have to normalize the vectors $\widetilde{\mathbf{P}}^0 = (\widetilde{P}_1^{(0)}, \widetilde{P}_2^{(0)})$, $\widetilde{\mathbf{R}}^0 = (\widetilde{R}_1^{(0)}, \widetilde{R}_2^{(0)})$, i.e., to work with stochastic vectors.

We use the following notation for normalization: $\mathcal{N}\{\widetilde{\mathbf{P}}^0, \widetilde{\mathbf{R}}^0\} = \{\mathbf{p}^0, \mathbf{r}^0\}$, where the coordinates of the stochastic vectors $\mathbf{p}^0, \mathbf{r}^0$ are determined by the formulae

$$p_1^{(0)} = \frac{\widetilde{P}_1^{(0)}}{\widetilde{z}_P^{(0)}}, p_2^{(0)} = \frac{\widetilde{P}_2^{(0)}}{\widetilde{z}_P^{(0)}}, r_1^{(0)} = \frac{\widetilde{R}_1^{(0)}}{\widetilde{z}_R^{(0)}}, r_2^{(0)} = \frac{\widetilde{R}_2^{(0)}}{\widetilde{z}_R^{(0)}},$$

where $\widetilde{z}_P^{(0)} = \widetilde{P}_1^{(0)} + \widetilde{P}_2^{(0)}, \widetilde{z}_R^{(0)} = \widetilde{R}_1^{(0)} + \widetilde{R}_2^{(0)}$.

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The next step exactly corresponds to the conflict interaction between the subsystems. We introduce new stochastic vectors $\{\mathbf{p}^1, \mathbf{r}^1\}$ with coordinates:

$$p_j^{(1)} = \frac{p_j^{(0)}(1 - \alpha r_j^{(0)})}{1 - \alpha \sum_{i=1}^2 p_i^{(0)} r_i^{(0)}}, r_j^{(1)} = \frac{r_j^{(0)}(1 - \alpha p_j^{(0)})}{1 - \alpha \sum_{i=1}^2 p_i^{(0)} r_i^{(0)}}, j = 1, 2.$$

Finally, we have to come back to the non-normalized vectors, which characterize quantitatively populations in both regions after inner and outer conflicts operations. So, at time $n = 1$ we have the following vectors $\mathcal{N}^{-1}\{\mathbf{p}^1, \mathbf{r}^1\} = \{\mathbf{P}^1, \mathbf{R}^1\}$, where

$$\mathbf{P}^1 = (P_1^{(1)}, P_2^{(1)}), \mathbf{R}^1 = (R_1^{(1)}, R_2^{(1)}),$$

and where

$$P_j^{(1)} = p_j^{(1)} \widetilde{z}_P^{(0)}, R_j^{(1)} = r_j^{(1)} \widetilde{z}_R^{(0)}, j = 1, 2.$$

We can repeat this procedure starting from $\{\mathbf{P}^1, \mathbf{R}^1\}$. So we get $\{\mathbf{P}^2, \mathbf{R}^2\}$. And so on for the nth step.

To find the equilibrium points in the case of the complex conflict interaction described above, we have to solve the following system of equations for P_1, P_2, R_1, R_2:

$$\begin{cases} (a + 1 - bR_1 - cP_1)(Z_2 - \alpha R_2(-d + 1 + eP_2 - fR_2))Z_1 = Z, \\ (-d + 1 + eP_1 - fR_1)(Z_2 - \alpha P_2(a + 1 - bR_2 - cP_2))Z_1 = Z, \\ (a + 1 - bR_2 - cP_2)(Z_1 - \alpha R_1(-d + 1 + eP_1 - fR_1))Z_2 = Z, \\ (-d + 1 + eP_2 - fR_2)(Z_1 - \alpha P_1(a + 1 - bR_1 - cP_1))Z_2 = Z, \end{cases}$$

where

$$Z_1 = P_1(a + 1 - bR_1 - cP_1) + R_1(-d + 1 + eP_1 - fR_1),$$

$$Z_2 = P_2(a + 1 - bR_2 - cP_2) + R_2(-d + 1 + eP_2 - fR_2),$$

$$Z = Z_1 Z_2 - \alpha[P_1 P_2(a + 1 - bR_2 - cP_2)(a + 1 - bR_1 - cP_1) +$$

$$+ R_1 R_2(-d + 1 + eP_1 - fR_1)(-d + 1 + eP_2 - fR_2)].$$

We note that in the case $\alpha = 0$ we have two copies of pure Lotka-Volterra models and the corresponding system of equations has at least three equilibrium points (trivial, axial, inner positive).

For the case $\alpha \neq 0$ it is difficult to obtain exact solutions. Let us obtain some insights by numerical approximation.

In particular, we found that there exist equilibrium points and the limit cycles for a wide set of parameter values and initial data (see Figure 6-9).

Moreover, we established the shift effect for the equilibrium point. Namely, we observed that the inner positive equilibrium point (it exist in any system and
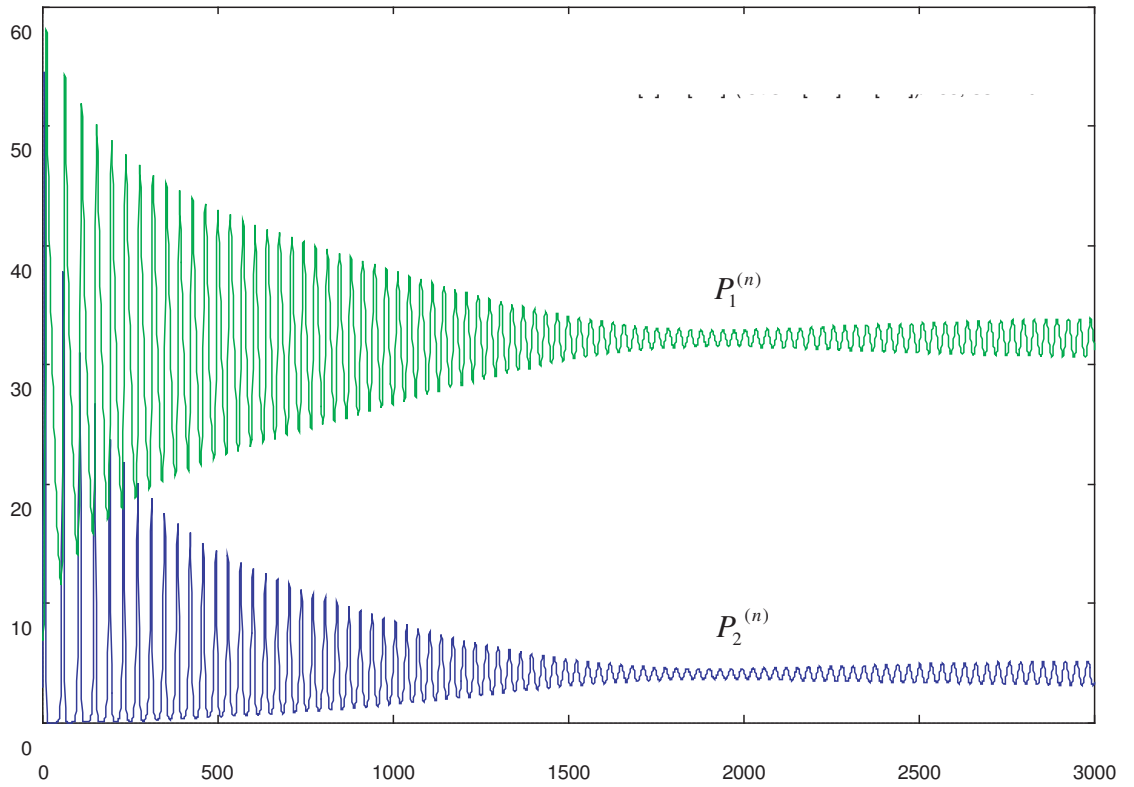
Figure 6: *The existence of the strong bifurcation produces oscillations of the large amplitude. The starting parameters are the same as in Figure 5, the conflict interaction coefficient $\alpha = 0.01$,*

$$P_1^{(0)} = 3, P_2^{(0)} = 5,$$
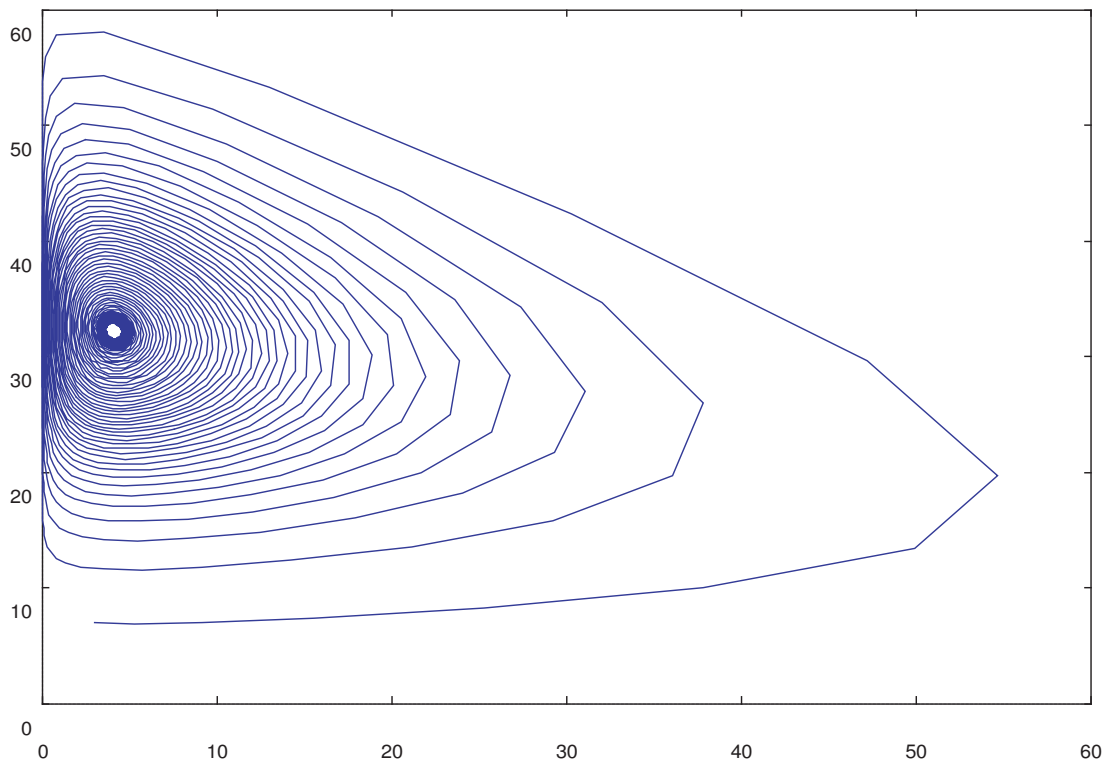
$$R_1^{(0)} = 7, R_2^{(0)} = 10.$$

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Figure 7: *The corresponding phase-space* $(P_1^{(0)}, P_2^{(0)})$.
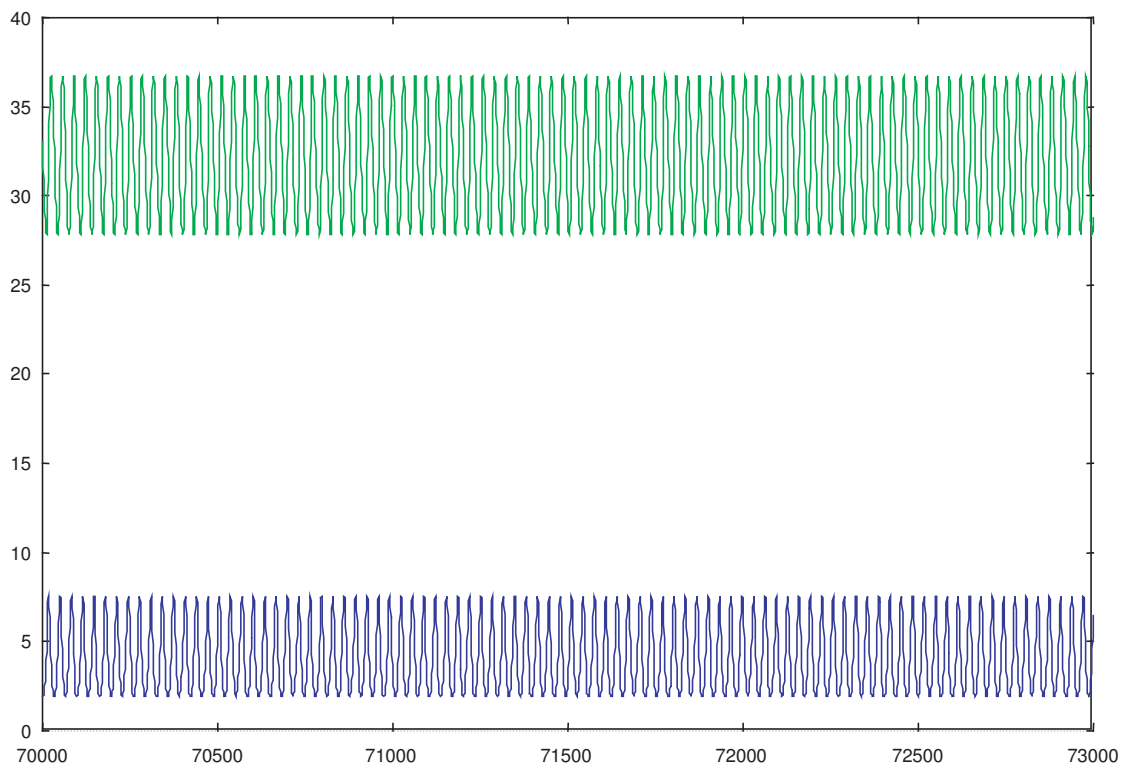
Figure 8: *The existence of the stable oscillations of the conflict interaction between Lotka-Volterra systems after 70000 steps of iteration.*
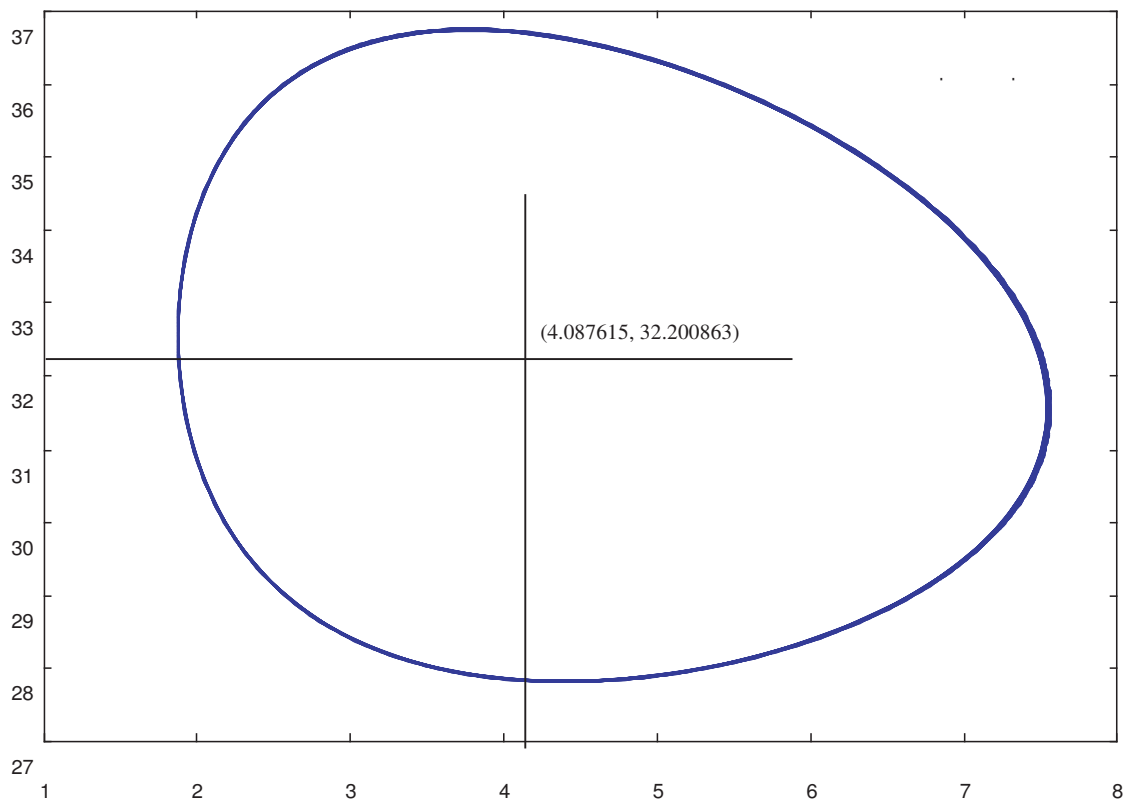
Figure 9: *The limiting cycle in the corresponding phase-space* $\left(P_1^{(0)}, P_2^{(0)}\right)$ *after 70000 steps of iteration. The unstable equilibrium point is shown.*

may be found by formula (4.1)) is shifted after the application of the conflict interaction between the systems. We see by (4.1) that stabilization of discrete Lotka-Volterra model with parameters like in Figure 5 occurs when $P_1 = 4, P_2 = 32$. This may be easily verified by putting these initial data into the corresponding equations. In this case we have a trivial dynamics.

Let us consider the case of a discrete Lotka-Volterra model with the conflict interaction between the subsystems. We take the same values of the coefficients a, b, c, d, e, f, α as in the situation described in Figure 4. Now the equilibrium point has the coordinates $P_1 = 4.043507, P_2 = 32.100629$. The dynamics is constant with these initial data.

In case of a larger α, when oscillations appear, the equilibrium point may also be easily found if we put the initial data in both systems to be equal. In this case the behavior is like in the case of a pure Lotka-Volterra model, and stabilization occurs. However, the stable point is shifted, for example, when $\alpha = 0.01$ (see Figure 6-9) the equilibrium point is $P_1 = R_1 = 4.087615, P_2 = R_2 = 32.200863$.

Thus, if we have some prey-predator system and want to change the population inside this system, we may create an analogous "artificial" system, introduce the conflict interaction and obtain the desired shift of the equilibrium point. Apparently a stronger shift of the stable equilibrium point occurs appears in the case of an "ensemble" of a larger amount of Lotka-Volterra systems. So, we observed the following interesting phenomenon that the equilibrium point of an isolated system is shifted in the case where identical systems are united as an "ensemble".

However, this equilibrium point is unstable, any perturbation of the initial data causes the receding of the system from the equilibrium point.

One of the more interesting observations concerns the limit cycles. It is known that no such kind of orbits in the discrete Lotka-Volterra model is possible. But under the effect of the outer conflict, as we see in the pictures, the dynamical system reaches a limit cycle starting both from an inside or outside point with respect to the orbit. Partially, in Figures 10,11 we present the model, that starts at $P_1^{(0)} = 4, P_2^{(0)} = 32$. As it was pointed above, in the case of a pure Lotka-Volterra model, with these initial data there is no non-trivial dynamical evolution. However, in the case of the model with an outer conflict the process tends to a limit cycle.

5   Interpretation

In many works on mathematical biology and economics [3, 5, 6, 11, 12, 13, 16, 17, 18] the modelling of population dynamics or economical processes is based on
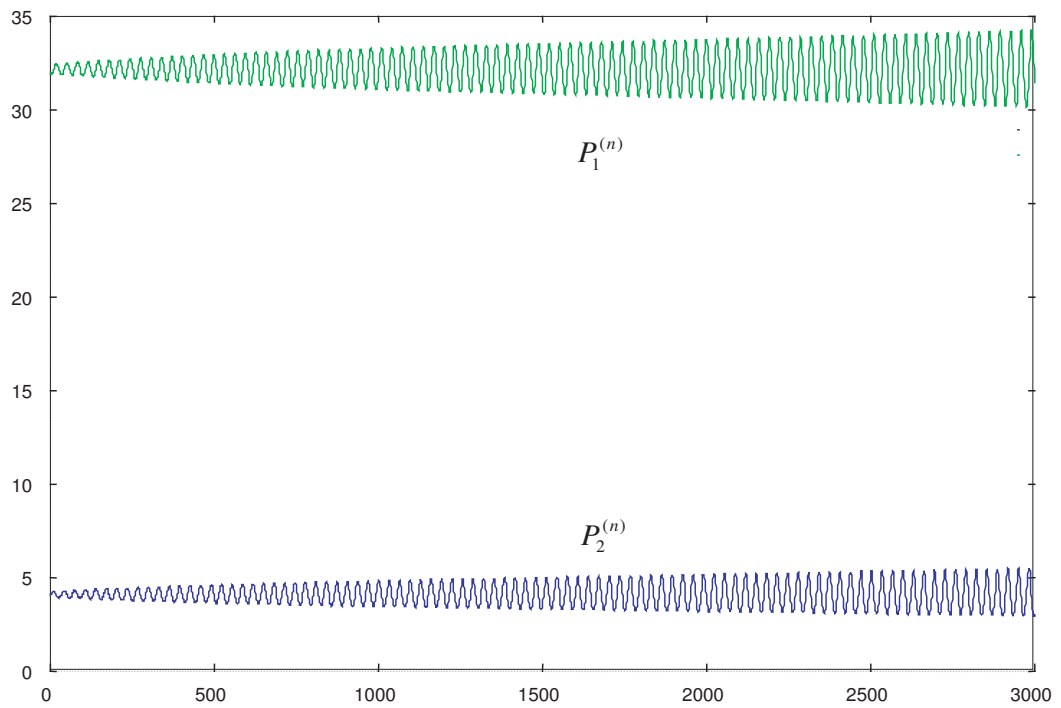
Figure 10: *Conflict interaction between Lotka-Volterra systems. The starting parameters are the same as in Figure 6.*
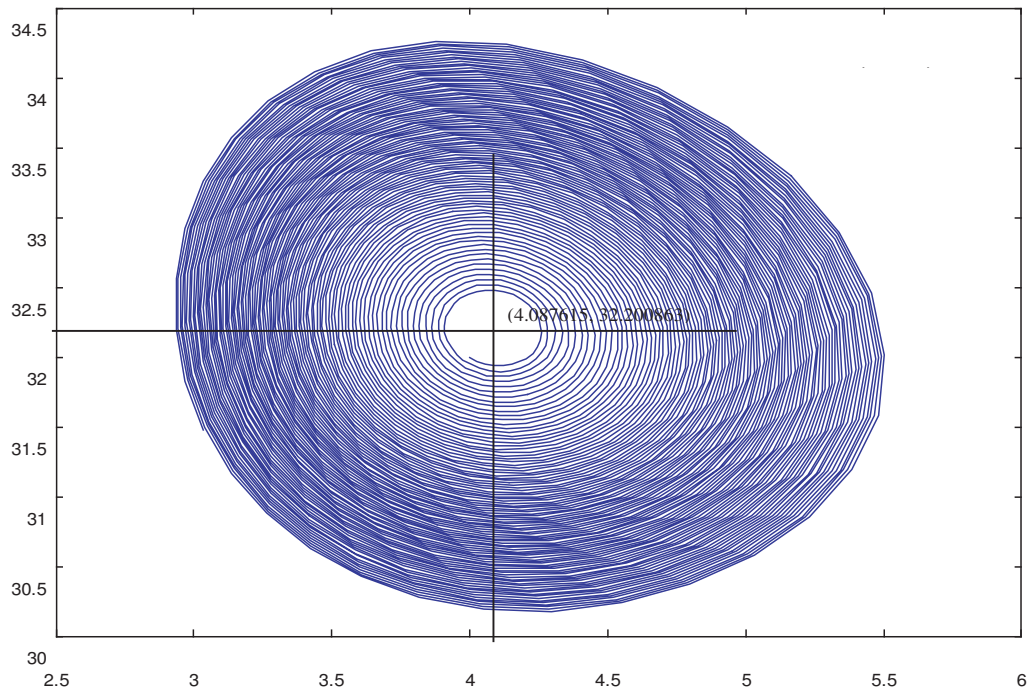*Initial data are inside the limit cycle (Figure 9).*

(4.037615, 32.209861)

Figure 11: *The corresponding phase-space* $(P_1^{(0)}, P_2^{(0)})$. *Trajectory tends to the limit cycle, which is an attractor.*
Initial data are inside the limit cycle (Figure 9).

Lotka-Volterra equations. As a rule, continuous, not discrete, models are studied. In some works the migration process is considered. It takes place between different regions, inside which an interaction of the Lotka-Volterra type is present. For example, in [5] the migration rate between regions has some fixed probability.

We study discrete Lotka-Volterra models with an additional interaction between them. That may be interpreted as some kind of correlation between the inhabitants of different regions. We suppose that discrete models are more natural since, for example, birth and death of individuals happen at discrete moments of time.

It is well known that in the classical discrete prey-predator model a stable point exist. The amount of preys and predators tends to this point in the phase-space. In this case we observe that after several periods of oscillations the populations stabilize (see Figure 2). Thus, we have an attracting point in an phase-space. Such a dynamics exists inside every region when "migration" is absent.

When we introduce an additional interaction between the inhabitants of different regions a redistribution process appears which we interpret as a migration. In some of our complex models there is no stable point, the amount of preys and predators in both regions oscillates along fixed orbits. Apparently these orbits in phase-space are attractors.

We note that explicit formulas of conflict interaction between non-annihilating opponents which describe the redistribution of populations are given by (3.1). The individuals of a certain kind migrate to the region in which they are more numerous.

Is the "migration strategy" which is described in our model a natural one? We suppose that in many cases individuals may be right behaving in such a way. If we consider a prey-predator model, it is clear that every separated individual is unable to estimate all factors that have an influence on the population dynamics like vital resources inside a given region, real amount of own and alternative population, current population dynamics. In other words, the individual "does not know" the parameters of the Lotka-Volterra equations and their current influence on the population dynamics.

However a given individual by virtue of the group reflex will migrate to the region where, as she/he supposes, the vital conditions are best (her/his population should be concentrated there because of better resources, possibility for reproduction, better conditions organize large groups). Formula (3.1) just describes this tendency.

Similar motivations may be proposed in the case of the work migration. Here the unemployed may be regarded as playing the role of "preys", employees as playing the role of "predators". People who seek for work and migrate to another

country, do not know, as a rule, the real situation in this other region. They prefer to migrate to the country where the majority of their friends migrated before (group reflex).

An opposite picture happens with employees who convert their capital into regions with a higher profit.

So, migration accelerates the increase of one of species population in one of the regions. But at the same time there is an effect of the inner prey-predator "fight" inside every subsystem. Partially, the population influences itself at the cost of the last term in Lotka-Volterra equation. As a result, some time later a backward migration starts.

In Figure 8 we may see the effect of delay, when the amount of preys inside the region decreases, but the predators continue migration to this region, until their amount starts decreasing by following the Lotka-Volterra model.

We emphasize, that in our model, in comparison with the discrete Lotka-Volterra model, a cyclic oscillation of populations is observed. Moreover, a cyclic attractor exists in the phase-space, and the prey-predator trajectory tends to this orbit both from an inside or outside point with respect to this cycle (Figure 10,11).

We remark that in our model the normalization was fulfilled by the amount of inhabitants of the region, so the component of the corresponding vector may be large both at the cost of a large population of fixed individuals and at the cost of a small total population in the region. In this case, a migration to a region with a lot of "free space" is also possible.

We also studied another model with an attracting interaction ($\alpha < 0$). In this case we obtained formally a similar dynamics, but with the individuals migrating to the region where they are less numerous. Such a migration strategy might also be natural for some species, e.g., for individuals who hunt separately, control large territory and have confrontations with relatives.

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