On the point spectrum of $\mathcal{H}_{-2}-$singular perturbations

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We prove that for any self-adjoint operator A in a separable Hilbert space $\mathcal{H}$ and a given countable set $\Lambda = \{\lambda_i\}_{i\in\mathbf{N}}$ of real numbers, there exist $\mathcal{H}_{-2}-$singular perturbations $\tilde{A}$ of A such that $\Lambda \subset \sigma_p(\tilde{A})$. In particular, if $\Lambda = \{\lambda_1, ..., \lambda_n\}$ is finite, then the operator $\tilde{A}$ solving the eigenvalues problem, $\tilde{A}\psi_k = \lambda_k\psi_k$, $k = 1, ..., n$, is uniquely defined by a given set of orthonormal vectors $\{\psi_k\}_{k=1}^n$ satisfying the condition $\mathrm{span}\{\psi_k\}_{k=1}^n \cap \mathrm{dom}(|A|^{1/2}) = \{0\}$.

1 Introduction

Let $A = A^*$ be a self-adjoint unbounded operator defined on $\mathcal{D}(A) \equiv \mathrm{Dom}(A)$ in a separable Hilbert space $\mathcal{H}$ with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$. A self-adjoint operator $\tilde{A} \neq A$ in $\mathcal{H}$ is called [6, 17] a (pure) singular perturbation of A if the set

$$\mathcal{D} := \{f \in \mathcal{D}(A) \cap \mathcal{D}(\tilde{A}) | Af = \tilde{A}f\} \tag{1}$$

is dense in $\mathcal{H}$. We shall denote by $\mathcal{P}_s(A)$ the family of all singular perturbations of A. For each $\tilde{A} \in \mathcal{P}_s(A)$ there exists a densely defined symmetric operator $\dot{A} := A|\mathcal{D} = \tilde{A}|\mathcal{D},\ \mathcal{D}(\dot{A}) = \mathcal{D}$ with non-trivial deficiency indices $\mathbf{n}^{\pm}(\dot{A}) = \mathrm{dimKer}(\dot{A} \pm i)^* \neq 0$. Thus, both A and $\tilde{A}$ are different self-adjoint extensions of $\dot{A}$. We use the notation $\tilde{A} \in \mathcal{P}_s^n(A)$, where $n = \mathbf{n}^{\pm}(\dot{A}) \leq \infty$.

Since each operator $\tilde{A} \in \mathcal{P}_s^n(A)$ is a self-adjoint extension of some symmetric operator, it is uniquely fixed by Krein's formula for resolvents (see [19, 4, 14, 20]),

$$(\tilde{A} - z)^{-1} = (A - z)^{-1} + B(z),\ \mathrm{Im}z \neq 0,$$

where $B : \mathbf{C} \setminus \mathbf{R} \to \mathcal{B}(\mathcal{H})$ is a certain analytic operator-valued function of rank $n \leq \infty$ such that

$$(\mathrm{Ran}B(z))^{\mathrm{cl}} \cap \mathcal{D}(A) = \{0\},\ \ \mathrm{cl} = \mathrm{closure}. \tag{2}$$

Here $\mathcal{B}(\mathcal{H})$ is the space of bounded linear operators in $\mathcal{H}$. Moreover the set $\mathcal{D}$ defined by (1) is dense in $\mathcal{H}$ if and only if condition (2) holds (the proof follows from Theorem A.1 in [5] or Lemma 13.1 in [17]).

Let

$$\mathcal{H}_{-2} \supset \mathcal{H}_{-1} \supset \mathcal{H}_0 \equiv \mathcal{H} \supset \mathcal{H}_1 \supset \mathcal{H}_2,$$

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denote a part of the A-scale of Hilbert spaces, where $\mathcal{H}_k = \mathcal{D}(|A|^{k/2})$, $k = 1, 2$, in the norm $\|\varphi\|_k := \|(|A| + 1)^{k/2}\varphi\|$, and $\mathcal{H}_{-k}$ is the dual space to $\mathcal{H}_k$. Obviously $\mathcal{D}(A) = \mathcal{H}_2$. For more technical details in scales of Hilbert spaces see [10].

We say that $\tilde{A} \in \mathcal{P}_s^n(A)$ is an $\mathcal{H}_{-2}-$*singular perturbation* of A of rank n if the set $\mathcal{D}$ defined in (1) is dense in $\mathcal{H}_1$. In turn (see again [5, 17]), the set $\mathcal{D}$ is dense in $\mathcal{H}_1$ if and only if

$$(\mathrm{Ran}B(z))^{\mathrm{cl}} \cap \mathcal{H}_1 = \{0\}, \quad \mathrm{Im}z \neq 0. \tag{3}$$

In this paper we study the problem of existence and construction of operators $\tilde{A}$ which are $\mathcal{H}_{-2}-$singular perturbations of A and solve the eigenvalue problem

$$\tilde{A}\psi_k = \lambda_k\psi_k,\ k = 1, 2, ... \tag{4}$$

for a given sequence $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ of real numbers.

The first detailed investigation of the point spectrum of self-adjoint extensions of symmetric operators in the general case was carried out by M.Krein [19]. The detailed study of the spectral properties of self-adjoint extensions of a symmetric operator with a gap was given in [14, 2, 1, 11, 12]. In particular in [12] (see, also [2]), the existence of a self-adjoint extension with a given point spectrum inside the corresponding gap is proved. In [1, 11, 14] spectral properties of appropriate self-adjoint extensions are characterized in terms of boundary value spaces and corresponding Weyl functions. We refer also to the survey [21] where a general theory of rank-one perturbations of self-adjoint operators is presented.

Here we consider the eigenvalue problem (4) for self-adjoint extensions of symmetric operators in the framework of the singular perturbation theory (see, [6, 7, 8, 9, 13, 17, 3, 20, 22] and and references therein). We note that for finite sequences $\{\lambda_k\}_{k=1}^n$, $\{\psi_k\}_{k=1}^n$ the corresponding problem was studied in [18, 15]. In [18] it was additionally assumed that A is positive operator and $\lambda_k \leq 0$. We also remark that the case of $\mathcal{H}_{-2}-$singular perturbations was not specified in [18, 15]. The main result of the present work is given by the following theorem.

Theorem 1.1. *Let A be a self-adjoint unbounded operator in a separable Hilbert space $\mathcal{H}$. Given a sequence of real numbers $\Lambda = \{\lambda_k : k \in \mathbf{N}\}$ (each λ_k may be repeated with an arbitrary multiplicity) there exists an $\mathcal{H}_{-2}-$singular perturbation $\tilde{A}$ of A such that*

$$\Lambda \subset \sigma_p(\tilde{A}).$$

If $\Lambda = \{\lambda_k\}_{k=1}^n$ is finite, $n < \infty$, then the $\mathcal{H}_{-2}-$singular perturbation $\tilde{A}$ of A of rank n solving the eigenvalue problem

$$\tilde{A}\psi_k = \lambda_k\psi_k,\ k = 1, ..., n,$$

is uniquely defined by the given orthonormal system of vectors ψ_k satisfying the condition

$$\mathrm{span}\{\psi_k\}_{k=1}^n \cap \mathcal{H}_1 = \{0\}.$$

The validity of this theorem follows from Theorems 2.1-5.1 presented below.

2 Preliminaries

Denote by $R(z) := (A - z)^{-1}$ the resolvent of an operator A. The following theorem gives a version of the Krein's formula for the resolvents. In particular the function b below is the Weyl function in the sense of [14, 11].

Theorem 2.1. *The operator function*

$$\tilde{R}(z) := R(z) + B(z),\ \mathrm{Im}z \neq 0 \tag{5}$$

defines the resolvent of a self-adjoint operator $\tilde{A} \in \mathcal{P}_s^1(A)$ if and only if the operator function $B(z)$ admits the representation

$$B(z) = b^{-1}(z)(\cdot, \eta(\bar{z}))\eta(z), \tag{6}$$

where the vector valued function $\eta(z) \in \mathcal{H} \setminus \mathcal{D}(A)$ satisfies the equation

$$R(\xi)\eta(z) = R(z)\eta(\xi),\ \mathrm{Im}z, \mathrm{Im}\xi \neq 0, \tag{7}$$

and the scalar function $b(z)$ satisfies the equation

$$\frac{b(z) - b(\xi)}{z - \xi} + (\eta(z), \eta(\bar{\xi})) = 0,\ and\ \bar{b}(z) = b(\bar{z}). \tag{8}$$

The operator $\tilde{A}$ is a rank one $\mathcal{H}_{-2}-$singular perturbation of A if

$$\eta(z) \in \mathcal{H} \setminus \mathcal{H}_1 \tag{9}$$

at least for one point z (and therefore for all points) on the complex plane with $\mathrm{Im}z \neq 0$.

Proof. It is well known (see [16], Chapter VIII) that an operator function $\tilde{R}(z)$ is the resolvent of a closed operator if and only if $\tilde{R}(z)$ is a pseudo-resolvent, i.e., it satisfies the Hilbert identity

$$\tilde{R}(z) - \tilde{R}(\xi) = (z - \xi)\tilde{R}(z)\tilde{R}(\xi),\ \mathrm{Im}z, \mathrm{Im}\xi \neq 0, \tag{10}$$

and

$$\mathrm{Ker}\tilde{R}(z) = \{0\},\ \mathrm{Im}z \neq 0. \tag{11}$$

Let us show that both these conditions are fulfilled for $\tilde{R}(z)$ defined by (5). By the Hilbert identity for $R(z)$ and (6) we find that (10) is equivalent to

$$\begin{aligned} b^{-1}(z)(\cdot, \eta(\bar{z}))\eta(z) - & b^{-1}(\xi)(\cdot, \eta(\bar{\xi}))\eta(\xi) \\ &= (z - \xi)b^{-1}(\xi)(\cdot, \eta(\bar{\xi}))R(z)\eta(\xi) + (z - \xi)b^{-1}(z)(\cdot, R(\bar{\xi})\eta(\bar{z}))\eta(z) \\ &+ (z - \xi)b^{-1}(z)b^{-1}(\xi)(\cdot, \eta(\bar{\xi}))(\eta(\xi), \eta(\bar{z}))\eta(z),\ \mathrm{Im}z, \mathrm{Im}\xi \neq 0. \end{aligned} \tag{12}$$

In turn the relation (12) can be rewritten in the form

$$\begin{aligned} b^{-1}(z)(\cdot, \eta(\bar{z}))\eta(z) - & b^{-1}(\xi)(\cdot, \eta(\bar{\xi}))\eta(\xi) \\ &= b^{-1}(\xi)(\cdot, \eta(\bar{\xi}))[\eta(z) - \eta(\xi)] + b^{-1}(z)(\cdot, [\eta(\bar{z}) - \eta(\bar{\xi})])\eta(z) \\ &+ (z - \xi)b^{-1}(z)b^{-1}(\xi)(\cdot, \eta(\bar{\xi}))(\eta(\xi), \eta(\bar{z}))\eta(z), \end{aligned} \tag{13}$$

where we have used the relation

$$\eta(z) = \eta(\xi) + (z - \xi)R(z)\eta(\xi)$$

which follows from (7). One can easily reduce (13) to the equality

$$b^{-1}(z)(\cdot, \eta(\bar{\xi}))\eta(z) - b^{-1}(\xi)(\cdot, \eta(\bar{\xi}))\eta(z) = (z - \xi)b^{-1}(z)b^{-1}(\xi)(\eta(\xi), \eta(\bar{z}))(\cdot, \eta(\bar{\xi}))\eta(z)$$

which is implied by the first part of (8). This proves that $\tilde{R}(z)$ is a pseudo-resolvent. Let us check (11). By (5), for $f \in \mathcal{H} \setminus \{0\}$, we have

$$\tilde{R}(z)f = R(z)f + b^{-1}(z)(f, \eta(\bar{z}))\eta(z) \neq 0,$$

since $0 \neq R(z)f \in \mathcal{D}(A)$ and $\eta(z) \notin \mathcal{D}(A)$ by (9). Thus (10) and (11) are true and therefore the operator function $\tilde{R}(z)$ in (5) is the resolvent of a closed operator $\tilde{A}$. To show that $\tilde{A}$ is self-adjoint we only need to check that $(\tilde{R}(z))^* = \tilde{R}(\bar{z})$. Clearly this relation is equivalent to the second equality in (8):

$$(\tilde{R}(z))^* = R(\bar{z}) + \overline{b^{-1}(z)}(\cdot, \eta(z))\eta(\bar{z}) = \tilde{R}(\bar{z}).$$

Let us to show that $\tilde{A} \in \mathcal{P}_s^1(A)$. Denote by $\mathcal{N}_z$ the one dimensional subspace in $\mathcal{H}$ spanned by $\eta(\bar{z})$. Put $\mathcal{M}_z = \mathcal{H} \ominus \mathcal{N}_z$ and define

$$\mathcal{D} := R(z)\mathcal{M}_z \equiv \tilde{R}(z)\mathcal{M}_z.$$

By (5) the operator A coincides with $\tilde{A}$ on $\mathcal{D}$. We assert that $\mathcal{D}$ is dense in $\mathcal{H}$. Assume for a moment that its closure $\mathcal{D}^{cl}$ satisfies $\mathcal{D}^{cl} \neq \mathcal{H}$. Then there exists a vector $\varphi \in \mathcal{H}$, such that

$$0 = (\mathcal{D}, \varphi) = (R(z)\mathcal{M}_z, \varphi) = (\mathcal{M}_z, R(\bar{z})\varphi),$$

i.e., we get that $R(\bar{z})\varphi \in \mathcal{N}_z$ and $R(\bar{z})\varphi \in \mathcal{D}(A)$. This contradicts the definition of $\mathcal{N}_z$ and (9). Finally we remark that due to conditions (6), (7), and (9), $\mathrm{Ran}B(z) \cap \mathcal{H}_1 = \{0\}$ for all z with Im$z \neq 0$. Thus, $\tilde{A}$ is a rank one $\mathcal{H}_{-2}-$singular perturbation of A.

Vice versa, if $\tilde{A}$ is a rank one perturbation of A, then the resolvent of $\tilde{A}$ has the form (5), (6), where the function b satisfies the second equality in (8). Repeating arguments based on Hilbert identity for the resolvents $R(z)$ and $\tilde{R}(z)$ it is easy to check the validity of (7) and the first part of (8). As above, condition (9) means that $\tilde{A}$ is $\mathcal{H}_{-2}-$singular perturbation of A. □

3 Rank one singular perturbations with an additional eigenvalue

Theorem 3.1. *For any self-adjoint unbounded operator A in $\mathcal{H}$, a given vector $\psi_1 \in \mathcal{H} \setminus \mathcal{H}_1$, $\|\psi_1\| = 1$, and any real number $\lambda_1 \in \mathbf{R}$, there exists a uniquely defined rank one $\mathcal{H}_{-2}-$singular perturbation $\tilde{A} \equiv A_1$ of A, solving the eigenvalue problem*

$$A_1\psi_1 = \lambda_1\psi_1. \tag{14}$$

P r o o f. Given $\psi_1 \in \mathcal{H} \setminus \mathcal{H}_1$, $\|\psi_1\| = 1$, and $\lambda_1 \in \mathbf{R}$, define

$$\eta_1(z) := (A - \lambda_1)R(z)\psi_1,\ \mathrm{Im}z \neq 0, \tag{15}$$

and

$$b_1(z) := (\lambda_1 - z)(\psi_1, \eta_1(\bar{z})), \tag{16}$$

where $R(z) = (A - z)^{-1}$. Rewriting (15) in the form

$$\eta_1(z) = \psi_1 + (z - \lambda_1)R(z)\psi_1 \tag{17}$$

we see that $\eta_1(z) \in \mathcal{H} \setminus \mathcal{H}_1$, since $\psi_1 \in \mathcal{H} \setminus \mathcal{H}_1$ and $R(z)\psi_1 \in \mathcal{D}(A)$. Let us show that $\eta_1(z)$ and $b_1(z)$ satisfy equations (7) and (8) resp. Indeed, by (17) we get

$$\begin{aligned}\eta_1(z) =& \eta_1(\xi) + (A - \lambda_1)R(z)\psi_1 - (A - \lambda_1)R_0(\xi)\psi_1 \\ =& \eta_1(\xi) + (z - \xi)R(z)\eta_1(\xi) = (A - \xi)R(z)\eta_1(\xi)\end{aligned}$$

which is equivalent to (7). Further we will prove (8). Using (16) and (17) we have

$$b_1(z) - b_1(\xi) = (\xi - z) + (\xi - \lambda_1)^2(R(\xi)\psi_1, \psi_1) - (z - \lambda_1)^2(R(z)\psi_1, \psi_1), \tag{18}$$

where we took into account that $\|\psi_1\| = 1$. Similarly we get

$$\begin{aligned}(\xi - z)(\eta_1(z), \eta_1(\bar{\xi})) =& (\xi - z)[(\psi_1, \psi_1) + (z - \lambda_1)(R(z)\psi_1, \psi_1) + (\xi - \lambda_1)(R(\xi)\psi_1, \psi_1) \\ &+ (z - \lambda_1)(\xi - \lambda_1)(R(z)R(\xi)\psi_1, \psi_1)].\end{aligned}$$

From the latter relation, using the Hilbert identity for the resolvent of A, we obtain

$$(\xi - z)(\eta_1(z), \eta_1(\bar{\xi})) = (\xi - z) - (z - \lambda_1)^2(R(z)\psi_1, \psi_1) + (\xi - \lambda_1)^2(R(\xi)\psi_1, \psi_1). \tag{19}$$

Comparing (18) and (19) we get the first equality in (8). Therefore, by Theorem 2.1 the operator function

$$R_1(z) = R(z) + B_1(z) \equiv R(z) + \frac{1}{(\lambda_1 - z)(\psi_1, \eta_1(\bar{z}))}(\cdot, \eta_1(\bar{z}))\eta_1(z) \tag{20}$$

is the resolvent of some operator $A_1 \in \mathcal{P}_s^1(A)$. Moreover A_1 is a rank one $\mathcal{H}_{-2}-$singular perturbation of A, since $\eta_1(z) \in \mathcal{H} \setminus \mathcal{H}_1$ due to $\psi_1 \in \mathcal{H} \setminus \mathcal{H}_1$.

Now we will check that A_1 solves the eigenvalue problem (14). Indeed, due to (17), (20) we have

$$R_1(z)\psi_1 = R(z)\psi_1 + \frac{1}{(\lambda_1 - z)(\psi_1, \eta_1(\bar{z}))}(\psi_1, \eta_1(\bar{z}))(\psi_1 + (z - \lambda_1)R(z)\psi_1) = \frac{1}{\lambda_1 - z}\psi_1.$$

Finally we have to prove the uniqueness of the operator A_1. Assume that there exists another operator $\hat{A}_1 \in \mathcal{P}_s^1(A)$ such that $\hat{A}_1\psi_1 = \lambda_1\psi_1$. By Theorem 2.1 its resolvent admits the representation

$$\hat{R}_1(z) = R(z) + B(z),\ \mathrm{Im} z \neq 0,$$

where $B(z)$ is a rank one operator function of the form $b^{-1}(z)(\cdot, \eta(\bar{z}))\eta(z)$. Since

$$\hat{R}_1(z)\psi_1 = R_1(z)\psi_1 = \frac{1}{\lambda_1 - z}\psi_1,$$

we see that

$$B(z)\psi_1 = (\lambda_1 - z)^{-1}\psi_1 - R(z)\psi_1 = (\lambda_1 - z)^{-1}(A - \lambda_1)R(z)\psi_1. \tag{21}$$

In particular, $B(z)\psi_1 \neq 0$ (recalling that $\psi_1 \notin \mathcal{D}(A)$). On the other hand

$$B(z)\psi_1 = b^{-1}(z)(\psi_1, \eta(\bar{z}))\eta(z). \tag{22}$$

Therefore for some $c(z) \neq 0$

$$\eta(z) = c(z)(A - \lambda_1)R(z)\psi_1 = c(z)\eta_1(z),\ \mathrm{Im} z \neq 0,$$

It easily follows from (7) that $c := c(z)$ does not depend on z and by (21), (22) $b(z) = |c|^2 b_1(z)$. This proves that $B(z) = B_1(z)$. □

4 Finite rank perturbations solving the eigenvalue problem

Theorem 4.1. *For any self-adjoint unbounded operator A in $\mathcal{H}$, a given finite sequence $\Lambda = \{\lambda_i\}_{i=1}^n$ of real numbers, and a family of orthonormal vectors $\{\psi_i\}_{i=1}^n$ such that*

$$\mathrm{span}\{\psi_i\}_{i=1}^n \cap \mathcal{H}_1 = \{0\}, \tag{23}$$

there exists a unique $\mathcal{H}_{-2}-$singular perturbation $\tilde{A} \equiv A_n \in \mathcal{P}_s^n(A)$ solving the eigenvalue problem

$$A_n\psi_i = \lambda_i\psi_i,\ i = 1, ..., n. \tag{24}$$

P r o o f. The theorem is already proved in the case $n = 1$ (see Theorem 3.1). We will prove the general case by induction. Let $n = 2$ and let the operator A_1 be defined by (20). We will show that A_2 is uniquely defined by the similar formula

$$R_2(z) = (A_2 - z)^{-1} = (A_1 - z)^{-1} + b_2^{-1}(z)(\cdot, \eta_2(\bar{z}))\eta_2(z),\ \mathrm{Im} z \neq 0,$$

where

$$\eta_2(z) := (A_1 - \lambda_2)R_1(z)\psi_2 = \psi_2 + (z - \lambda_2)R_1(z)\psi_2, \tag{25}$$

and

$$b_2(z) := (\lambda_2 - z)(\psi_2, \eta_2(\bar{z})). \tag{26}$$

To this aim we use Theorem 3.1, where A is replaced by A_1. But at first we have to prove that $\psi_2 \in \mathcal{H} \setminus \mathcal{H}_1$, where $\mathcal{H}_1 \equiv \mathcal{H}_1(A_1)$ is now defined by the operator A_1. From (20) it follows that for each fixed z, $\mathrm{Im} z \neq 0$, the domain of the operator A_1 has the representation

$$\mathcal{D}(A_1) = \{h \in \mathcal{H} \mid h = f + b_1^{-1}(z)((A - z)f, \eta_1(\bar{z}))\eta_1(z),\ f \in \mathcal{D}(A)\}.$$

By (17) we see that each $h \in \mathcal{D}(A_1)$ has the form $h = c\psi_1 + \varphi$ with some $c \in \mathbf{C}$, $\varphi \in \mathcal{D}(A)$. Therefore (see, (23)) we have that $\psi_2 \notin \mathcal{D}(A_1)$. In fact $\psi_2 \notin \mathcal{D}(\mid A_1 \mid^{1/2}) = \mathcal{H}_1(A_1)$ by similar arguments. Thus, by Theorem 3.1 the operator A_2 is a rank one $\mathcal{H}_{-2}-$singular perturbation of A_1 solving the problem $A_2\psi_2 = \lambda_2\psi_2$. By a direct calculation we can check that $A_2\psi_1 = \lambda_1\psi_1$. Indeed using (25), (26) we have

$$R_2(z)\psi_1 = R_1(z)\psi_1 + b_2^{-1}(z)(\psi_1, \eta_2(\bar{z}))\eta_2(z) = (\lambda_1 - z)^{-1}\psi_1,$$

as $(\psi_1, \eta_2(\bar{z})) = 0$, due to $\eta_2(z) = \psi_2 + (z - \lambda_2)R_1(z)\psi_2$, $\psi_1 \perp \psi_2$, and $R_1(z)\psi_1 = (\lambda_1 - z)^{-1}\psi_1$. In the class of rank two singular perturbations $\mathcal{P}_s^2(A)$ the constructed operator A_2 is uniquely defined. This easily follows from Krein's formula for $(A_2 - z)^{-1}$, the equalities $A_2\psi_i = \lambda_i\psi_i, i = 1, 2$, and the conditions: $\psi_i \notin \mathcal{H}_1,\ \psi_1 \perp \psi_2$.

Thus, we proved the theorem in the case $n = 2$. One can easily repeat the above construction for the next step with a pair λ_3, ψ_3 and continue the procedure up to any finite n. We omit the detailed description and limit ourselves to presenting the main formulae. The resolvent of A_n is defined by induction and has the form

$$R_n(z) := R(z) + B_n(z) = R_{n-1}(z) + b_n^{-1}(z)(\cdot, \eta_n(\bar{z}))\eta_n(z), \mathrm{Im} z \neq 0, \tag{27}$$

where we recall that $R(z) := (A - z)^{-1}$,

$$B_n(z) = \sum_{k=1}^{n} b_k^{-1}(z)(\cdot, \eta_k(\bar{z}))\eta_k(z),$$

$$\eta_k(z) := (A_{k-1} - \lambda_k)R_{k-1}(z)\psi_k = \psi_k + (z - \lambda_k)R_{k-1}(z)\psi_k,$$

and

$$b_k(z) := (\lambda_k - z)(\psi_k, \eta_k(\bar{z})).$$

The uniqueness of A_n in the class of finite rank singular perturbations $\mathcal{P}_s^n(A)$ easily follows from Krein's formula (27) for $(A_n - z)^{-1}$, (23), (24), and the conditions: $\psi_k \notin \mathcal{H}_1,\ \ \psi_k \perp \psi_j, k \neq j$. □

5 Infinite rank perturbations with an arbitrary point spectrum

Theorem 5.1. *Let $\Lambda = \{\lambda_k, k \in \mathbf{N}\}$ be a sequence of real numbers (each λ_k may be repeated with an arbitrary multiplicity). Then for any self-adjoint unbounded operator A in a Hilbert space $\mathcal{H}$ there exists an $\mathcal{H}_{-2}-$singular perturbation $\tilde{A}$ such that*

$$\Lambda \subset \sigma_p(\tilde{A}).$$

P r o o f. First we construct an appropriate sequence of vectors $\psi_k,\ k \in \mathbf{N}$, which satisfies the equations $\tilde{A}\psi_k = \lambda_k\psi_k$. Let $g \in \mathcal{H} \setminus \mathcal{H}_1$ and therefore

$$\int_{-\infty}^{+\infty} |\lambda| d(E_\lambda g, g) = \infty.$$

Here E_λ denotes the spectral measure of A. Then we decompose the real line into an infinite family of bounded Borel mutually disjoint sets δ_{ik} such that

$$\int\limits_{\delta_{ik}} |\lambda| d(E_\lambda g, g) = a_{ik} \geq 1, \ \ i,k = 1,2,...$$

Obviously $\sum\limits_{i=1}^{\infty} a_{ik} = \infty$ for all $k = 1,2,....$ Set $\Delta_k := \bigcup\limits_i \delta_{ik}$ and put $\psi_k := E(\Delta_k)g$. By this construction all ψ_k belong to $\mathcal{H} \setminus \mathcal{H}_1$ and $\psi_k \perp \psi_l$, $k \neq l$. Moreover, the subspace $\Psi := (\mathrm{span}\{\psi_k\}_{k=1}^{\infty})^{\mathrm{cl}}$ has a zero intersection with $\mathcal{H}_1$.

Let us introduce the orthogonal decompositions,

$$\mathcal{H} = \mathcal{H}_{(1)} \oplus \mathcal{H}_{(2)} \oplus \cdots \oplus \mathcal{H}_{(k)} \oplus \cdots$$

and

$$A = A_{(1)} \oplus A_{(2)} \oplus \cdots \oplus A_{(k)} \oplus \cdots$$

where $\mathcal{H}_{(k)} := E(\Delta_k)\mathcal{H}$ and $A_{(k)} := A|\mathcal{H}_{(k)}$. By the construction

$$\psi_k \in \mathcal{H}_{(k)} \setminus \mathcal{H}_{1,(k)},$$

where $\mathcal{H}_{1,(k)} \equiv \mathcal{H}_1(A_{(k)})$ is the $\mathcal{H}_1$-space in the $A_{(k)}-$scale of spaces constructed using the operator $A_{(k)}$. So, by Theorem 3 for each pair λ_k and ψ_k there exists an $\mathcal{H}_{-2}-$singular perturbation $\tilde{A}_{(k)} \in \mathcal{P}_s^1(A_{(k)})$ such that $\lambda_k \in \sigma_{\mathrm{p}}(\tilde{A}_{(k)})$. Now we define the operator $\tilde{A}$ as the orthogonal sum of the $\tilde{A}_{(k)}$,

$$\tilde{A} := \tilde{A}_{(1)} \oplus \tilde{A}_{(2)} \oplus ... \oplus \tilde{A}_{(k)} \oplus ...$$

The resolvent of $\tilde{A}$ has the representation,

$$\tilde{R}(z) = R_0(z) + \sum_{k=1}^{\infty} b_k^{-1}(z)(\cdot, \eta_k(\bar{z}))\eta_k(z)$$

with

$$\eta_k(z) := (A - \lambda_k)R(z)\psi_k = \psi_k + (z - \lambda_k)R(z)\psi_k \in \mathcal{H}_{(k)} \setminus \mathcal{H}_{1,(k)}$$

and $b_k(z) := (\lambda_k - z)(\psi_k, \eta_k(\bar{z}))$. The domain of $\tilde{A}$ has the following description,

$$\mathcal{D}(\tilde{A}) = \{h \in \mathcal{H} \mid h = f + \sum_{k=1}^{\infty} b_k^{-1}(z)((A-z)f, \eta_k(\bar{z}))\eta_k(z), \ f \in \mathcal{D}(A)\}.$$

Both, A and $\tilde{A}$ are different self-adjoint extensions of the symmetric operator $\dot{A} := A|\mathcal{D} = \tilde{A}|\mathcal{D}$, where (see (1))

$$\mathcal{D}(\dot{A}) = \mathcal{D} := \{f \in \mathcal{D}(A) \mid ((A-z)f, \eta_k(\bar{z})) = 0, \ k = 1,2,...\}.$$

By the above construction the range of the operator $B(z) = \tilde{R}(z) - R_0(z)$ satisfies the condition (3). Therefore the set $\mathcal{D}$ is dense in $\mathcal{H}_1$ and the operator $\tilde{A}$ is the $\mathcal{H}_{-2}-$singular perturbation of A solving the eigenvalue problem $\tilde{A}\psi_k = \lambda_k\psi_k, \ k = 1,2,....$ □

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