DYNAMICS OF DISCRETE CONFLICT INTERACTIONS BETWEEN NON-ANNIHILATING OPPONENTS

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ABSTRACT. Discrete time dynamical systems modeling the conflict interaction between *a priori* non-annihilating opponents are introduced and investigated. Starting from a vector version of the logistics equation, a new mathematical models describing the conflict interactions is given. This model intends to give tools for the description of the "compromise" redistribution of a conflict space instead of reaching the maximal increase of the amount of selected species. The existence of invariant limiting states for the associated dynamical system is proven in the case of a purely repulsive interaction, as well as in the one of a purely attractive interaction.

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1. Introduction

In this paper we develop the mathematical tools for the study of dynamical systems describing the phenomenon of conflicts between *a priori* non-annihilating opponents. Our approach is related to the mathematical theory of population dynamics (see e.g. [8]-[11]), which describes the quantitative changes of conflicting species and is based on different variants of the well-known Lotka-Volterra equations. However, due to particular features of our model, we are able to describe the attraction limit behavior.

We start with the vector version of the Lotka-Volterra equation, which involves at least two opponents. To each opponent we associate a stochastic vector in Euclidean space $\mathbb{R}^n$, i.e., we start with a couple of vectors $\mathbf{p}, \mathbf{r} \in \mathbb{R}^n, n > 1,\ \|\mathbf{p}\|_1 = \|\mathbf{r}\|_1 = 1$ ($\|\cdot\|_1$ denoting the l_1 norm on $\mathbb{R}^n$, i.e., $\|\mathbf{p}\|_1 = \sum_{i=1}^n |p_i|$. The evolution of $\mathbf{p}, \mathbf{r}$ in time under the conflict interaction is governed by the vector version of the Lotka-Volterra equation

$$\begin{cases} \dot{\mathbf{p}} = \mathbf{p} \divideontimes (\mathbf{1} - A\mathbf{r}) \\ \dot{\mathbf{r}} = \mathbf{r} \divideontimes (\mathbf{1} - A\mathbf{p}), \end{cases}$$

where $\mathbf{1}$ is the unit vector, A denotes the interaction matrix acting in $\mathbb{R}^n$, and $\circledast$ stands for a certain kind of vector composition defined below.

In fact we use here only the simplest difference analogue of these equations, i.e., we replace continuous time by a discrete time $t \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$. In terms of coordinates these equations have form

$$\begin{cases} p_i(t+1) = \frac{1}{z(t)} p_i(t)(1 - \alpha r_i(t)) \\ r_i(t+1) = \frac{1}{z(t)} r_i(t)(1 - \alpha p_i(t)) \ , \end{cases} \quad i = 1, ..., n, \ t \in \mathbb{N}_0,$$

where $p_i(0) = p_i, \ r_i(0) = r_i$ are the coordinates of the starting vectors $\mathbf{p}, \mathbf{r}, \ -1 \leq \alpha \leq 1, \alpha \neq 0$ denotes the interaction coupling constant (now $A = \alpha I, \ I$ being the unit $n \times n$ matrix), and the normalizing coefficient function $z(t)$ is determined by the non-annihilating condition $\|\mathbf{p}(t)\|_1 = \|\mathbf{r}(t)\|_1 = 1$. Depending on the sign of α, we have two different cases of the interaction: repulsive with $\alpha > 0$, and attractive with $\alpha < 0$. For $\alpha = 0$ we have a trivial evolution.

The sequence of states

$$\mathbf{p}(t) = \{p_i(t)\}_{i=1}^n, \ \mathbf{r}(t) = \{r_i(t)\}_{i=1}^n, \ t = 0, 1, 2, ...$$

generates a trajectory of a dynamical system in $\mathbb{R}^n \times \mathbb{R}^n$. We study the behaviour of such trajectories, prove the existence of the invariant limiting states $\mathbf{p}(\infty), \mathbf{r}(\infty)$, and describe the distribution of the limiting vectors $\mathbf{p}(\infty), \mathbf{r}(\infty)$ in $\mathbb{R}^n$. For applications and examples see [1]-[4],[7].

2. **The conflict dynamical system**

In this section we introduce the dynamical system describing the discrete time conflict interaction between two non-annihilating opponents distributed on a finite set of controversial positions.

Let $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}, \ n > 1$, denote a finite space of controversial positions for a pair of opponents which are represented by discrete probability measures μ and ν on Ω. The starting distributions of $\mu, \ \nu$ along Ω are given by two sets of numbers:

$$\begin{cases} p_i := \mu(\omega_i) \geq 0, & \sum_{i=1}^n p_i = 1 \\ r_i := \nu(\omega_i) \geq 0, & \sum_{i=1}^n r_i = 1. \end{cases}$$

By this we associate with the opponents two stochastic vectors $\mathbf{p} = \{p_i\}_{i=1}^n$ and $\mathbf{r} = \{r_i\}_{i=1}^n$ from $\mathbb{R}_+^n$, with positive coordinates, $p_i, r_i \geq 0$, and unit l_1-norms, $\| \mathbf{p} \|_1 = \| \mathbf{r} \|_1 = 1$.

The conflict interaction between opponents is represented in the form of a non-linear and non-commutative conflict composition (denoted by $\circledast$) between the vectors $\mathbf{p}, \mathbf{r}$. $\circledast$ is defined as follows

$$\mathbf{p} \circledast \mathbf{r} = \mathbf{p}^{\circledast,1}, \quad \mathbf{r} \circledast \mathbf{p} = \mathbf{r}^{\circledast,1},$$

where the coordinates of the new vectors $\mathbf{p}^{\circledast,1}, \mathbf{r}^{\circledast,1}$ from $\mathbb{R}_+^n$ are defined as follows:

$$p_i^{\circledast,1} := \frac{1}{z} p_i(1 - \alpha r_i), \quad r_i^{\circledast,1} := \frac{1}{z} r_i(1 - \alpha p_i), \ i = 1, 2, ..., n, \tag{1}$$

where $-1 \leq \alpha \leq 1, \ \alpha \neq 0$, denotes a coupling constant, and a normalizing coefficient z is chosen such that the vectors $\mathbf{p}^{\circledast,1}, \mathbf{r}^{\circledast,1}$ are stochastic too (this demand exactly reflects our intention to study the conflict interaction between non-annihilating opponents). One can easily find that $z = 1 - \alpha(\mathbf{p}, \mathbf{r})$ where $(\cdot, \cdot)$ stands for the inner product in $\mathbb{R}^n$. We observe that formulae (1) are well-defined only if the starting vectors and the coupling constant satisfy the condition:

$$\alpha \neq \frac{1}{(\mathbf{p}, \mathbf{r})}. \tag{2}$$

Given a starting pair of vectors $\mathbf{p}, \mathbf{q}$ the iteration of the mapping

$$f^{\circledast} : \left\{ \begin{array}{c} \mathbf{p} \\ \mathbf{r} \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \mathbf{p}^{\circledast,1} \\ \mathbf{r}^{\circledast,1} \end{array} \right\}$$

generates a discrete trajectory of the conflict dynamical system in the direct product of spaces $\mathbb{R}^n_+ \times \mathbb{R}^n_+$:

$$(f^{\circledast})^N : \left\{ \begin{array}{c} \mathbf{p} \\ \mathbf{r} \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \mathbf{p}^{\circledast,N} \\ \mathbf{r}^{\circledast,N} \end{array} \right\}, \ N = 1, 2, ...$$

The couple of vectors $\mathbf{p}^{\circledast,N}$, $\mathbf{r}^{\circledast,N}$ is called the state of the conflict dynamical system at the N-step of interaction. The problem is to study the behavior of these states as $N \rightarrow \infty$. Our main result which, we might call the Theorem of conflicts, reads as follows.

Theorem 1. *For each pair of stochastic vectors* $\mathbf{p}, \mathbf{r} \in \mathbb{R}^n_+$, *with* $(\mathbf{p}, \mathbf{r}) > 0$, *and any fixed coupling constant* $\alpha \neq 0$, $-1 \leq \alpha \leq 1$ *with condition (2), the limiting vectors*

$$\mathbf{p}^{\circledast,\infty} = \lim_{N\rightarrow\infty} \mathbf{p}^{\circledast,N}, \quad \mathbf{r}^{\circledast,\infty} = \lim_{N\rightarrow\infty} \mathbf{r}^{\circledast,N},$$

exist in $\mathbb{R}^n_+$, *and are invariant with respect to the action of conflict composition:*

$$\mathbf{p}^{\circledast,\infty} = \mathbf{p}^{\circledast,\infty} \circledast \mathbf{r}^{\circledast,\infty}, \quad \mathbf{r}^{\circledast,\infty} = \mathbf{r}^{\circledast,\infty} \circledast \mathbf{p}^{\circledast,\infty}. \tag{3}$$

Moreover

$$\left\{ \begin{array}{ll} \mathbf{p}^{\circledast,\infty} \perp \mathbf{r}^{\circledast,\infty}, & \textit{if} \quad \mathbf{p} \neq \mathbf{r} \quad \textit{and} \quad 0 < \alpha \leq 1 \\ \mathbf{p}^{\circledast,\infty} = \mathbf{r}^{\circledast,\infty}, & \qquad \textit{otherwise.} \end{array} \right. \tag{4}$$

Here we will prove this theorem only in the two special cases $\alpha = \pm 1$, i.e., when the conflict interaction is purely repulsive, $\alpha = 1$, resp., purely attractive, $\alpha = -1$. For the cases $-1 < \alpha < 1$ see [3].

2.1. **The purely repulsive case.** Let $\mathbf{p} \neq \mathbf{r}$ and $\alpha = 1$. Here we reproduce only a sketch of the proof (for more details see [5, 6]). Assume $0 \leq r_i < p_i \leq 1$ for some i. Then

$$\lim_{N\rightarrow\infty} r_i^{\circledast,N} = r_i^{\circledast,\infty} = 0 \tag{5}$$

and

$$\lim_{N\rightarrow\infty} p_i^{\circledast,N} = p_i^{\circledast,\infty} > 0. \tag{6}$$

(5), (6) are obviously true if $0 = r_i$ or $p_i = 1$. So we have to prove (5), (6) only under the condition $0 < r_i < p_i < 1$. To this aim we consider the sequence of ratios

$$R_i^{(N)} := \frac{p_i^{\circledast,N}}{r_i^{\circledast,N}}, \ N \geq 1.$$

Using (1) one can easily show (for details see [5, 6]) that the sequence $R_i^{(N)}$ is monotone increasing as $N \to \infty$, and moreover

$$R_i^{(N)} = \frac{p_i^{\circledast,N}}{r_i^{\circledast,N}} \to \infty, \quad N \to \infty. \tag{7}$$

This implies $r_i^{\circledast,N} \to 0$, since $p_i^{\circledast,N} < 1$, that proves (5). To prove (6) we consider the sequence of differences

$$0 < d_i := p_i - r_i, \quad d_i^{(N)} := p_i^{\circledast,N} - r_i^{\circledast,N}, \ N \in \mathbb{N} = \{1, 2, ...\}$$

From (1) it follows that

$$0 < d_i^{(1)} = p_i^{\circledast,1} - r_i^{\circledast,1} = \frac{p_i(1-q_i) - r_i(1-p_i)}{1 - (\mathbf{p}, \mathbf{r})} = \frac{d_i}{1 - (\mathbf{p}, \mathbf{r})}. \tag{8}$$

Hence $0 < d_i < d_i^{(1)} < 1$. By induction, we prove $0 < d_i^{(N)} < d_i^{(N+1)} < 1$ for all $N \in \mathbb{N}$. Therefore, the sequence $d_i^{(N)}$ is monotone increasing too, as $N \to \infty$, and moreover there exists a finite limit

$$0 < d_i^{\infty} = \lim_{N\to\infty} d_i^{(N)} \leq 1, \quad i = 1, 2, ... \tag{9}$$

Besides, we see that due to $r_i^{\circledast,N} \to 0$,

$$\lim_{N\to\infty} p_i^{\circledast,N} = p_i^{\circledast,\infty} = d_i^{\infty} > 0. \tag{10}$$

Similarly, in the case $0 < p_k < r_k < 1$ for some k we get,

$$\lim_{N\to\infty} p_k^{\circledast,N} = p_k^{\circledast,\infty} = 0, \tag{11}$$

and

$$\lim_{N\to\infty} r_k^{\circledast,N} = r_k^{\circledast,\infty} = -d_k^{\infty} > 0. \tag{12}$$

Further, if $\mathbf{p} \neq \mathbf{r}$ but $0 \neq p_j = r_j$ for some j, then it is not hard to understand (for details see Lemma 2 in [6]) that in this case both coordinates converge to zero,

$$\lim_{N\to\infty} p_j^{\circledast,N} = \lim_{N\to\infty} r_j^{\circledast,N} = 0. \tag{13}$$

In turn, (5), (11), and (13) imply that $(\mathbf{p}^{\circledast,N}, \mathbf{r}^{\circledast,N}) \to 0$ and therefore the limiting vectors, which exist due to (10), (12), and (13) are orthogonal. By (1) any orthogonal vectors are invariant with respect to the action of the conflict composition.

Let $\mathbf{p} = \mathbf{r}$ and $\alpha = 1$. Then obviously $p_i^{\circledast,N} = r_i^{\circledast,N}$ for all i and all N, and we get in the limit (see Proposition 6 in [6]) the invariant equilibrium state, $\mathbf{p}^{\circledast,\infty} = \mathbf{r}^{\circledast,\infty}$, with coordinates

$$\begin{cases} p_i^{\circledast,\infty} = r_i^{\circledast,\infty} = 1/m, & \text{if} \quad p_i = r_i \neq 0 \\ p_i^{\circledast,\infty} = r_i^{\circledast,\infty} = 0, & \text{otherwise}, \end{cases} \tag{14}$$

where $m \leq n$ denotes the amount of non-zero coordinates in the starting vectors.

2.2. **The purely attractive case.** Let $\alpha = -1$ and $(\mathbf{p}, \mathbf{r}) > 0$. The coordinates of the new stochastic vectors $\mathbf{p}^{\circledast,1}, \mathbf{r}^{\circledast,1} \in \mathbb{R}^n_+$ are defined as follows:

$$p_i^{\circledast,1} := \frac{p_i(1+r_i)}{1+(\mathbf{p},\mathbf{r})}, \quad r_i^{\circledast,1} := \frac{r_i(1+p_i)}{1+(\mathbf{p},\mathbf{r})}, \ i = 1, ..., n \tag{15}$$

We will study the behaviour for $N \to \infty$ of the coordinates $p_i^{\circledast,N}, r_i^{\circledast,N}, \ N \in \mathbb{N}$, defined by induction,

$$p_i^{\circledast,N} := \frac{p_i^{\circledast,N-1}(1+r_i^{\circledast,N-1})}{1+(\mathbf{p}^{\circledast,N-1},\mathbf{r}^{\circledast,N-1})}, \quad r_i^{\circledast,N} := \frac{r_i^{\circledast,N-1}(1+p_i^{\circledast,N-1})}{1+(\mathbf{p}^{\circledast,N-1},\mathbf{r}^{\circledast,N-1})}, \ i = 1, ..., n \tag{16}$$

Our arguments are based on the following Propositions.

For fixed i let us consider the sequence of differences

$$d_i := p_i - r_i, \quad d_i^{(N)} := p_i^{\circledast,N} - r_i^{\circledast,N}, \quad N \in \mathbb{N}$$

Proposition 1. *In the case $\alpha = -1$ the sequence $d_i^{(N)}$ is monotone decreasing as $N \to \infty$ and moreover*

$$\lim_{N\to\infty} d_i^{(N)} = d_i^{\infty} = 0, \ i = 1, ..., n. \tag{17}$$

Proof. If $d_i = 0$, then by (15) $p_i^{\circledast,1} - r_i^{\circledast,1} = d_i^{(1)} = 0$ too. Therefore by induction, $d_i^{(N)} = 0$ for all N. Consider the case $d_i \neq 0$. Then we assert that the sequence $|d_i^{(N)}|$ is monotone decreasing. Indeed, due to $(\mathbf{p}, \mathbf{r}) > 0$ on the first step we have

$$\mid d_i^{(1)} \mid = \mid p_i^{\circledast,1} - r_i^{\circledast,1} \mid = \frac{\mid d_i \mid}{1+(\mathbf{p},\mathbf{q})} < |d_i|. \tag{18}$$

By induction we get

$$|d_i^{(N+1)}| < \mid d_i^{(N)} \mid, \ N = 1, 2, ... \tag{19}$$

since obviously $(\mathbf{p}^N, \mathbf{r}^N) > 0$ for all N. Therefore, there exists the limit

$$\lim_{N\to\infty} d_i^{(N)} = d_i^{\infty} < 1. \tag{20}$$

Moreover it is easily seen that this limit is zero since

$$d_i^{(N)} = d_i \prod_{l=1}^{N} (1 + (\mathbf{p}^{\circledast,l}, \mathbf{q}^{\circledast,l})^{-1}$$

and

$$\prod_{l=1}^{N} (1 + (\mathbf{p}^{\circledast,l}, \mathbf{q}^{\circledast,l})^{-1} \to 0, N \to \infty. \tag{21}$$

In fact (21) follows from the existence of a constant c, independent of l, such that $(\mathbf{p}^{\circledast,l}, \mathbf{q}^{\circledast,l}) > c > 0$. The latter inequality is true since the starting vectors $\mathbf{p}, \mathbf{r}$ are non-orthogonal and the difference between the stochastic vectors $\mathbf{p}^{\circledast,N}, \mathbf{r}^{\circledast,N}$ is monotone decreasing thanks to (19). Thus, we proved that $d_i^{(N)} \to 0, \ N \to \infty$. $\square$

Proposition 2. Let $p_i \neq 0 \neq r_i$. Then

$$R_i^{(N)} := \frac{p_i^{\circledast,N}}{r_i^{\circledast,N}} \to 1, \ N \to \infty. \tag{22}$$

Proof. Clearly, if $p_i/r_i = 1$, i.e., $p_i = r_i$, then obviously $p_i^{\circledast,N} = r_i^{\circledast,N}$ and therefore by induction $R_i^{(N)} = 1$ for all N. Let now $0 \neq p_i \neq r_i \neq 0$. For example, let $p_i > r_i$. Then from (15) it follows that $1 < R_i^{(1)} < p_i/r_i$ since

$$R_i^{(1)} = \frac{p_i^{\circledast,1}}{r_i^{\circledast,1}} = \frac{p_i + p_i r_i}{r_i + p_i r_i} = \frac{p_i}{r_i} \cdot \frac{1 + r_i}{1 + p_i}.$$

Therefore $p_i > r_i$ implies $p_i^{\circledast,1} > r_i^{\circledast,1}$ and using (16) by induction we have

$$1 < R_i^{(N)} < R_i^{(N-1)}, \quad N = 1, 2, ... \tag{23}$$

This ensures (22), since we have already proved that $p_i^{\circledast,N} - r_i^{\circledast,N} \to 0$ (see (17)). □

Proposition 3. *Assume*

$$p_i > p_k \geq 0 \text{ and } \quad r_i > r_k \geq 0 \quad \text{for some } i, k.$$

Then

$$p_i^{\circledast,N} > p_k^{\circledast,N} \text{ and } \quad r_i^{\circledast,N} > r_k^{\circledast,N} \text{ for all } N, \tag{24}$$

and moreover $p_k^{\circledast,\infty} = r_k^{\circledast,\infty} = 0$.

Proof. Indeed, by the assumption and due to (15) we easily get,

$$\frac{p_i^{\circledast,1}}{p_k^{\circledast,1}} = \frac{p_i}{p_k} \cdot \frac{1 + r_i}{1 + r_k} > \frac{p_i}{p_k} > 1.$$

Similarly,

$$\frac{r_i^{\circledast,1}}{r_k^{\circledast,1}} > \frac{r_i}{r_k} > 1.$$

Therefore

$$p_i^{\circledast,1} > p_k^{\circledast,1}, \ r_i^{\circledast,1} > r_k^{\circledast,1},$$

and by induction using (16),

$$1 < \frac{p_i}{p_k} < \frac{p_i^{\circledast,1}}{p_k^{\circledast,1}} \cdots < \frac{p_i^{\circledast,N}}{p_k^{\circledast,N}} \cdots ,$$

$$1 < \frac{r_i}{r_k} < \frac{r_i^{\circledast,1}}{r_k^{\circledast,1}} \cdots < \frac{r_i^{\circledast,N}}{r_k^{\circledast,N}} \cdots , \quad N = 1, 2, ... \tag{25}$$

Thus, sequences of the ratios

$$\frac{p_i^{\circledast,N}}{p_k^{\circledast,N}}, \ \frac{r_i^{\circledast,N}}{r_k^{\circledast,N}}$$

are monotone increasing as $N \to \infty$, which proves (24). Assume for a moment that there exists a finite limit,

$$1 < \lim_{N\to\infty} \frac{p_i^{\circledast,N}}{p_k^{\circledast,N}} = \frac{p_i^{\circledast,\infty}}{p_k^{\circledast,\infty}} \equiv \frac{p_i^{\circledast,\infty}}{p_k^{\circledast,\infty}} \cdot \frac{1 + r_i^{\circledast,\infty}}{1 + r_k^{\circledast,\infty}} = M < \infty.$$

This is only possible if $r_i^{\circledast,\infty} = r_k^{\circledast,\infty}$, which contradicts (25). Thus $M = \infty$ and therefore $p_k^{\circledast,\infty} = 0$, as well as $r_k^{\circledast,\infty} = 0$. □

Proposition 4. *Assume* $p_i = r_i$ *for all* i, *then*

$$\left\{ \begin{array}{ll} p_i^{\divideontimes,\infty} = r_i^{\divideontimes,\infty} = 1/m, & \textit{if} \quad i \in \mathbb{S}_=^\infty \\ p_i^{\divideontimes,\infty} = r_i^{\divideontimes,\infty} = 0, & \textit{otherwise}, \end{array} \right. \tag{26}$$

where $\mathbb{S}_=^\infty := \{i : p_i = \max_j\{p_j\}\}$ *and* $m = |\mathbb{S}_=^\infty|$ *denotes the cardinality of the set* $\mathbb{S}_=^\infty$.

Proof. If $p_i > p_k$ for some i, k then $p_k^{\divideontimes,\infty} = r_k^{\divideontimes,\infty} = 0$ due to Proposition 3. If $p_i = p_k = r_i = r_k$, then obviously $p_k^{\divideontimes,N} = r_k^{\divideontimes,N} = p_i^{\divideontimes,N} = r_i^{\divideontimes,N}$ for all N. That is, the non-zero limits $\lim_{N\to\infty} p_k^{\divideontimes,N} = \lim_{N\to\infty} r_k^{\divideontimes,N} = p_k^{\divideontimes,\infty} = r_k^{\divideontimes,\infty} \neq 0$ appear only iff both i and k belong to $\mathbb{S}_=^\infty$. From $\|\mathbf{p}^{\divideontimes,\infty}\|_1 = \|\mathbf{r}^{\divideontimes,\infty}\|_1 = 1$ and the fact $p_i = r_i, \forall i$, it follows that $p_k^{\divideontimes,\infty} = r_k^{\divideontimes,\infty} = 1/m$. □

Proposition 5. *For a pair i, k at least one of the following two possibilities holds:*

(a) both coordinates $p_i^{\divideontimes,N}$ and $r_i^{\divideontimes,N}$, or $p_k^{\divideontimes,N}$ and $r_k^{\divideontimes,N}$, converge to zero:

$$\lim_{N\to\infty} p_i^{\divideontimes,N} = \lim_{N\to\infty} r_i^{\divideontimes,N} = 0, \ \textit{or} \ \lim_{N\to\infty} p_k^{\divideontimes,N} = \lim_{N\to\infty} r_k^{\divideontimes,N} = 0,$$

(b) both differences $d_{ik}^{(N)} := p_i^{\divideontimes,N} - r_k^{\divideontimes,N}$ and $d_{ki}^{(N)} := p_k^{\divideontimes,N} - r_i^{\divideontimes,N}$ converge to zero:

$$\lim_{N\to\infty} d_{ik}^{(N)} = \lim_{N\to\infty} d_{ki}^{(N)} = 0.$$

Proof. Since the differences $d_i^{(N)}$ converge to zero (see (17)), there are only two possibilities:

(a) The inequalities (24) (or the opposite ones) hold for some N_0. And then by Proposition 3 the corresponding relations are fulfilled for all $N > N_0$ and moreover $p_k^{\divideontimes,\infty} = r_k^{\divideontimes,\infty} = 0$ (or resp., $p_i^{\divideontimes,\infty} = r_i^{\divideontimes,\infty} = 0$).

(b) In this case there does not exist an N_0 as in the proof of case (a) and therefore for each $N \in \mathbb{N}$ the segments $[p_i^{\divideontimes,N}, r_i^{\divideontimes,N}]$ and $[p_k^{\divideontimes,N}, r_k^{\divideontimes,N}]$ have non void intersection. Then the differences $d_{ik}^{(N)}$, $d_{ki}^{(N)}$ converge with necessity to zero together with $d_i^{(N)}$, $d_k^{(N)}$ as proven by Proposition 1. □

Let us introduce the notation:

$$\mathbb{S}_0 := \{k : p_k^{\divideontimes,\infty} = r_k^{\divideontimes,\infty} = 0\}, \quad \mathbb{S}^\infty := \{1, 2, ..., n\} \setminus \mathbb{S}_0 \tag{27}$$

Proposition 6. *For all $i, k \in \mathbb{S}^\infty$ the limiting coordinates exist, are non-zero, and equal:*

$$\lim_{N\to\infty} p_i^{\divideontimes,N} = p_i^{\divideontimes,\infty} = r_i^{\divideontimes,\infty} = \lim_{N\to\infty} r_k^{\divideontimes,N} = r_k^{\divideontimes,\infty} = p_k^{\divideontimes,\infty} \neq 0. \tag{28}$$

Proof. Since all vectors $\mathbf{p}^{\divideontimes,N}, \mathbf{r}^{\divideontimes,N}$, $N = 1, 2, ...$ are stochastic from Proposition 5 it follows, there exist only one non-zero point $0 < a \leq 1$ such that

$$a \in \lim_{N\to\infty} \bigcap_{i\in\mathbb{S}^\infty} [p_i^{\divideontimes,N}, r_i^{\divideontimes,N}].$$

Therefore due to $d_{ik}^{(N)}$, $d_{ki}^{(N)}$, $d_i^{(N)}$, $d_k^{(N)} \to 0$ as $N \to \infty$ (28) is true. □

This completes the proof of Theorem 1.

3. **Description of the limiting distributions**

3.1. **The purely repulsive case,** $\alpha = 1$. Given a couple of stochastic vectors $\mathbf{p}, \mathbf{r} \in \mathbb{R}^n_+,\ (\mathbf{p}, \mathbf{r}) > 0$, define

$$D_+ := \sum_{i \in \mathbb{N}_+} d_i,\ D_- := \sum_{i \in \mathbb{N}_-} d_i,$$

where

$$d_i = p_i - r_i, \quad \mathbb{N}_+ := \{i : d_i > 0\},\ \mathbb{N}_- := \{i : d_i < 0\}.$$

Obviously

$$0 < D_+ = -D_- \leq 1,$$

since $\mathbf{p} \neq \mathbf{r}$, and $\sum_i p_i - \sum_i r_i = 0 = D_+ + D_-$.

Theorem 2. *In the purely repulsive case, $\alpha = 1$, the coordinates of the limiting vectors $\mathbf{p}^{*,\infty}, \mathbf{q}^{*,\infty}$ have the following explicit representations*

$$p_i^{*,\infty} = \begin{cases} d_i/D, & i \in \mathbb{N}_+ \\ 0, & \textit{otherwise} \end{cases}, \quad r_i^{*,\infty} = \begin{cases} -d_i/D, & i \in \mathbb{N}_- \\ 0, & \textit{otherwise,} \end{cases} \tag{29}$$

where $D := D_+ = -D_-$.

Proof. By (8), (10), (12) the coordinates of the limiting vectors $\mathbf{p}^{*,\infty}, \mathbf{r}^{*,\infty}$ admit the representation:

$$p_i^{*,\infty} = \begin{cases} d_i^{\infty}, & i \in \mathbb{N}_+ \\ 0, & \text{otherwise} \end{cases}, \quad r_i^{*,\infty} = \begin{cases} -d_i^{\infty}, & i \in \mathbb{N}_- \\ 0, & \text{otherwise.} \end{cases} \tag{30}$$

Further, due to (8) we get

$$\frac{d_i^{(1)}}{d_j^{(1)}} = \frac{d_i}{d_j},\ i, j \in \mathbb{N}_+ \cup \mathbb{N}_-.$$

By induction,

$$\frac{d_i^{(N)}}{d_j^{(N)}} = \frac{d_i}{d_j},\ N = 1, 2, ...,$$

and therefore

$$\frac{d_i^{\infty}}{d_j^{\infty}} = \frac{d_i}{d_j}.$$

Thus, thanks to (30), the coordinates of the vectors $\mathbf{p}^{*,\infty}, \mathbf{r}^{*,\infty}$ satisfy the system of equations:

$$\frac{p_i^{*,\infty}}{p_j^{*,\infty}} = \frac{d_i}{d_j}, \quad \sum_{i \in \mathbb{N}_+} p_i^{*,\infty} = 1 \quad i, j \in \mathbb{N}_+, \tag{31}$$

$$\frac{r_i^{*}}{r_j^{*}} = \frac{d_i}{d_j}, \quad \sum_{i \in \mathbb{N}_-} r_i^{*,\infty} = 1 \quad i, j \in \mathbb{N}_-. \tag{32}$$

It is easy to see that the latter system has the unique solution of the form (29). Indeed, (31) implies that $p_i^{*,\infty} = k_p d_i,\ i \in \mathbb{N}_+$ with some coefficient k_p independent of i. By the condition $\sum_i p_i^{*,\infty} = 1$ we easily find that $k_p = 1/D$. Similarly, thanks to (32), $r_i^{*,\infty} = k_r d_i,\ i \in \mathbb{N}_-$ with $k_r = -1/D$, since $d_i < 0$ for $i \in \mathbf{N}_-$. $\square$

Remark. From (29) it follows that any transformation $\mathbf{p}, \mathbf{r} \to \mathbf{p}', \mathbf{r}'$, which does not change the values d_i and D, preserves the same limiting distribution as for the vectors $\mathbf{p}^{\circledast}, \mathbf{r}^{\circledast}$. A class of such transformations may be presented by a shift transformation of coordinates, $p_i \to p_i' = p_i + a_i$, $r_i \to r_i' = p_i + a_i$ with appropriated $a_i's$.

3.2. **Attractive case,** $\alpha = -1$.

Theorem 3. *In the purely attractive case, $\alpha = -1$, the limiting vectors $\mathbf{p}^{\circledast,\infty}, \mathbf{q}^{\circledast,\infty}$ are equal and their coordinates have the following representations:*

$$p_i^{\circledast,\infty} = r_i^{\circledast,\infty} = \begin{cases} 1/m, & i \in \mathbb{S}^\infty \\ 0, & otherwise \end{cases}, \tag{33}$$

where $\mathbb{S}^\infty$ is defined by (27) and $m = |\mathbb{S}^\infty|$ denotes the cardinality of the set $\mathbb{S}^\infty$.

Proof. The proof directly follows from Propositions 5 and 6. We only remark that m is determined by the amount of coordinates selected by the condition $\lim_{N\to\infty} p_k^{\circledast,N} = \lim_{N\to\infty} r_k^{\circledast,N} = 0$, i.e., it is the cardinality of the set $\mathbb{S}^\infty$ defined in (27). $\square$

Below we represent several sufficient conditions for k to belong to $\mathbb{S}_0$. Simultaneously these conditions give some characterization for the points in $\mathbb{S}^\infty$.

We will use the following notations:

$$\sigma_i := p_i + r_i, \;\; \rho_i := p_i r_i, \;\; \sigma_i^1 := p_i^{\circledast,1} + r_i^{\circledast,1} \;\; \rho_i^1 := p_i^{\circledast,1} r_i^{\circledast,1} \tag{34}$$

Proposition 7. *If*

$$\sigma_i \geq \sigma_k, \quad \rho_i > \rho_k, \quad \text{or} \quad \sigma_i > \sigma_k, \quad \rho_i \geq \rho_k, \tag{35}$$

then

$$p_k^{\circledast,\infty} = r_k^{\circledast,\infty} = 0 \tag{36}$$

Proof. By (34) we have

$$\sigma_k^1 = p_k^{\circledast,1} + r_k^{\circledast,1} = 1/z(p_k + r_k + 2p_k r_k) = 1/z(\sigma_k + 2\rho_k)$$

where we recall that $z = 1 + (\mathbf{p}, \mathbf{r})$. Therefore each of the conditions (35) implies that $\sigma_i^1 > \sigma_k^1$. Further, since

$$\rho_k^1 = 1/z^2(\rho_k + (\rho_k)^2 + \rho_k \sigma_k), \tag{37}$$

again from (35) it also follows that $\rho_i^1 > \rho_k^1$. Thus, by induction $\sigma_i^N > \sigma_k^N$ and $\rho_i^N > \rho_k^N$ for all $N \geq 1$. Therefore, by similar arguments as in the proof of Proposition 3 we get (36). $\square$

Let us consider now the critical situation, when for a fixed pair of indices, say i and k, the values $\sigma_k - \sigma_i$, $\rho_k - \rho_i$ have opposite signs, for example, $\sigma_k - \sigma_i > 0$, $\rho_k - \rho_i < 0$. In such a case it is not clear what the behavior of coordinates $p_i^{\circledast,N}, r_i^{\circledast,N}$ and $p_k^{\circledast,N}, r_k^{\circledast,N}$ will be when $N \to \infty$. We will show that the limits depend on which the two values, $2\rho_i + \sigma_i$ or $2\rho_k + \sigma_k$, is larger. Moreover we will show that even if p_k is the largest coordinate, it may happen that $p_k^{\circledast,\infty} = 0$. Let for example, $p_k = \max_j\{p_j, r_j\}$ and $\sigma_k = p_k + r_k > p_i + r_i = \sigma_i$, however the value of r_k is such that $\rho_k = p_k r_k < p_i r_i = \rho_i$. Then under some additional condition it is possible that $p_k^{\circledast,\infty} = 0$. In fact we have:

Lemma 1. *Let for the coordinates $p_i, r_i, p_k, r_k,\ i \neq k$, the following conditions be fulfilled:*

$$\sigma_k > \sigma_i \tag{38}$$

but

$$\rho_k < \rho_i. \tag{39}$$

Assume

$$2\rho_k + \sigma_k \leq 2\rho_i + \sigma_i. \tag{40}$$

Then

$$p_k^{\divideontimes,\infty} = r_k^{\divideontimes,\infty} = 0. \tag{41}$$

Proof. We will show that (38), (39), and (40) imply,

$$p_k^{\divideontimes,1} + r_k^{\divideontimes,1} = \sigma_k^1 \leq \sigma_i^1 = p_i^{\divideontimes,1} + r_i^{\divideontimes,1} \tag{42}$$

and

$$p_k^{\divideontimes,1} r_k^{\divideontimes,1} = \rho_k^1 < \rho_i^1 = p_i^{\divideontimes,1} r_i^{\divideontimes,1}. \tag{43}$$

Then (41) follows from Proposition 7. In reality (42) follows from (40) directly, without condition (39), see Proposition 8 below. So we have only to prove (43).

With this aim we find the representation for ρ_i^1 in terms σ_i and σ_i^1. Since $\sigma_i^1 = 1/z(\sigma_i + 2\rho_i)$ we have

$$\rho_i = 1/2(z\sigma_i^1 - \sigma_i) \tag{44}$$

By (37) and (44) we get

$$\rho_i^1 = 1/z^2(\rho_i + \rho_i^2 + \rho_i\sigma_i) = \frac{1}{2z^2}(z\sigma_i^1 - \sigma_i)[1 + 1/2(z\sigma_i^1 - \sigma_i) + \sigma_i]$$

$$= \frac{1}{4z^2}(z\sigma_i^1 - \sigma_i)(2 + z\sigma_i^1 + \sigma_i) = \frac{1}{4z^2}[2z\sigma_i^1 + z^2(\sigma_i^1)^2 + z\sigma_i^1\sigma_i - 2\sigma_i - z\sigma_i^1\sigma_i - \sigma_i^2]$$

$$= \frac{1}{4z^2}[2z\sigma_i^1 + z^2(\sigma_i^1)^2 - \sigma_i^2 - 2\sigma_i]$$

Therefore

$$\rho_k^1 - \rho_i^1 = 1/z^2[\rho_k(1 + \rho_k + \sigma_k) - \rho_i(1 + \rho_i + \sigma_i)] \tag{45}$$

Thus we have

$$\rho_k^1 - \rho_i^1 = 1/4z^2[2z(\sigma_k^1 - \sigma_i^1) + z^2((\sigma_k^1)^2 - (\sigma_i^1)^2) + ((\sigma_i)^2 - (\sigma_k)^2) + 2(\sigma_i - \sigma_k)] < 0 \tag{46}$$

due to starting condition (39), and (42). Thus $\rho_k^1 < \rho_i^1$, i.e., (43) is true. $\square$

We stress that (41) is true in spite of $\sigma_k > \sigma_i$. Of course, if $\sigma_k < \sigma_i$ and $\rho_k < \rho_i$, then (41) holds without any additional condition of the form (40).

Proposition 8. *The conditions (40) and (42) are equivalent.*

Proof. By (34)

$$\sigma_i^1 = p_i^{\divideontimes,1} + r_i^{\divideontimes,1} = 1/z(p_i + r_i + 2p_i r_i) = 1/z(\sigma_i + 2\rho_i)$$

Therefore

$$\sigma_k^1 - \sigma_i^1 = 1/z[\sigma_k + 2\rho_k - (\sigma_i + 2\rho_i)] \leq 0$$

if and only if (40) is fulfilled. □

What about the case

$$\sigma_k > \sigma_i,\ \rho_k < \rho_i,\ 2\rho_k + \sigma_k := \kappa_k > \kappa_i =: 2\rho_i + \sigma_i? \tag{47}$$

Proposition 9. *Let* $\kappa_k^N := 2\rho_k^N + \sigma_k^N$. *Under the initial conditions (47) the sign*$(\kappa_k^N - \kappa_i^N)$ *may at most change one time as* $N \to \infty$.

Proof. On the first step by Proposition 8 we get $\sigma_k^1 > \sigma_i^1$, since $\kappa_k > \kappa_i$.

(a) If we assume that $\rho_k^1 \geq \rho_i^1$, then obviously $\kappa_k^1 > \kappa_i^1$, and $\kappa_k^N > \kappa_i^N$ for all N. Therefore the sign$(\kappa_k^N - \kappa_i^N)$ is the same for all N.

(b) If we assume that $\rho_k^1 < \rho_i^1$, then we have to consider two subcases.

(b') $\kappa_k^1 > \kappa_i^1$. Since now we have $\sigma_k^1 > \sigma_i^1$ and $\rho_k^1 < \rho_i^1$, this means that after the first step we get again the starting situation and the sign$(\kappa_k^N - \kappa_i^N)$ is not changed.

(b") Finally if $\kappa_k^1 \leq \kappa_i^1$, i.e., the sign$(\kappa_k^1 - \kappa_i^1)$ is opposite to sign$(\kappa_k - \kappa_i)$, then we have: $\sigma_k^1 > \sigma_i^1$, $\rho_k^1 < \rho_i^1$, $\kappa_k^1 \leq \kappa_i^1$, and by Lemma 1, $\kappa_k^N < \kappa_i^N$, for all $N = 1, 2, \ldots.$ □

We remark that, as a consequence of the Proposition 9 and its proof, there is only one chance to observe the changing of the sign$(\kappa_k^N - \kappa_i^N)$ in the case where (47) holds (i.e., it is the sub-case (b")).

The hypotheses is: this sub-case in general does not meet. However we do not have a proof of this at the moment.

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