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Theorem of conflicts for a pair of probability measures

Volodymyr Koshmanenko

University of Bonn, Institute of Mathematics, 53115 Bonn, Germany
(e-mail: koshman03@yahoo.com)

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Abstract. We develop mathematical tools suitable for the construction of conflict models with non-annihilating adversaries. In a set of probability measures we introduce a non-commutative conflict composition and consider the associated dynamical system. We prove that for each couple of non-identical mutually nonsingular measures, the corresponding trajectory of the dynamical system converges to an invariant point represented by a pair of mutually singular measures. The disjoint supports of the limiting measures determine the final re-distribution of the starting area of conflict as a result of an "infinite war" for existence space (the pure repelling effect).

Key words: Probability measure, Conflict composition, Discrete measure, Stochastic vector, Dynamical system, Mutually singular measures

AMS Subject Classifications: 37M, 90A58, 90B, 90D15, 91D, 92B, 92D

1 Introduction

This paper is aimed to develop mathematical tools for constructing of conflict models in a situation where none of the opponents have any strategic priority. The conflicting interaction among the opponents only produces a certain re-distribution of the common area of interests. In other words we assume that each adversary is *a priori* non-annihilating.

In fact we develop an alternative approach to the well-known mathematical theory of population dynamics (see e.g. [3]-[6]) based on a modified Lotka-Volterra equation and aimed to describe the quantitative changes of conflicting species.

We assign to each opponent a probability measure on the same metric space, which is interpreted as the existence space or area of common interests.

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The competition interaction between opponents we express in a form of a conflict composition between probability measures. The iteration of this composition generates a certain dynamical system. We investigate its trajectories and prove the existence of the limiting states.

Here is a more detailed explanation of our ideas.

Let us assign to adversaries, here we consider only a pair of them, say A and B, a couple of probability measures μ_0 and ν_0 on some metric space X. Independently, A and B occupy a subset $E \subset X$ with probabilities $\mu_0(E)$ and $\nu_0(E)$, respectively. We assume that $\mu_0,\ \nu_0$ are non-identical and are mutually nonsingular. Hence $\mathrm{supp}\mu_0 \cap \mathrm{supp}\nu_0 \neq \emptyset$. Incompatibility of A and B generates a conflicting interaction. We write this fact mathematically in a form of a non-commutative conflict composition, notation $\divideontimes$, between measures μ_0 and ν_0. In other words, we construct a new pair of probability measures, $\mu_1 = \mu_0 \divideontimes \nu_0$ and $\nu_1 = \nu_0 \divideontimes \mu_0$ in terms of the conditional probability to occupy a subset E by A (or B) when B (A) is absent in E. So a value $\mu_1(E)$ is proportional to the product of $\mu_0(E)$ and $\nu_0(X \setminus E)$, the starting probability for A to occupy E and the probability for B to be absent in the set E. Similarly for the side B. Thus, measures μ_1, ν_1 describe the re-distribution of the conflicting area between A and B after the first step of interaction. However the conflict is not solved provided that measures μ_1, ν_1 are mutually nonsingular. So one can repeat the above described procedure for infinite times and get two sequences of probability measures $\mu_N, \nu_N,\ N = 1, 2, \ldots$ which generates a trajectory of a certain dynamical system.

In the present paper we consider mainly the case of discrete measures. We prove the existence of the limiting pair $\mu_\infty = \lim_{N\to\infty} \mu_N, \nu_\infty = \lim_{N\to\infty} \nu_N$ and show that $\mu_\infty,\ \nu_\infty$ are mutually singular and invariant with respect to the action of $\divideontimes$. The disjoint supports of the limiting measures establish the final re-distribution of the starting conflict area, i.e., we observe the pure repelling effect for non-identical adversaries.

In [1, 2] we extend our results for a class of so-called image measures and investigate the fractal structure of the limiting supports.

2 The conflict composition for stochastic vectors

We start with a simplest case.

Let $X \equiv \Omega$ be a finite set, $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$, $n > 1$, and μ_0, ν_0 be a pair of probability discrete measures,

$$\mu_0(\omega_i) = p_i \geq 0, \quad \nu_0(\omega_i) = q_i \geq 0, \quad i = 1, 2, \ldots, n,$$

$$\mu_0(\Omega) = p_1 + \cdots + p_n = q_1 + \cdots + q_n = \nu_0(\Omega) = 1.$$

So measures μ_0, ν_0 are associated with a couple of stochastic vectors, say $\mathbf{p}, \mathbf{q} \in \mathbf{R}^n_+$. We recall that a vector $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ with coordinates $p_i \geq 0$ is called stochastic if $\|\mathbf{p}\|_1 := p_1 + \cdots + p_n = 1$.

Given a pair of stochastic vectors $\mathbf{p}, \mathbf{q}$, we introduce the non-linear non-commutative conflict composition, notation $\divideontimes$, by

$$\mathbf{p}^{\divideontimes}, 1 = \mathbf{p} \divideontimes \mathbf{q}, \quad \mathbf{q}^{\divideontimes}, 1 = \mathbf{q} \divideontimes \mathbf{p},$$

where the coordinates of the vectors $\mathbf{p}^{\divideontimes}, 1, \mathbf{q}^{\divideontimes}, 1 \in \mathbf{R}^n_+$ are defined as follows:

$$p_i^{*,1} := \frac{p_i(1-q_i)}{z}, \quad q_i^{*,1} := \frac{q_i(1-p_i)}{z}, \; i = 1,2,...,n. \tag{1}$$

The normalizing coefficient z is determined by the condition $\|\mathbf{p}^{*,1}\|_1 = \|\mathbf{q}^{*,1}\|_1 = 1$, and it follows that

$$z = 1 - (\mathbf{p},\mathbf{q}), \quad 0 \le z \le 1, \tag{2}$$

where $(\cdot,\cdot)$ stands for the inner Euclidean product in $\mathbf{R}^n$.

Remark. The conflict composition is well defined only if $(\mathbf{p},\mathbf{q}) \neq 1$, and it acts as the identical transformation if $(\mathbf{p},\mathbf{q}) = 0$. So we will suppose that $0 < (\mathbf{p},\mathbf{q}) < 1$.

The N-fold iteration of the conflict composition $\circledast$ produces a couple of stochastic vectors $\mathbf{p}^{*,N}, \; \mathbf{q}^{*,N} \in \mathbf{R}^n_+$ with coordinates

$$p_i^{*,N} := \frac{1}{z_{N-1}} p_i^{*,N-1}(1-q_i^{*,N-1}), \; q_i^{*,N} := \frac{1}{z_{N-1}} q_i^{*,N-1}(1-p_i^{*,N-1}), \; N = 1,2,..., \tag{3}$$

where $p_i^{(0)} \equiv p_i, q_i^{(0)} \equiv q_i, \; z_0 = z$, and

$$0 < z_{N-1} = 1 - (\mathbf{p}^{*,N-1}, \mathbf{q}^{*,N-1}) < 1. \tag{4}$$

We are interested in the existence of the limits $\mathbf{p}^{*,\infty} = \lim_{N\to\infty} \mathbf{p}^{*,N}$, $\mathbf{q}^{*,\infty} = \lim_{N\to\infty} \mathbf{q}^{*,N}$.

Example. Let $n = 2$. Consider a couple of vectors $\mathbf{p} = (p_1, p_2)$, $\mathbf{q} = (q_1, q_2) \in \mathbf{R}^2_+$, with coordinates $0 < p_1, p_2, q_1, q_2 < 1$, $p_1 + p_2 = q_1 + q_2 = 1$. We observe that already on the first step, $p_1^{*,1} = q_2^{*,1}$, $p_2^{*,1} = q_1^{*,1}$. So one can start at once with the case:

$$\mathbf{p} = (a,b), \; \mathbf{q} = (b,a), \; 0 < a,b < 1, a + b = 1.$$

Then by (1) and (2) we get, $\mathbf{p}^{*,1} = (a_1, b_1)$, $\mathbf{q}^{*,1} = (b_1, a_1)$, $\quad a_1 + b_1 = 1$, where

$$a_1 = \frac{1}{z} a(1-b) = \frac{a^2}{z}, \quad b_1 = \frac{1}{z} b(1-a) = \frac{b^2}{z}, \quad z = 1 - 2ab = a^2 + b^2.$$

Thus $a_1 = a^2(a^2+b^2)^{-1}$, $b_1 = b^2(a^2+b^2)^{-1}$. If we assume that $a < b$, i.e., $a < 1/2 < b$, then we get $a_1 = ak_1 < a$ since $k_1 := a(a^2+b^2)^{-1} < 1$. For $\mathbf{p}^{*,2} = (a_2, b_2)$ we find $a_2 = a_1 k_2$ with $k_2 := a_1(a_1^2 + b_1^2)^{-1} < 1$. Thus $a_2 = ak_1k_2$, with $k_1, k_2 < 1$. By induction, for $\mathbf{p}^{*,N} = (a_N, b_N)$ we get $a_N = ak_1 \cdots k_N$ with $k_N := a_{N-1}(a_{N-1}^2 + b_{N-1}^2) < 1$. We see that $a_N \to 0$ since in the opposite case, $k_N \to 1$, $2a_N^2 - 3a_N + 1 \to 0$, and $a_N \to 1/2$, which is contradictory to $a_N < a < 1/2$. So the limiting vectors are $\mathbf{p}^{*,\infty} = (0,1)$ and $\mathbf{q}^{*,\infty} = (1,0)$ provided that $a < b$.

In the case $\mathbf{p} = \mathbf{q}$ we get $\mathbf{p}^{*,N} = \mathbf{q}^{*,N} = (1/2, 1/2)$ for any $N \geq 1$. Thus for $\mathbf{p}, \mathbf{q} \in \mathbf{R}^2_+$ only three limiting cases are possible,

$$\begin{pmatrix} \mathbf{p}^{*,\infty} \\ \mathbf{q}^{*,\infty} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

Theorem 1. (Theorem of conflicts for stochastic vectors) *For each pair of stochastic vectors* $\mathbf{p}, \mathbf{q} \in \mathbf{R}^n_+, n > 1, \quad 0 < (\mathbf{p}, \mathbf{q}) < 1$, *there exist limits*

$$\mathbf{p}^{*,\infty} = \lim_{N\to\infty} \mathbf{p}^{*,N}, \quad \mathbf{q}^{*,\infty} = \lim_{N\to\infty} \mathbf{q}^{*,N},$$

where $\mathbf{p}^{*,N}$, $\mathbf{q}^{*,N}$ *are given by (3) and (4). The limiting vectors* $\mathbf{p}^{*,\infty}$, $\mathbf{q}^{*,\infty}$ *are invariant with respect to the action of the conflict composition*:

$$\mathbf{p}^{*,\infty} = \mathbf{p}^{*,\infty} \circledast \mathbf{q}^{*,\infty}, \quad \mathbf{q}^{*,\infty} = \mathbf{q}^{*,\infty} \circledast \mathbf{p}^{*,\infty}. \tag{5}$$

If $\mathbf{p} \neq \mathbf{q}$, *then the limiting vectors are orthogonal,*

$$\mathbf{p}^{*,\infty} \perp \mathbf{q}^{*,\infty}. \tag{6}$$

If $\mathbf{p} = \mathbf{q}$, *then* $\mathbf{p}^{*,\infty} = \mathbf{q}^{*,\infty}$ *and* $p_i^{*,\infty} = q_i^{*,\infty} = 1/m$ *for all* i *such that* $p_i = q_i \neq 0$, *where* m $(m \leq n)$ *denotes the number of non-zero coordinates.*

For the proof of this theorem we use the following lemmas and propositions.

Lemma 1. *Let* $\mathbf{p} \neq \mathbf{q}$ *and* $0 \leq q_i < p_i \leq 1$ *for some* i. *Then*

$$\lim_{N\to\infty} q_i^{*,N} = 0 \tag{7}$$

and

$$\lim_{N\to\infty} p_i^{*,N} = p_i^{*,\infty} > 0. \tag{8}$$

Proof. If $q_i = 0$ or $p_i = 1$, then obviously $q_i^{*,N} = 0$ and $p_i^{*,N} = 1$ for all $N \geq 1$. So we have to prove only the case $0 < q_i < p_i < 1$. Denote

$$R_i^{(0)} := \frac{p_i}{q_i} \text{ and } R_i^{(N)} := \frac{p_i^{*,N}}{q_i^{*,N}} \text{ for } N \geq 1.$$

Clearly,

$$1 < R_i^{(0)} < R_i^{(1)},$$

since due to (1) and (2), $R_i^{(1)} = R_i^{(0)} k_i^{(0)}$ with $k_i^{(0)} := \frac{1-q_i}{1-p_i} > 1$. Therefore $0 < q_i^{*,1} < p_i^{*,1} < 1$. By induction we get $1 < R_i^{(N)} < \infty$ for all N, which is equivalent to $0 < q_i^{*,N} < p_i^{*,N} < 1$. We note that

$$R_i^{(N)} = \frac{p_i^{*,N-1}}{q_i^{*,N-1}} \cdot \frac{1 - q_i^{*,N-1}}{1 - p_i^{*,N-1)}} = R_i^{(N-1)} k_i^{(N-1)} = R_i^{(0)} \cdot k_i^{(0)} \cdots k_i^{(N-1)}, \tag{9}$$

where

$$k_i^{(N)} := \frac{1 - q_i^{*,N}}{1 - p_i^{*,N}}.$$

Let us now show that

$$1 < k_i^{(0)} < k_i^{(1)} < \cdots < k_i^{(N)} < \cdots \tag{10}$$

Indeed, using $z = 1 - (\mathbf{p}, \mathbf{q}) > 0$ we have

$$k_i^{(1)} = \frac{1 - q_i^{*,1}}{1 - p_i^{*,1}} = \frac{1 - \frac{1}{z} q_i(1 - p_i)}{1 - \frac{1}{z} p_i(1 - q_i)} = \frac{z - q_i(1 - p_i)}{z - p_i(1 - q_i)} = \frac{1 - q_i - (\mathbf{p}, \mathbf{q}) + q_i p_i}{1 - p_i - (\mathbf{p}, \mathbf{q}) + q_i p_i} = \frac{1 - q_i - I_i}{1 - p_i - I_i},$$

where $I_i := (\mathbf{p}, \mathbf{q}) - q_i p_i$. Obviously $0 < I_i < 1 - p_i < 1 - q_i$ due to $q_i < p_i$ and

$$I_i = (\mathbf{p}, \mathbf{q}) - q_i p_i = \sum_{k \neq i} p_k q_k < \sum_{k \neq i} p_k = 1 - p_i.$$

This implies that $k_i^{(0)} = \frac{1 - q_i}{1 - p_i} < k_i^{(1)}$. By induction we get (10), since $1 - p_i^{*,N} < 1 - q_i^{*,N}$ for all N. In turn, (9) and (10) imply

$$R_i^{(N)} = \frac{p_i^{*,N}}{q_i^{*,N}} \to \infty, \quad N \to \infty. \tag{11}$$

This yields $q_i^{*,N} \to 0$ since $p_i^{*,N} < 1$, which proves (7). Let us show (8). Define

$$D_i^{(0)} := p_i - q_i, \quad D_i^{(N)} := p_i^{*,N} - q_i^{*,N}, \ N = 1, 2, \ldots$$

Obviously $D_i^{(0)} > 0$ and by (1)

$$D_i^{(1)} = p_i^{*,1} - q_i^{*,1} = \frac{1}{z} [p_i(1 - q_i) - q_i(1 - p_i)] = \frac{1}{z} D_i^{(0)}.$$

Hence $D_i^{(0)} < D_i^{(1)}$ since $0 < z < 1$. By induction, $D_i^{(N)} < D_i^{(N+1)}$ for all N. Therefore, there exists the limit $D_i^{(\infty)} = \lim_{N \to \infty} D_i^{(N)} \leq 1$, since $D_i^{(N)} = p_i^{*,N} - q_i^{*,N} < 1$, and $q_i^{*,N} \to 0$ by (7). Moreover, we see that due to $q_i^{*,N} \to 0$,

$$\lim_{N \to \infty} p_i^{*,N} = p_i^{*,\infty} = D_i^{(\infty)} = \sup_N D_i^{(N)} > 0. \qquad \square$$

In the case $0 \leq p_k < q_k \leq 1$, similarly to (7) we get

$$\lim_{N \to \infty} p_k^{*,N} = 0, \tag{12}$$

and hence

$$\lim_{N \to \infty} q_k^{*,N} = q_k^{*,\infty} = D_k^{(\infty)} = \sup_N D_k^{(N)} > 0,$$

where $D_k^{(N)} = q_k^{*,N} - p_k^{*,N}$.

Proposition 1. *Let* $1 > p_i = q_i > 0$ *for some* i, *then*

$$1 > p_i^{*,N} = q_i^{*,N} > 0 \ \textit{for all} \ N = 1, 2, \ldots$$

Proof. By (3) we have $p_i^{*,N} = q_i^{*,N}$ if and only if

$$p_i^{*,N-1}(1 - q_i^{*,N-1}) = q_i^{*,N-1}(1 - p_i^{*,N-1}),$$

i.e., if and only if $p_i^{*,N-1} = q_i^{*,N-1}$. $\square$

Lemma 2. *Let* $\mathbf{p} \neq \mathbf{q}$, but $p_j = q_j$ *for some* j. *Then*

$$p_j = q_j \Longrightarrow p_j^{*,N} = q_j^{*,N} \to 0, \ N \to \infty. \tag{13}$$

Proof. If $1 > p_j = q_j > 0$ for some j, then $1 > p_j^{*,N} = q_j^{*,N} > 0$ for all N due to Proposition 1. Assume for a moment that $p_j^{*,N}$ does not converge to zero. Then one can choose a subsequence N' such that $p_j^{*,N'} \longrightarrow c > 0$. This yields a contradiction. Indeed, since $\mathbf{p} \neq \mathbf{q}$, there exists i such that $0 \leq q_i < p_i \leq 1$, and then due to (7) and (8) the right- and the left-hand sides of the relation

$$p_i^{*,N'+1} = \frac{p_i^{*,N'}(1 - q_i^{*,N'})}{z_{N'}}$$

have different limiting behaviour. Indeed, $q_i^{*,N'} \to 0$, but $z_{N'} = 1 - (\mathbf{p}^{*,N'}, \mathbf{q}^{*,N'}) = 1 - (p_j^{*,N'})^2 - \sum_{k \neq j} p_k^{*,N'} q_k^{*,N'} \leq 1 - (p_j^{*,N'})^2 \to 1 - c^2 < 1$ by assumption. □

Proposition 2. *If $p_i \geq q_i$ for some i, then*

$$p_i^{*,N} \geq q_i^{*,N} \text{ for all } N \geq 1. \tag{14}$$

Proof. By (1), (2) we get $p_i^{*,1} \geq q_i^{*,1}$. By induction we have

$$p_i^{*,N+1} = \frac{1}{z_N} p_i^{*,N}(1 - q_i^{*,N}) \geq q_i^{*,N+1} = \frac{1}{z_N} q_i^{*,N}(1 - p_i^{*,N}) \geq q_i^{*,N+1},$$

since $p_i^{*,N} \geq q_i^{*,N}$ implies $(1 - q_i^{*,N}) \geq (1 - p_i^{*,N})$. □

Proposition 3. *If* $\mathbf{p} \perp \mathbf{q}$, *then*

$$\mathbf{p} = \mathbf{p} \circledast \mathbf{q}, \quad \mathbf{q} = \mathbf{q} \circledast \mathbf{p},$$

i.e., vectors $\mathbf{p}$, $\mathbf{q}$ *are invariant with respect to the action of the conflict composition.*

Proof. The condition $\mathbf{p} \perp \mathbf{q}$ means that either p_i or q_i is equal to zero for each i. This yields that all coordinates $p_i^{*,1} = p_i$ and $q_i^{*,1} = q_i$. Indeed, if $p_i \neq 0$, then $q_i = 0$, and $p_i^{*,1} = \frac{1}{z} p_i(1 - q_i) = p_i$ since $z = 1$ due to $\mathbf{p} \perp \mathbf{q}$. And if $p_i = 0$, then $p_i^{*,1} = 0$ too. □

Assume now that vectors $\mathbf{p}, \mathbf{q} \in \mathbf{R}_+^n$, $n > 2$, coincide and

$$p_i = q_i \neq 0, \text{ for all } i = 1, 2, ..., n. \tag{15}$$

Then without loss of generality one can assume that

$$0 < p_1 \leq 1/n \leq p_2 \leq \cdots \leq p_{n-1} \leq 1/n \leq p_n < 1. \tag{16}$$

Setting $q_i^c := 1 - q_i = 1 - p_i = p_i^c$ we have

$$1 > q_1^c \geq 1/n \geq q_2^c \geq \cdots \geq q_{n-1}^c \geq 1/n \geq q_n^c > 0. \tag{17}$$

Proposition 4. *Under assumption (15) with $n > 2$ the normalizing coefficient z satisfies the inequalities*

$$q_1^c \geq z \geq q_n^c. \tag{18}$$

Proof. If $\mathbf{p} = \mathbf{q} = (1/n, 1/n, ..., 1/n)$, then $z = \frac{n-1}{n} = q_1^c = q_n^c$, and (18) is true. In a general case we note that

$$z = (\mathbf{p}, \mathbf{q}^c) = p_1 q_1^c + p_2 q_2^c + ... + p_n q_n^c,$$

where $\mathbf{q}^c := (q_1^c, q_2^c, ..., q_n^c)$. If each term q_i^c is replaced by the maximal term q_1^c (see (17)), then obviously

$$z = \sum_{i=1}^{n} p_i q_i^c \leq q_1^c \sum_{i=1}^{n} p_i = q_1^c.$$

Similarly, if all q_i^c are replaced by the minimal term q_n^c, then

$$z = \sum_{i=1}^{n} p_i q_i^c \geq q_n^c \sum_{i=1}^{n} p_i = q_n^c.$$

This proves (18). □

Proposition 5 *Under assumption (15) with* $n > 2$ *the following inequalities hold*:

$$p_1 \leq p_1^{*,1}, \quad p_n^{*,1} \leq p_n, \tag{19}$$

$$p_1^{*,1} \leq p_i^{*,1} \leq p_n^{*,1}, \quad i = 1, ..., n, \tag{20}$$

$$p_1 \leq p_1^{*,1} \leq p_1^{*,N} \leq p_i^{*,N} \leq p_n^{*,N} \leq p_n^{*,1} \leq p_n, \quad N \geq 1. \tag{21}$$

Proof. For $p_1^{*,1} = p_1 q_1^c / z_0$ we have $p_1 \leq p_1^{*,1}$ due to (18). Similarly, for $p_n^{*,1} = p_n q_n^c / z_0$ we have $p_n^{*,1} \leq p_n$ again due to (18). Thus (19) is proved.

Further, each coordinate $p_i^{*,1} = p_i(1 - q_i)/z_0$ satisfies inequalities (20) because $q_i = p_i$ and $p_1(1 - p_1) \leq p_i(1 - p_i) \leq p_n(1 - p_n)$ due to $p_1 = \min\{p_i\} \leq 1/n$, and $p_n = \max\{p_i\} \geq 1/n$. For more details one can consider a function $y = x(1 - x), x \in (0, 1)$. It has the maximum in the point $x_0 = 1/2$. From the graphic of this function we see that for any point $x = p_i \leq p_n$, the value $y(p_i) \leq y(p_n)$ since $p_i \leq \min\{p_n, 1 - p_n\}$, $n \geq 3$. By the similar arguments $y(p_1) \leq y(p_i)$. By induction (19), (20) hold for all $N = 1, 2, ...$ This yields (21). □

Proposition 6 *Under assumption (15) with* $n > 2$ *there exist the limiting vectors*,

$$\mathbf{p}^{*,\infty} = \lim_{N \to \infty} \mathbf{p}^{*,N} = \lim_{N \to \infty} \mathbf{q}^{*,N} = \mathbf{q}^{*,\infty},$$

which have the form,

$$\mathbf{p}^{*,\infty} = \mathbf{q}^{*,\infty} = (1/n, 1/n, ..., 1/n). \tag{22}$$

Proof. It is easy to see that (21) and (18) imply

$$q_1^{*,N,c} \geq z_N \geq q_n^{*,N,c}, \quad N = 1, 2, ... \tag{23}$$

where $q_i^{*,N,c} = 1 - q_i^{*,N}$. In turn, (21), (23) imply that sequences $p_i^{*,N}$ with $i = 1$ and $i = n$ are monotonic on N and therefore there exist the limits $p_1^{*,\infty} = \lim_{N \to \infty} p_1^{*,N}$, $p_n^{*,\infty} = \lim_{N \to \infty} p_n^{*,N}$. Further, since $p_1^{*,\infty} = p_1^{*,\infty}(1 - p_1^{*,\infty})/z_\infty$ and $p_n^{*,\infty} = p_n^{*,\infty}(1 - p_n^{*,\infty})/z_\infty$, we conclude that $z_\infty = 1 - p_1^{*,\infty} = 1 - p_n^{*,\infty}$. By (16) and (21) this is only possible if

$$p_1^{*,\infty} = p_n^{*,\infty} = 1/n.$$

Hence $p_i^{*,\infty} = \lim_{N\to\infty} p_i^{*,N,}$ exist for all i and due to (21) $p_i^{*,\infty} = 1/n$. This proves (22). □

Proof of Theorem 1. For the case $\mathbf{p},\mathbf{q} \in \mathbf{R}_+^2$ see Example 1. Let $\mathbf{p},\mathbf{q} \in \mathbf{R}_+^n,\ n \geq 3$. If $\mathbf{p} \neq \mathbf{q}$, then the existence of the limiting vectors $\mathbf{p}^{*,\infty}, \mathbf{q}^{*,\infty}$ is proved by Lemma 1 and Lemma 2. Moreover due to (7), (12), and (13) we have $(\mathbf{p}^{*,N}, \mathbf{q}^{*,N}) \to 0$ as $N \to \infty$. Thus, the limiting vectors are orthogonal, i.e., (6) is proved. Hence $z_N \to 1$, i.e.,

$$z_\infty = 1. \tag{24}$$

In turn (24) imply (5).

If $\mathbf{p} = \mathbf{q}$, then by Proposition 1, $\mathbf{p}^{*,N} = \mathbf{q}^{*,N}$ for all N and therefore $\mathbf{p}^{*,\infty} = \mathbf{q}^{*,\infty}$. Under condition (15) all coordinates of vectors $\mathbf{p}^{*,\infty}, \mathbf{q}^{*,\infty}$ are equal to $1/n$ (see Proposition 6). Clearly that if $p_i = q_i \neq 0$ only for $i = 1, ..., m < n$, then $p_i^\infty = q_i^\infty = 1/m$. In any case (5) and (6) are true too. □

3 The conflict composition for discrete measures on a countable space

Let $X = \Omega$ be a countable set of points, $\Omega = \{\omega_1, \omega_2, ...\}$, endowed by the discrete topology. Let μ_0, ν_0 be a pair of the discrete probability measures on Ω,

$$\mu_0(\omega_i) = p_i^{(0)} \geq 0, \quad \nu_0(\omega_i) = q_i^{(0)} \geq 0, \ \ i = 1, 2, ...$$

$$\mu_0(\Omega) = \sum_{i=1}^{\infty} p_i^{(0)} = \nu_0(\Omega) = \sum_{i=1}^{\infty} q_i^{(0)} = 1.$$

We assume that the measures μ_0, ν_0 are mutually nonsingular and exclude the situation with $\mu_0(\omega_i) = \nu_0(\omega_i) = 1$ for some ω_i.

Given measures μ_0, ν_0 we introduce a new pair of discrete probability measures μ_1, ν_1 on Ω by the conflict composition $*$ defined as follows:

$$\mu_1 = \mu_0 * \nu_0, \quad \nu_1 = \nu_0 * \mu_0,$$

where

$$\mu_1(\omega_i) := p_i^{(1)} := \frac{1}{z_0} p_i^{(0)} (1 - q_i^{(0)}) \equiv \frac{1}{z_0} p_i^{(0)} \cdot \sum_{k=1, k\neq i}^{\infty} q_k^{(0)},$$

$$\nu_1(\omega_i) := q_i^{(1)} := \frac{1}{z_0} q_i^{(0)} (1 - p_i^{(0)}) \equiv \frac{1}{z_0} q_i^{(0)} \cdot \sum_{k=1, k\neq i}^{\infty} p_k^{(0)},$$

and where the coefficient z_0 is calculated using the normalizing condition:

$$\mu_0(\Omega) = \sum_{i=1}^{\infty} p_i^{(1)} = \nu_0(\Omega) = \sum_{i=1}^{\infty} q_i^{(1)} = 1.$$

Thus

$$z_0 = \sum_{i=1}^{\infty} \left(p_i^{(0)} \cdot \sum_{k=1, k\neq i}^{\infty} q_k^{(0)} \right) = 1 - \sum_{i=1}^{\infty} p_i^{(0)} q_i^{(0)} < 1 = 1 - (\mathbf{p}^0, \mathbf{q}^0),$$

where $(\cdot,\cdot)$ stands for the inner product in the Hilbert space l^2 between vectors $\mathbf{p}^0 := (p_1^{(0)}, p_2^{(0)}, \ldots)$, $\mathbf{q}^0 := (q_1^{(0)}, q_2^{(0)}, \ldots)$ which in fact belong to l^1.

Similarly we can define the pair of probability measures μ_2 and ν_2 as a result of the second step of the conflict interaction:

$$\mu_2 = \mu_1 \circledast \nu_1, \quad \nu_2 = \nu_1 \circledast \mu_1,$$

where

$$\mu_2(\omega_i) = p_i^{(2)} := \frac{1}{z_1} p_i^{(1)}(1 - q_i^{(1)}), \quad \nu_1(\omega_i) = q_i^{(2)} := \frac{1}{z_1} q_i^{(1)}(1 - p_i^{(1)}),$$

with the normalizing coefficient $z_1 = 1 - \sum_{i=1}^{\infty} p_i^{(1)} q_i^{(1)} < 1$. And so on, for any $N = 1, 2, \ldots$, up to infinity,

$$\begin{aligned} \mu_N(\omega_i) = p_i^{(N)} &:= \frac{1}{z_{N-1}} p_i^{(N-1)}(1 - q_i^{(N-1)}), \\ \nu_N(\omega_i) = q_i^{(N)} &:= \frac{1}{z_{N-1}} q_i^{(N-1)}(1 - p_i^{(N-1)}), \end{aligned} \tag{25}$$

where

$$z_{N-1} = 1 - (\mathbf{p}^{N-1}, \mathbf{q}^{N-1}) \tag{26}$$

with $\mathbf{p}^{N-1} = (p_1^{(N-1)}, p_2^{(N-1)}, \ldots)$, $\mathbf{q}^{N-1} := (q_1^{(N-1)}, q_2^{(N-1)}, \ldots)$.

The problem is to prove the existence of the limiting measures $\mu_\infty, \quad \nu_\infty$:

$$\mu_\infty(\omega_i) = p_i^{(\infty)} = \lim_{N\to\infty} p_i^{(N)}, \quad \nu_\infty(\omega_i) = q_i^{(\infty)} = \lim_{N\to\infty} q_i^{(N)}, \tag{27}$$

and investigate their distributions on Ω.

Theorem 2. (Theorem of conflicts for discrete measures) *Let $\mu_0 \neq \nu_0$ be a pair of mutually nonsingular discrete probability measures on a space $\Omega = \{\omega_i\}_{i=1}^{\infty}$. The case $\mu_0(\omega_i) = \nu_0(\omega_i) = 1$ for some ω_i is excluded. Then all limits in (27) exist and thus determine two probability measures*

$$\mu_\infty = \lim_{N\to\infty} \mu_N, \quad \nu_\infty = \lim_{N\to\infty} \nu_N,$$

which are mutually singular,

$$\mu_\infty \perp \nu_\infty,$$

and are both invariant with respect to the action of the conflict composition:

$$\mu_\infty = \mu_\infty \circledast \nu_\infty, \quad \nu_\infty = \nu_\infty \circledast \mu_\infty. \tag{28}$$

Proof. If $0 \leq \mu_0(\omega_i) < \nu_0(\omega_i) \leq 1$ for some i, then by the same arguments as in Lemma 1 we get $\mu_N(\omega_i) \to 0$, and $\nu_N(\omega_i) \to \nu_\infty(\omega_i) = \sup_N (q_i^{(N)} - p_i^{(N)}) \leq 1$ in notations of (25). Similarly, if $0 \leq \nu_0(\omega_k) < \mu_0(\omega_k) \leq 1$ for some k, then $\nu_N(\omega_k) \to 0$, and $\mu_N(\omega_k) \to \mu_\infty(\omega_k) = \sup_N (p_k^{(N)} - q_k^{(N)}) \leq 1$. Moreover, if $\mu_0(\omega_j) = \nu_0(\omega_j)$ for some j, then both sequences $\mu_N(\omega_j)$ and $\nu_N(\omega_j)$ converge to zero, as $N \to \infty$. Indeed, $\mu_0 \neq \nu_0$ implies the existence at least a point ω_k such that $\mu_0(\omega_k) \neq \nu_0(\omega_k)$. Suppose $\nu_0(\omega_k) < \mu_0(\omega_k)$. Then in notations of (25) we get $\mu_\infty(\omega_k) = p_k^{(\infty)} = p_k^{(\infty)}(1 - q_k^{(\infty)})/z_\infty$, where by Lemma 1, $p_k^{(\infty)} > 0$ and $\nu_\infty(\omega_k) \equiv q_k^{(\infty)} = 0$. This yields $z_\infty = 1$. Therefore, (see (26)) $(\mathbf{p}^N, \mathbf{q}^N) \to 0$. In particular,

$\mu_N(\omega_j) = \nu_N(\omega_j) \to 0$ for all indices j such that $\mu_0(\omega_j) = \nu_0(\omega_j)$. Thus there exist two non-trivial discrete measures μ_∞, ν_∞ on Ω, which are mutually singular due to $(\mathbf{p}^\infty, \mathbf{q}^\infty) = 0$, where $\mathbf{p}^\infty = (p_1^{(\infty)}, p_2^{(\infty)}, \ldots)$, $\mathbf{q}^\infty = (q_1^{(\infty)}, q_2^{(\infty)}, \ldots)$. These measures are probability measures since $\mu_N(\Omega) = \nu_N(\Omega) = 1$ for each N. Finally, (28) directly follows from $(\mathbf{p}^\infty, \mathbf{q}^\infty) = 0$. □

4 Discussion

The above results admit natural extensions to a general case where X is a metric space with a σ-algebra $\mathcal{B}$ of Borel subsets. Here we present a brief sketch. So let μ_0, ν_0 be a pair of Borel mutually nonsingular probability measures on X. Assume there exists a countable ε-covering, $\varepsilon > 0$, of X,

$$X = \cup_i B_i, \quad B_i \in \mathcal{B}, \; diam(B_i) \le \varepsilon,$$

such that

$$\mu_0(B_i \cap B_k) = \nu_0(B_i \cap B_k) = 0, \quad i \ne k. \tag{29}$$

Then we introduce a new pair of probability measures $\mu_1 \equiv \mu_{1,\varepsilon}, \nu_1 \equiv \nu_{1,\varepsilon}$ as follows, $\mu_1 = \mu_0 \circledast \nu_0$, $\nu_1 = \nu_0 \circledast \mu_0$, where for any $A \in \mathcal{B}$, $A = \cup_i A_i$, $A_i = A \cap B_i$,

$$\mu_1(A) = \sum_{i=1}^{\infty} \mu_1(A_i), \quad \nu_1(A) = \sum_{i=1}^{\infty} \nu_1(A_i),$$

and where

$$\mu_1(A_i) := \frac{1}{z_0} \mu_0(A_i) \nu_0(B_i^c), \quad \nu_1(A_i) := \frac{1}{z_0} \nu_0(A_i) \mu_0(B_i^c),$$

with $B_i^c = X \setminus B_i$. The normalizing coefficient $z_0 \equiv z_{0,\varepsilon}$ is determined by the probability condition: $1 = \mu_1(X) = \nu_1(X)$. It is easy to check using (29) that μ_1 and ν_1 are Borel measures on X. Of course we have to exclude a blow-up situation when $\mu_0(B_i) = \nu_0(B_i) = 1$ for some i. Clearly we can repeat the above construction $N \ge 1$ times and obtain two sequences of probability measures: $\mu_{N,\varepsilon} \equiv \mu_N = \mu_{N-1} \circledast \nu_{N-1}$, $\nu_{N,\varepsilon} \equiv \nu_N = \nu_{N-1} \circledast \mu_{N-1}$. By Theorem 2 there exist two limiting probability measures

$$\mu_{\infty,\varepsilon} = \lim_{N \to 0} \mu_{N,\varepsilon}, \quad \nu_{\infty,\varepsilon} = \lim_{N \to 0} \nu_{N,\varepsilon},$$

which are invariant with respect to the action of the conflict composition and which are mutually singular provided that $\mu_0(B_i) \ne \nu_0(B_i)$ for some B_i.

Thus under condition (29) we are able to describe the conflict interaction between a couple of measures μ_0, ν_0 on a metric space X with any $\varepsilon-$accuracy.

The open problem is to prove the existence of the limiting measures $\mu_\infty = \lim_{\varepsilon \to 0} \mu_{\infty,\varepsilon}$ and $\nu_\infty = \lim_{\varepsilon \to 0} \nu_{\infty,\varepsilon}$.

We note that the above version of the conflict composition $\circledast$ is not unique. The existence of the limiting invariant measures μ_∞, ν_∞ may be ensured by various modifications of $\circledast$. A specific choice of $\circledast$ is determined by applications.

For example, assume that the conflicting sides do not want to leave positions with a starting non-zero parity, $p_j = q_j \ne 0$. We recall that according to Lemma 2 these coordinates converge to zero under the infinite time action of the conflict composition. However one can improve the construction of the composition $\ast$ in such a way that $p_j^{*,N} = q_j^{*,N}$ will not converge to zero, which means preservation of the non-zero parity with respect to j position. For instance, in order to reach this one can decompose each measure into two parts: $\mu_0 = \mu_{0,p} + \mu_{0,c}$, $\nu_0 = \nu_{0,p} + \nu_{0,c}$, where

$$\mu_{0,p} := \mu_0 \restriction \Omega_= = \nu_{0,p} := \nu_0 \restriction \Omega_=, \quad \mu_{0,c} := \mu_0 \restriction \Omega_=^c \neq \nu_{0,c} := \nu_0 \restriction \Omega_=^c,$$

with $\Omega_= := \{\omega_k \in \Omega : p_k^{(0)} = q_k^{(0)} \neq 0\}$ and $\Omega_=^c = \Omega \setminus \Omega_=$. Then we leave measures $\mu_{0,p}, \nu_{0,p}$ without any change and apply the previous version of the composition $\ast$, with obvious modifications, only to measures $\mu_{0,c}, \nu_{0,c}$, which in general are not probabilistic. We note that in [2], a version of the conflict composition suitable to arbitrary normalized measures was developed. In fact on this way one obtains a new conflict composition which preserves nontrivial parity positions.

Further, one can construct a more complex conflict composition in the following way:

$$\mu_0 \circledast \nu_0 := \alpha(\mu_{0,c} \ast \nu_{0,c}) + \beta(\mu_{0,p} \leftrightsquigarrow \nu_{0,p}), \quad \alpha, \beta \in [0, 1],$$

where $\ast$ is defined as above and the new term involves interaction, possibly in some power, which explicitly depends on the starting distributions on all positions. Computer simulations of such conflict composition exhibit some new effects including the blow-up, a chaotic behaviour, and infinite oscillations of values $p_i^{*,N} = q_i^{*,N}$ as $N \to \infty$. The latter may be interpreted as an "infinite war" without a winner.

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