Additive spectral problem
(brief survey and some recent results)

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1. Weyl’s problem
2. Additive spectral problem
3. Quivers. Algebras associated to quivers
4. Coxeter transformation
5. Extended Dynkin case
Let $A = A^*$, $B = B^*$ and $C = C^*$ be hermitian $n \times n$ matrices. For hermitian matrix $X$ we denote its eigenvalues by

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\[ \sigma_{i+j-1}(A + B) \leq \sigma_{i}(A) + \sigma_{j}(B), \quad i + j \leq n + 1, \]
\[ \sigma_{i+j-n}(A + B) \geq \sigma_{i}(A) + \sigma_{j}(B), \quad i + j \geq n + 1, \]
\[ \sum_{i \leq p} \sigma_{i}(A + B) \leq \sum_{j \leq p} \sigma_{j}(A) + \sum_{k \leq p} \sigma_{k}(B), \]
\[ \sum_{i \in I} \sigma_{i}(A + B) \leq \sum_{j \in I} \sigma_{j}(A) + \sum_{k \leq p} \sigma_{k}(B), \]

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**Conjecture 1.1 (Alfred Horn)**

*These inequalities form complete list of the restrictions on the spectrums*
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$\uparrow$

there exists a solution for

$$A_1 + \ldots + A_n = \gamma I,$$

$$A_i = A_i^*, \text{ with given } \sigma(A_i).$$
Let $H$ be separable Hilbert space.
Let $M_i = \{0 = \alpha_0^{(i)} < \alpha_1^{(i)} < \ldots < \alpha_{m_i}^{(i)}\} \subset \mathbb{R}^+$ and $\gamma \in \mathbb{R}^+$. 

Remark 1
An essential difference with classical Weyl’s problem is that we do not fix the dimension of Hilbert space and we do not fix spectral multiplicities.
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$$\mathcal{P}_{M_1, M_2, \ldots, M_n; \gamma} = \mathbb{C}\langle p_1^{(1)}, \ldots, p_{m_n}^{(n)} \mid p_i^{(k)} = p_i^{(k)2} = p_i^{(k)*}, \sum_{i=1}^{n} \sum_{k=1}^{m_i} \alpha_k^{(i)} p_k^{(i)} = \gamma e, p_j^{(i)} p_k^{(i)} = 0 \rangle.$$
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- weight $\chi: \Gamma \to \mathbb{R}_+, \chi = (\alpha_1^{(1)}, \ldots, \alpha_{m_1}^{(1)}; \ldots; \alpha_1^{(n)}, \ldots, \alpha_{m_n}^{(n)})$
For example if

\[ \mathcal{P}(\alpha_{1}^{(1)}, \ldots, \alpha_{m_{1}}^{(1)}; \alpha_{1}^{(2)}, \ldots, \alpha_{m_{2}}^{(2)}; \alpha_{1}^{(3)}, \ldots, \alpha_{m_{3}}^{(3)}), \gamma \]

\[ \subset \mathbb{C}[p_{1}^{(1)}, \ldots, p_{m_{1}}, \ldots, p_{1}^{(3)}, \ldots, p_{m_{3}}^{(3)} | p_{i}^{(k)} = p_{i}^{(k)2} = p_{i}^{(k)*}, \]

\[ \sum_{i=1}^{3} \sum_{k=1}^{m_{i}} \alpha_{k}^{(i)} p_{k}^{(i)} = \gamma e, p_{j}^{(i)} p_{k}^{(i)} = 0 \}, \]
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\[ P(\alpha_1^{(1)}, \ldots, \alpha_{m_1}^{(1)}; \alpha_1^{(2)}, \ldots, \alpha_{m_2}^{(2)}; \alpha_1^{(3)}, \ldots, \alpha_{m_3}^{(3)}), \gamma = \mathbb{C}\langle p_1^{(1)}, \ldots, p_{m_1}^{(1)}, \ldots, p_1^{(3)}, \ldots, p_{m_3}^{(3)} | p_i^{(k)} = p_i^{(k)2} = p_i^{(k)*} , \]

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[Diagram of a graph with labeled vertices indicating the structure based on the given conditions.]
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- for each weight $\chi$ to describe the set
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- for each weight $\chi$ to describe the set $\Sigma_{\Gamma,\chi} = (\text{all possible } \gamma \text{ for which there are representations of } \mathcal{P}_{\Gamma,\chi,\gamma});$

- for each appropriated pair $(\chi; \gamma)$ to describe all irreducible $\ast$-representation of $\mathcal{P}_{\Gamma,\chi,\gamma}.$
If $\Gamma$ is Dynkin graph
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$A_n$

\[ \begin{array}{c}
\bullet & \bullet & \cdots & \bullet \\
\end{array} \]

$D_n$

\[ \begin{array}{c}
\bullet \\
\end{array} \begin{array}{c}
\cdots \\
\end{array} \]

$E_6$

\[ \begin{array}{c}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array} \]

$E_7$

\[ \begin{array}{c}
\bullet \\
\end{array} \begin{array}{c}
\bullet \\
\end{array} \begin{array}{c}
\cdots \\
\end{array} \]

$E_8$

\[ \begin{array}{c}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
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If $\Gamma$ is Dynkin graph

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\[ \cdots \]

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\[ \cdots \]

$E_8$

\[ \cdots \]

then $\mathcal{P}_{\Gamma,\chi,\gamma}$ is finite dimensional, and complete answers for posed problem are known for all possible weights $\chi$;
if $\Gamma$ is extended Dynkin graph
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$\tilde{D}_4$

$\tilde{E}_6$

$\tilde{E}_7$

$\tilde{E}_8$
if $\Gamma$ is extended Dynkin graph

then the algebra $P_{\Gamma, \chi, \gamma}$ is infinite dimensional and of polynomial growth
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M. A. Vlasenko, A. S. Mellit, and Yu. S. Samoilenko, *On algebras generated with linearly dependent generators that have given spectra*, 2005

A quiver $Q$ consists of a finite set $Q_0$ of vertices, a finite sets $Q_1$ of arrows, and two maps $s, t : Q_1 \to Q_0$: $s(a) \rightarrow^a t(a)$
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A path in $Q$ is the sequence

$$\xi = \xi_r \ldots \xi_1$$

of arrows s.t. $t(\xi_p) = s(\xi_{p+1})$, $1 \leq p < r$.

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Path algebra $\mathbb{C}Q$ of a quiver $Q$ is algebra spanned by all paths in $Q$ with multiplication given by composition

$$xy = \begin{cases} 	ext{obvious composition (if } t(y) = s(x)) & \text{(otherwise)} \\ 0 & \end{cases}$$
A representation $X$ of a quiver $Q$ is given by a vector space $X_i$ for each vertex $i \in Q_0$ and linear operator $X_\rho : X_{s(\rho)} \rightarrow X_{t(\rho)}$ for each arrow $\rho \in Q_1$. 
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A morphism $\theta : X \rightarrow X'$ is given by linear maps $\theta_i : X_i \rightarrow X_i$ for each $i \in Q_0$, satisfying $X'_\rho \theta_{s(\rho)} = \theta_{t(\rho)} X_\rho$ for each $\rho \in Q_1$. 
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**Proposition 1** (see for example Crawley-Boevey)

*Representations of quiver $Q \iff$ left $\mathbb{C}Q$-modules.*
Theorem 1 (Gabriel ~ 1973)

*The classification of all indecomposable representations of Q is*

- **finite problem**, if Q is Dynkin quiver;
- **tame problem**, if Q is an extended Dynkin quiver;
- **wild problem**, in all other cases
The double quiver of $\overline{Q}$ is the quiver obtained by adjoining an arrow $a^* : j \to i$ for each arrow $a : i \to j$ in $Q$. 
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The deformed preprojective algebra of weight $\lambda \in \mathbb{C}^{Q_0}$ is

$$\Pi^\lambda(Q) = \mathbb{C} \overline{Q} / \left( \sum_{a \in Q} [a, a^*] - \sum_{i \in Q_0} \lambda_i e_i \right).$$
Let $Q\ (\Gamma$ its underlying graph) be star-shaped quiver with orientation to root vertex $c$, then
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**Theorem 2**

*For given weight $\chi$ it is possible to determine $\lambda$ s.t. algebras $\mathcal{P}_{\Gamma,\chi,\gamma}$ and $e_c \Pi^\lambda(Q)e_c$ are isomorphic.*
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**Theorem 2**

*For given weight $\chi$ it is possible to determine $\lambda$ s.t. algebras $P_{\Gamma, \chi, \gamma}$ and $e_c \Pi^\lambda(Q)e_c$ are isomorphic.*

There also exist interconnection between $P_{\Gamma, \chi, \gamma}$ and orthoscalar representation of quivers.

A powerful tool to investigate representations of quiver are Coxeter functors which allow to build series of representations starting from simplest representation

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Similar functors were built for algebras $\mathcal{P}_{\Gamma,\chi,\gamma}$ by Kruglyak and Roiter. Namely there are exist two functors linear $S$ (which generate representation in the same space) and hyperbolical $T$ (which, strictly speaking, build representation in new space).
More precisely if functors $S$ and $T$ are applicable they establish the equivalence between categories:
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$S: \text{Rep}(\mathcal{P}_{\Gamma,\chi,\gamma}) \rightarrow \text{Rep}(\mathcal{P}_{\Gamma,\chi',\gamma'});$  
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On the pairs $(\chi, \gamma)$ they act as follows

$S : (\chi; \gamma) \longmapsto (\chi'; \gamma'),$

$\chi' = (\alpha_{m_1}^{(1)} - \alpha_{m_1-1}^{(1)}, \ldots, \alpha_{m_1}^{(1)}; \ldots; \alpha_{m_n}^{(n)} - \alpha_{m_{n-1}}^{(n)}, \ldots, \alpha_{m_n}^{(n)}),$  
$\gamma' = \alpha_{m_1}^{(1)} + \cdots + \alpha_{m_n}^{(n)} - \gamma;$

$T : (\chi; \gamma) \longmapsto (\chi''; \gamma'),$

$\chi'' = (\gamma - \alpha_{m_1}^{(1)}, \ldots, \gamma - \alpha_1^{(1)}; \ldots; \gamma - \alpha_{m_n}^{(n)}, \ldots, \gamma - \alpha_1^{(n)}).$
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$$\gamma' = \alpha^{(1)}_{m_1} + \cdots + \alpha^{(n)}_{m_n} - \gamma;$$

$$T : (\chi; \gamma) \mapsto (\chi''; \gamma),$$
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Now we are going to study the dynamic of Coxeter functors for the case where $\Gamma$ is an extended Dynkin graph.
Let us denote

\[ \tilde{\omega}_{D_4}(\chi) = \frac{1}{2}(\alpha_{1}^{(1)} + \alpha_{1}^{(2)} + \alpha_{1}^{(3)} + \alpha_{1}^{(4)}) \]

\[ \tilde{\omega}_{E_6}(\chi) = \frac{1}{3}(\alpha_{1}^{(1)} + \alpha_{1}^{(2)} + \alpha_{1}^{(3)} + \alpha_{1}^{(2)} + \alpha_{1}^{(3)} + \alpha_{2}^{(1)} + \alpha_{2}^{(2)}) \]

\[ \tilde{\omega}_{E_7}(\chi) = \frac{1}{4}(\alpha_{1}^{(1)} + \alpha_{1}^{(2)} + \alpha_{1}^{(3)} + \alpha_{1}^{(2)} + \alpha_{1}^{(3)} + \alpha_{1}^{(4)} + 2\alpha_{2}^{(1)}) \]

\[ \tilde{\omega}_{E_8}(\chi) = \frac{1}{6}(\alpha_{1}^{(1)} + \alpha_{1}^{(2)} + \alpha_{1}^{(3)} + \alpha_{1}^{(4)} + \alpha_{1}^{(5)} + 2\alpha_{1}^{(2)} + 2\alpha_{1}^{(3)} + 3\alpha_{1}^{(2)}) \]

these hyperplanes are invariant in the sense

\[ S: (\chi; \omega(\chi)) \mapsto -\rightarrow (\chi'; \omega(\chi')) \]

\[ T: (\chi; \omega(\chi)) \mapsto -\rightarrow (\chi''; \omega(\chi'')) \]
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\[
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\]

\[
\omega_{\tilde{E}_6}(\chi) = \frac{1}{3}(\alpha^{(1)}_1 + \alpha^{(2)}_1 + \alpha^{(2)}_1 + \alpha^{(3)}_1 + \alpha^{(3)}_2),
\]

\[
\omega_{\tilde{E}_7}(\chi) = \frac{1}{4}(\alpha^{(1)}_1 + \alpha^{(1)}_2 + \alpha^{(1)}_3 + \alpha^{(2)}_1 + \alpha^{(2)}_2 + \alpha^{(2)}_3 + 2\alpha^{(3)}_1),
\]

\[
\omega_{\tilde{E}_8}(\chi) = \frac{1}{6}(\alpha^{(1)}_1 + \alpha^{(1)}_2 + \alpha^{(1)}_3 + \alpha^{(1)}_4 + \alpha^{(1)}_5 + 2\alpha^{(2)}_1 + 2\alpha^{(2)}_2 + 3\alpha^{(3)}_1).
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\]

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V. L. Ostrovskyi, *Special characters on star graphs and representations of \(*\)-algebras*
Let us denote

\[ \omega_{\tilde{D}_4}(\chi) = \frac{1}{2}(\alpha_1^{(1)} + \alpha_1^{(2)} + \alpha_1^{(3)} + \alpha_1^{(4)}), \]

\[ \omega_{\tilde{E}_6}(\chi) = \frac{1}{3}(\alpha_1^{(1)} + \alpha_2^{(1)} + \alpha_1^{(2)} + \alpha_2^{(2)} + \alpha_1^{(3)} + \alpha_2^{(3)}), \]

\[ \omega_{\tilde{E}_7}(\chi) = \frac{1}{4}(\alpha_1^{(1)} + \alpha_2^{(1)} + \alpha_3^{(1)} + \alpha_1^{(2)} + \alpha_2^{(2)} + \alpha_3^{(2)} + 2\alpha_1^{(3)}), \]

\[ \omega_{\tilde{E}_8}(\chi) = \frac{1}{6}(\alpha_1^{(1)} + \alpha_2^{(1)} + \alpha_3^{(1)} + \alpha_4^{(1)} + \alpha_5^{(1)} + 2\alpha_1^{(2)} + 2\alpha_2^{(2)} + 3\alpha_1^{(3)}). \]

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Proposition 2

The action of \((ST)^k\) functor on the pair \((\chi; \gamma)\) could be written in the following way:

\[
(ST)^k(\chi; \gamma) = (f_{1,k}(\chi) - (\omega \Gamma(\chi) - \gamma)f_{2,k}(\chi \Gamma); \psi_{1,k} - (\omega \Gamma(\chi) - \gamma)\psi_{2,k}),
\]

where the characters \(f_{1,k}(\chi)\) and \(f_{2,k}(\chi \Gamma)\), and the numbers \(\psi_{1,k}\) and \(\psi_{2,k}\) satisfy the following properties:

(i) if \(k_1 \equiv k_2 \pmod{p \Gamma(p \Gamma - 1)}\) then \(f_{1,k_1}(\chi) = f_{1,k_2}(\chi)\) and \(\psi_{1,k_1} = \psi_{1,k_2}\);

(ii) the components of \(f_{2,k}(\chi \Gamma)\) and the numbers \(\psi_{2,k}\) are defined in the following way:

\[
\begin{align*}
&f_{2,k}(\chi \Gamma)_{i}^{(j)} = \left[\frac{\chi_{i}^{(j)}}{p \Gamma - 1} k\right], \\
&\psi_{2,k} = \left[\frac{p \Gamma}{p \Gamma - 1} k\right];
\end{align*}
\]

(iii) \(f_{1,p \Gamma(p \Gamma - 1)k} = \chi, f_{2,p \Gamma(p \Gamma - 1)k} = k p \Gamma \chi \Gamma, \psi_{1,k} = \gamma,\) and \(\psi_{2,k} = kp \Gamma^2\).
Theorem 3 (K.Y. 2006)

The set \( \Sigma_{D_4, \chi} \) is a union of the following sets

\[
\begin{align*}
\Sigma_1 &= \left\{ \frac{\alpha}{2} - \frac{\alpha}{2(4n-1)} \mid n < \frac{\alpha_4}{4\alpha_4 - \alpha}, \ n < \frac{\alpha - \alpha_1}{\alpha - 4\alpha_1}, \ n \in \mathbb{N} \right\}, \\
\Sigma_2 &= \left\{ \frac{\alpha}{2} - \frac{\alpha_i}{2n} \mid n < \frac{\alpha_i}{2\alpha_i + 2\alpha_4 - \alpha}, \ n < \frac{\alpha_i}{\alpha_i - \alpha_1}, \ n < \frac{\alpha_i}{4\alpha_i - \alpha}, \ n \in \mathbb{N} \right\}, \\
\Sigma_3 &= \left\{ \frac{\alpha}{2} - \frac{\alpha - 2\alpha_1}{2(2n + 1)} \mid n < \frac{\alpha - \alpha_1}{\alpha - 4\alpha_1}, \ n < \frac{\alpha_2 + \alpha_3}{2(\alpha_4 - \alpha_1)}, n(4\alpha_i - \alpha) < \alpha_i \right\}, \\
\Sigma_4 &= \left\{ \frac{\alpha}{2} - \frac{\alpha}{2(4n + 1)} \mid n < \frac{\alpha - \alpha_4}{4\alpha_4 - \alpha}, \ n < \frac{\alpha_1}{\alpha - 4\alpha_1}, \ n \in \mathbb{N} \right\}, \\
\Sigma_5 &= \left\{ \frac{\alpha}{2} - \frac{\alpha - 2\alpha_i}{2(2n + 1)} \mid n < \frac{\alpha_1}{\alpha - 2\alpha_i - 2\alpha_1}, \ n < \frac{\alpha_i}{\alpha - 4\alpha_i}, n < \frac{\alpha - \alpha_4 - \alpha_i}{2(\alpha_4 - \alpha_i)} \right\}, \\
\Sigma_\infty &= \left\{ \frac{\alpha}{2} - \frac{\alpha - 2\alpha_4}{2(2n - 1)} \mid n \in \mathbb{N} \right\}, \\
\Sigma_0 &= \left\{ \frac{\alpha}{2} - \frac{\alpha_1}{2n} \mid n < \frac{\alpha_1}{\alpha_1 + \alpha_4 - \alpha_2 - \alpha_3}, n \in \mathbb{N} \right\}.
\end{align*}
\]
There are exact formulas for representation of algebras $\mathcal{P}_{\tilde{D}_4;\chi,\gamma}$. In other words the description of all irreducible quadruples of projections s.t.

$$\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4 = \gamma I.$$
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K.Y. *On existence of $*$-representations of certain algebras related to extended Dynkin graphs*
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K.Y. On existence of $\ast$-representations of certain algebras related to extended Dynkin graphs

**Theorem 4**

The set $\Sigma_{\Gamma,\chi}$ is infinite if and only if all components of weight satisfies two conditions: $\chi_i < \omega_\Gamma(\chi)$ and $\chi'_i < \omega_\Gamma(\chi')$. 
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**Theorem 4**

The set $\Sigma_{\Gamma,\chi}$ is infinite if and only if all components of weight satisfies two conditions: $\chi_i < \omega_\Gamma(\chi)$ and $\chi'_i < \omega_\Gamma(\chi')$.

**Corollary 1**

Let $\chi$ be the weight on $\Gamma$ such that the conditions of previous theorem are satisfied. Then there is a representation of algebra $\mathcal{P}_{\Gamma,\chi,\omega_\Gamma(\chi)}$ on hyperplane $\gamma = \omega_\Gamma(\chi)$.
Theorem 5

If the set $\Sigma_{\Gamma,\chi}$ is infinite then it contains the only limit point.
Theorem 5

If the set $\Sigma_{\Gamma, \chi}$ is infinite then it contains the only limit point.

Corollary 2

Let $\Gamma$ be extended Dynkin graph. The algebras $\mathcal{P}_{\Gamma, \chi, \gamma}$ are of tame representation type when $\chi_i < \omega_\Gamma(\chi)$ and $\chi'_i < \omega_\Gamma(\chi')$ otherwise they are of finite representation type.
A few open problem:
Let $\Gamma$ be neither Dynkin graph nor extended Dynkin graph. Is there such weight $\chi$ on $\Gamma$ that
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A few open problem:
Let $\Gamma$ be neither Dynkin graph nor extended Dynkin graph. Is there such weight $\chi$ on $\Gamma$ that
- $\Sigma_{\Gamma,\chi}$ contain continuous part?
- algebra $\mathcal{P}_{\Gamma,\chi,\gamma}$ is of $\ast$-wild representation type?
Thank you very much for your attention.