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Lecture 1.

Unitary representations of groups.

Why do we need to study
∞-dimensional representations?

Consider the category $\mathcal{H}$ of Hilbert spaces
Morphism are supposed to be unitary morphisms
On $\mathcal{H}$ we have sum operation and tensor product.
$H_1 + H_2 \quad H_1 \otimes H_2$

Let G be locally compact group we define a functor.
$\mathrm{Rep}_G : \mathcal{H} \to \mathrm{Top}$ - cat of top. spaces.
$\mathrm{Rep}_G(H) = \mathrm{Hom}(G, U(H))$ - with compact open topology.
$U(H)$ caries strong topology

Ex. $\mathrm{Rep}_{\mathbb{Z}}(H) = U(H)$.
$\varphi \mapsto \varphi(1)$ - bijection.

There are natural operators.

$$\mathrm{Rep}_G(H) \times \mathrm{Rep}_G(H') \xrightarrow{\oplus} \mathrm{Rep}_G(H \oplus H')$$

$$\mathrm{Rep}_G(H) \times \mathrm{Rep}_G(H') \xrightarrow{\otimes} \mathrm{Rep}_G(H \otimes H')$$

$$U(H) \times \mathrm{Rep}_G(H) \to \mathrm{Rep}_G(H)$$

$$(u, \varphi) \mapsto \{ g \mapsto u\varphi(g)u^* \}$$

Theorem (Krein-?).

A natural transformation between functors $F: \mathcal{H} \to \mathrm{Top}$ and $F': \mathcal{H} \to \mathrm{Top}$ is an assignment $\mathcal{H} \mapsto f_{\mathcal{H}}: F(\mathcal{H}) \to F'(\mathcal{H})$

Let G be a locally compact group. Then
G = "set of natural transformation between $\mathrm{Rep}_G \to \mathrm{Rep}_{\not{\mathbb{Z}}}$ which are compatible with $\oplus$, $\otimes$ and conjugation.

Question Why is the ~~right side~~ RHS is the group?
Question2 How RHS related to G.

Claim: there exists a natural homomorphism
$\mathcal{E}_G: G \to$ "set of natural transf."
$\mathcal{E}_G(g)(\varphi) = \varphi(g)$.

Theorem this is an iso of top. of groups provided ~~one~~ one takes ~~of~~ compact open topology.

Theorem (Tannaka-Krein) If G is compact the same works if one consider only finite-dim representations.

Let G be a group which is ~~discrete~~ discrete and ~~countable~~ finitely generated

Theorem (Chu) Let G be virtually abelian (:$\Leftrightarrow \exists$ abelian subgroup $H \leq G$ with $[G:H] < \infty$).

G = "set of natural transformation $\mathrm{Rep}_G^{fin} \to \mathrm{Rep}_{\mathbb{Z}}^{fin}$"

Definition G is called maximaly almost periodic if for each finite set $F \subseteq G$, there exists a finite dimension unitary representation $\varphi: G \to \mathcal{U}(n)$ such that $\varphi|_F$ - is injective.

Remark. The free group is not virtually abelian but maximaly almost periodic.

The choice of the generic pair u, v of unitary in $\mathcal{U}(2)$ gives an injective $\mathbb{F}_2 \to \mathcal{U}(2)$.

Question Can one recover the free group of two generators from its finite dim. representation.

1) $\mathrm{Rep}_{\mathbb{F}_2}(\mathbb{F}_2, n) = \mathcal{U}(n) \times \mathcal{U}(n)$ $\quad \mathbb{F}_2 = \langle a, b \rangle$.

$\varphi \mapsto (\varphi(a), \varphi(b))$

2) we define $\mathbb{F}_2^{\#} := \left\{ \begin{array}{l} \Psi_n : \mathrm{Rep}_{\mathbb{F}_2}(n) \to \mathcal{U}(n) \\ \text{natural and compatible with} \\ \oplus \text{ and } \otimes \end{array} \right\}$

$\mathbb{F}_2^{\#}$ - carries the compact open topology inherited from the space of maps.

$$\mathrm{map}\Big(\coprod_{n\geq 1} \mathrm{Rep}_{\mathbb{F}_2}(n), \coprod_{n\geq 1} U(n)\Big).$$

Note $\exists$ $\varepsilon_{\mathbb{F}_2} : \mathbb{F}_2 \to \mathbb{F}_2^{\#}$

$$\varepsilon_{\mathbb{F}_2}(w)(\varphi) = \varphi(w) \quad \text{for } w \in \mathbb{F}_2.$$

$\varepsilon_{\mathbb{F}_2}$ - is not a isomorphism of top. groups.

$\exists$ sequence $(w_n)_{n\in\mathbb{N}}$, $w_n \in \mathbb{F}_2 \setminus \{e\}$.

$$\varepsilon_{\mathbb{F}_2}(w_n) \to 1 \quad \text{in } \mathbb{F}_2^{\#}$$

$\Leftrightarrow$ uniformly on n-dimensional representations

$$\varphi(w_n) = w_n(a_\varphi, b_\varphi) \to 1_n \in U(n)$$

$$\mathrm{Rep}_{\mathbb{F}_2}(n) \ni \varphi \leftrightarrow (a_\varphi, b_\varphi) \in U(n)\times U(n).$$

The analogue is not true for $\mathbb{Z}$!

$\nexists$ $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{Z}$ such that

$$\forall t \in S^1 = \mathrm{Rep}_{\mathbb{Z}}(1), \quad t^{n_k} \to 1.$$

Proof. $0 = \int t^{n_k} dz \longrightarrow \int 1\, dz = 2\pi$

bounded conv.

Contradiction

Remark For many elements $w \in \mathbb{F}_2$

$$w: U(n)U(n) \to U(n)$$
$$(a,b) \mapsto w(a,b) \quad \text{is surjective.}$$

It is true whenever sum of exponents of $\overset{\text{either}}{a}$ or b is non zero.

Examples $w = a b a^{-1} b^2 a^{15}$

$$\mathcal{U}(n) \times \mathcal{U}(n) \ni (u,v) \longmapsto u\, v\, u^{-1} v^2 u^{15} \in \mathcal{U}(n)$$
is surjective.

Remark. $\mathbb{F}_2^{\#}$ - is complete metrizable group.

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Now, n is fixed and we consider $\mathcal{U}(n)$.
We define $\ell(u) = \|1_n - u\| \leftarrow$ operator norm
$\ell(uv) \leq \ell(u) + \ell(v)$.

1). $\ell(vuv^{-1}) = \ell(u) = \ell(u^{-1})$

2) $\ell([u,v]) \leq 2\ell(u)\ell(v)$.

$$\ell([u,v]) = \|1_n - uvu^*v^*\| = \|vu - uv\| = \|(1-u)(1-v) - (1-v)(1-u)\| \leq$$
$$\leq 2\|1-v\|\|1-u\| = \ell(u)\ell(v).?$$

For each $\varepsilon > 0$, we have to find $w \in \mathbb{F}_2 \setminus \{e\}$
~~given~~ such that $\ell(w(u,v)) \leq \varepsilon$, $\forall u,v \in \mathcal{U}(n)$.

Lemma For $\forall \varepsilon > 0$, there exist such $q(\varepsilon) \in \mathbb{N}$ that for all $u \in \mathcal{U}(n)$ there exist $m(u) \in \{1, \dots q(\varepsilon)\}$ such that $\ell(u^{m(u)}) \leq \varepsilon$.

Proof. Assume the $\ell(u^m) > \varepsilon$ for all $m \in \{1, \dots q\}$.
then $\|u^m - u^{m'}\| > \varepsilon$ for all $m, m' \in \{1, \dots q\}$.

since $\|u^m - u^{m'}\| = \|1 - u^{m-m'}\|$ $m \neq m'$!

$V = \{v \in U(n) \mid \ell(v) \le \frac{\varepsilon}{2}\}$

$\Rightarrow$ the sets Vu^m are disjoint for $m \in \{1, \dots, q\}$.

$$q \cdot \mu(V) \le \mu(U(n)).$$

$$q \le \frac{\mu(U(n))}{\mu(V)}$$

we have a sequence

$$w_{-1} \quad w_0 \quad w_1 \quad \dots w_2 \dots w_3 \dots$$

$$w_k' = [w_{k-1}, [w_k, w_{k+1}]].$$

$$\to [w_{-1}, [w_0, w_1]].$$

$$\ell(w_k') \le 4 \cdot \ell(w_{k-1})\, \ell(w_k)\, \ell(w_{k+1}).$$

$$\ell(w_k'') \le C\, \ell(w_{k-2})\, \ell(w_{k-1})^2\, \ell(w_k)^3\, \ell(w_{k+1})^2\, \ell(w_{k+2}).$$

$$\ell(w_k'') \le c\, \ell(w_{k+1})^2.$$
$$\ell(w_k'') \le c\, \ell(w_k)^3.$$

Property $\ell(w_k) < \frac{1}{c}$ spreads out!

Starting out with $w_k = a^k b a^{-k}$.

<u>Theorem</u> the natural homomorphism
$\varepsilon_G : G \to G^\#$ is an isomorphism of topological groups $\Leftrightarrow$ G is virtually abelian.

Lecture 2

"Stability of unitary representations"

G - countable, discrete group.
H - Hilbert space.

Def. $\pi: G \to \mathcal{U}(H)$ is said to be "unitary ε-representation" if $\|\pi(gh) - \pi(g)\pi(h)\| < \varepsilon, \quad \forall g, h \in G.$

Def. We say that π, σ are ε-closed if
$$\|\pi(g) - \sigma(g)\| < \varepsilon \qquad \forall g \in G.$$

Observation: If π is ε-closed to unitary representation then it is 3ε-repr unitary rep.

Theorem (Kazhdan) Let G be amenable group, then for all $\varepsilon > 0$, $\exists\, \delta > 0$, such that unitary δ representation are ε-closed to unitary representations.

($\delta = \frac{\varepsilon}{100}$ - works for all G).

Def. G admits a paradoxical decomposition, if there are subsets $A_1, \dots A_n$ and $B_1 \dots B_n \subseteq G$ such that
$$\coprod_{i=1}^{n} A_i = \coprod_{i=1}^{n} B_i = G.$$

$\exists\, g_1 \dots g_n, h_1 \dots h_n \in G$ ~~that~~ such that
$$g_1 A_1, \dots g_n A_n, h_1 B_1 \dots h_n B_n \text{ - are disjoint.}$$

Definition G-admits invariant measure if $\exists\, m: P(G) \to [0,1]$ such that:
1) $m(G) = 1$.
2) $A, B \subseteq G$, $A \cap B = \emptyset \Rightarrow m(A \cup B) = m(A) + m(B)$

3) $\forall A \subseteq G,\ g \in G$.

$m(gA) = m(A)$ - invariancy.

Observation: if G admits invariant measure then G has no paradoxical decomposition.

$\square$:

$$2 = m(G) + m(G) = \sum m(A_i) + \sum m(B_i) =$$
$$= \sum m(g_i A_i) + \sum m(h_i B_i) \leq 1.$$

Definition: G - satisfies Følner's condition if for all finite subsets $S \subseteq G$ there exists finite subset $K \subseteq G$ and all $\varepsilon > 0$ such that $|S \cdot K| \leq (1+\varepsilon)|K|$.

Example: $\mathbb{Z}$ - satisfy Følner's conditions.

WLOG $S = \{-k, \dots, k\}$.

$K(n) = \{-n, \dots n\}$.

$$|S + K(n)| = 2(k+n) + 1, \quad \frac{2(k(n)+1)}{2n+1} \leq 1 + \varepsilon.$$

Ok for $\mathbb{Z}$.

Example: If is finitely generated by $F \subseteq G$. if G is subexponential growth.

$\Leftrightarrow$ ~~$\forall C > 0, \alpha > 0$ $|F^n| < C \cdot e^{\alpha n}$~~

$\nexists C, \alpha > 0$, such that $|F^n| \geq C e^{\alpha \cdot n}$

Exercise: Prove that G satisfies Følner conditions

Look at the ∂F^n!

Lemma If G satisfies Følner's condition then G admits invariant measure.

□: Assume that G is finitely gen. by set S

$\forall n \in \mathbb{N}$, $\exists K(n) \subseteq G$.

$$|S \cdot K(n)| < (1+\tfrac{1}{n})|K_n|.$$

Consider for $A \subseteq G$

$$m_n(A) = \frac{|A \cap K(n)|}{|K(n)|} \in [0,1].$$

1) $\mu(G) = 1$.

2) $\mu(A \cap B) = \emptyset$ $\quad m_n(A \cup B) = m_n(A) + m_n(B)$.

3) – ?

If $g \in S$ $\quad |m_n(A) - m_n(gA)| \leq \frac{2}{n}$.

$$\underbrace{|\{e,g\} \cdot K(n)|}_{K(n) \cup gK(n)} \leq (1+\tfrac{1}{n})|K(n)|$$

$\Rightarrow$ $K(n) \cap gK(n)$ is large in $K(n)$.

$m(A) = \underset{n \to \infty}{LIM}\, m_n(A)$. – Banach limits.
$\in [0,1]$. ■

Theorem

1) G has no paradoxical decompositions
2) G has an invariant measure
3) G satisfies Følner conditions

Proof. $3) \Rightarrow 2) \Rightarrow 1)$

We will show the $\neg 3) \Rightarrow \neg 1)$.

Definition G is amenable if it satisfies 1), 2) and/or 3).

$\forall \varepsilon > 0$, $\forall S \subseteq G$, $\exists K \subseteq G$:

$$|SK| \leq (1+\varepsilon)|K| \qquad 3)$$

$\neg 3)$ is

$\exists \varepsilon > 0$, $\exists S \subseteq S$, $\forall K \subseteq G$.

$$|SK| > (1+\varepsilon)|K|.$$

$$\frac{|S^n K|}{|K|} = \frac{|S^n K|}{|S^{n-1}K|} \cdots \frac{|SK|}{|K|} > (1+\varepsilon)^n$$

$\Rightarrow \exists S \subseteq G$ finite. $|SK| > 2 \cdot |K|$.

Consider the bipartite graph with

$V = (G \amalg G) \amalg G$.

$E = \{ (g, h) \mid g \in G \amalg G,\ h \in G.$

$h = s \cdot g$ for some $s \in S \}$.

We get a paradoxical decomposition of G if we find injective map $\varphi : G \amalg G \to G$.

such that $(g, \varphi(g)) \in E$ for all $g \in G \amalg G$.

$A_s = \{ g \in G \amalg \emptyset \mid \varphi(g) = s \cdot g \} \subseteq G$.

$B_s = \{ g \in \emptyset \amalg G \mid \varphi(g) = s \cdot g \} \subseteq G$.

G is $\coprod_{s\in S} A_s$ and $\coprod_{s\in S} B_s$

φ - is injective
and sets sA_s and sB_s are disjoint. since φ
is injective.

Such φ exists if and only if for all sets
$L \subseteq G \amalg G$. the set of group elements ~~$\{g\in L \dots\}$~~
$\cdot |\{g \in G \mid \exists e \in I$ $\}| \geq |L|$.

WLOG $L = K \amalg K$.

$|\{g \in G \mid \dots\}| = |S \cdot K| \geq 2 \cdot |K| = |K \amalg K|$.

If G is amenable the invariant measure gives
to a theory of integration. $\exists \int_G ? \, d\mu : \ell^{\infty}(G, B(H)) \to B(H)$.

$$\left\| \int_G T(g) \, d\mu(g) \right\| \leq \sup_{g\in G} \|T(g)\|.$$

Shleyn's algorithm.

Let. $\pi : G \to U(H)$ be unitary ε-representation.

1. $\pi(g^{-1}) = \pi(g)^{-1}$.
2. Replace π by
$$\tilde{\pi}(g) := \int_G \pi(gh^{-1}) \pi(h) \, d\mu(h).$$

$\tilde{\pi}$ - is $5\varepsilon^2$-representation which is ε-close to π.

$$\overset{\varepsilon}{\pi} \xrightarrow{\varepsilon} \overset{5\varepsilon^2}{\tilde{\pi}} \xrightarrow{5\varepsilon^2} \overset{5\cdot 25\varepsilon^4}{\tilde{\tilde{\pi}}} \rightsquigarrow \dots \text{ tends to unitary representation}$$

One defines a sequence a_n

$a_0 = \varepsilon, \quad a_{n+1} = \cancel{C} a_n^2$

1) if $c < \frac{1}{5}$ then $a_n \to 0$

2) if a_n is summable

$\Rightarrow \quad \pi_0 = \pi, \quad \pi_{n+1} = \tilde{\pi}_n \Rightarrow$ $\forall g \ \pi_n(g) \to \sigma(g)$

1) σ - is unitary

2) $\|\sigma - \pi\| < f(\varepsilon)$

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① how to perform?

If $g \neq g^{-1}$ then we replace $\pi(g^{-1})$ by $\pi(g)^{*}$

$$\|\pi(g^{-1}) - \pi(g)^{-1}\| = \|\pi(g^{-1})\pi(g) - 1\| \leq 2\varepsilon.$$

$\Rightarrow$ After ① one has 3ε- representation.

$g = g^{-1}, \Rightarrow \quad \|\pi(g)^2 - 1\| < \varepsilon.$

Replace $\pi(g)$ by $f(\pi(g))$ where f is some function $f: S^1 \to S^1$ such that.

$f(z) = \pm 1$ if $|z - \pm 1| < \varepsilon$.

$f(\pi(g))$ is a symmetry.

② $\tilde{\pi}(g) := \int \pi(gh^{-1})\pi(h) \, dm(h)$

$$\|\tilde{\pi}(g) - \pi(g)\| = \Big\| \int_G \underbrace{\big(\pi(gh^{-1})\pi(h) - \pi(g)\big)}_{\leq \varepsilon \text{ - in norm}} \, dm(h) \Big\| < \varepsilon.$$

$$\left((\pi(gk^{-1})\pi(k) - \pi(g e^{-1})\pi(e)\right) \cdot \left(\pi(k^{-1})\pi(kh) - \pi(e^{-1})\pi(eh)\right) =$$

$$= \pi(gk^{-1})\pi(kh) + \pi(ge^{-1})\pi(eh) - \pi(gk^{-1})\pi(k)\pi(e^{-1})\pi(eh) -$$

$$- \pi(ge^{-1})\pi(e)\pi(k^{-1})\pi(kh). = *$$

$$\int_G \pi(gk^{-1})\pi(kh)\,dm(k) = \quad (\text{Replace } k \text{ with } k \cdot h^{-1}).$$

$$= \int_G \pi(ghk^{-1})\pi(k)\,dm(k) = \bar{\pi}(gh).$$

$$\int_G\int_G * \; dm(k)\,dm(e) = 2\bar{\pi}(gh) - 2\bar{\pi}(g)\bar{\pi}(h). \Longrightarrow$$

$$\|\bar{\pi}(gh) - \bar{\pi}(g)\bar{\pi}(h)\| < \frac{(2\varepsilon)^2}{2} < 2\varepsilon^2.$$

Lecture 3

Theorem Let G be amenable then $\forall \varepsilon > 0$, $\exists \delta > 0$, any δ-representation is ε-close to unitary representation.

Let $G = \mathbb{F}_2$

Example 1 Fix $n \in \mathbb{N}$ and $u_k, w_k \in \mathcal{U}(n)$ for $k \in \mathbb{Z}$, such that

1) $\ell(u_k) \leq \frac{\varepsilon}{3}$, $\ell(w_k) \leq \frac{\varepsilon}{3}$

2) $u_k^{-1} = u_{-k}$, $w_k^{-1} = w_{-k}$, $u_0 = v_0 = 1_n$.

For any $v \in \mathbb{F}_2 = \langle a, b \rangle$ there exists a unique reduced word such that

$v = a^{n_1} b^{m_1} \cdots b^{m_k}$ such that all but n_1, m_k are non-zero.

We define $\varphi(v) = u_{n_1} w_{m_1} \cdots u_{n_k} v_{m_k}$.

For v, v' in $\mathbb{F}_2$ we have to determine how $\varphi(v \cdot v')$ is related to $\varphi(v)\varphi(v')$.

$v = s \cdot b^{m_k}$, $v' = b^{m_1'} s'$,

$v \cdot v' = s b^{m_k + m_1'} s' = (s \cdot b^{m_k})(b^{m_1'} \cdot s')$ – is ε-close to $\varphi(v v') - \varphi(v)\varphi(v')$

$\Rightarrow \varphi$ – is ε-representation.

Assume that φ is ε'-close to a unitary representation μ.

$$\ell(\mu(a^4)) = \ell(\mu(a)^4) \le \varepsilon + \varepsilon' \Rightarrow$$

$$\varepsilon + \varepsilon' < \frac{1}{\sqrt{2}} \Rightarrow \mu(a) = I_n.$$

$$\Downarrow$$

$$\mu(b) = 1_n \Rightarrow \mu \text{ is trivial.}$$

Choosing u_k, w_k such that $\varphi(\mathbb{F}_2)$ is dense in $U(n)$ provides an example of non-trivial unitary repres.

<u>Remark</u> Induction of ε-representations shows that any G that contains $\mathbb{F}_2$ as a subgroup admits ∞-dim ε-represents which are not trivial.

groups

$\mathbb{F}_2 \subseteq G$? ? ? amenable

$B(n,p) = \mathbb{F}_n / \langle \mathbb{F}_n^p \rangle$

if $p \ge 665$ – not amenable with torsion.

$B(n,p)$ is non amenable + infinite.

$$\begin{array}{ccc} & \Pi & \\ \varepsilon \swarrow & & \searrow \varepsilon \\ \text{unitary} - \mu & & \mu' - \text{unitary} \end{array}$$

$\mu \sim \mu'$ → are they conjugate?

$\Rightarrow$ if μ and μ' are ε-close to π
then μ is 2ε-close to μ' -?

<u>Question</u> for which groups is it true
that $\forall \varepsilon > 0$, $\exists \delta > 0$, such that
δ-close unitary representations are conjugated by
a unitary which is ε-close to identity.

<u>Theorem</u> this is true if G is amenable.

Pf: Let $\mu, \sigma : G \to \mathcal{U}(\mathcal{H})$ be unitary represent.
and assume that $\|\mu(g) - \sigma(g)\| < \varepsilon$ $\forall g \in G$.

Define an action ρ of G on $B(\mathcal{H})$ by setting

$$\rho(g)(T) = \mu(g)\, T\, \sigma(g)^{-1}$$

Note that $\|1_{\mathcal{H}} - \rho(g)(1_{\mathcal{H}})\| \leq \delta$

We set $Q := \int_G \rho(g)\,(1_{\mathcal{H}})\, dm(g)$

Q is ρ-fixed. $\rho(h)(Q) = \rho(h) \int_G \rho(g)(1_{\mathcal{H}})\, dm(g) =$

$$= \int_G \rho(hg)(1_{\mathcal{H}})\, dm(g) = \int \rho(g)\,(1_{\mathcal{H}})\, dm(g).$$

$\Rightarrow \|1_{\mathcal{H}} - Q\| \leq \delta$

Q intertwines μ and σ. $\blacksquare$

Again the result fails if $\mathbb{F}_2 \leq G$ by a result
of Pytlik-Szwarc.

They show: $\exists\, \pi_t : \mathbb{F}_2 \to B(\ell^2 \mathbb{F}_2)$ such that
$\| \sup \|\pi_t(g) - \pi_s(g)\| \leq$

Definition G is said to be deformation rigid if the answer to the question is yes.

Dixmier conjecture

Definition Let G be a group

$$\pi : G \to GL(H) = \{T \in B(H) \mid T^{-1} \in B(H)\}.$$

is said to be uniformly bounded

if $\sup_{g \in G} \|\pi(g)\| < \infty$.

π is said to be unitarizable if there exist $T \in GL(H)$, such that

$g \mapsto T\pi(g)T^{-1}$ - is a unitary.

Theorem (Dixmier) If G is amenable then every uniformly bounded is unitarizable.

$\square$: Define $\langle \xi, \eta \rangle_{new} = \int_G \langle \pi(g)\xi, \pi(g)\eta \rangle \, dm(g)$.

π is uniformly bounded.

$\Leftrightarrow \exists C > 0$, $\frac{1}{C}\|\xi\| \le \|\xi\|_{new} \le C\|\xi\|$.

$\Rightarrow T$ can be constructed ~~$\blacksquare$~~

Conjecture (Dixmier) this happens only for amenable group.

Definition $\pi : G \to \mathcal{U}(H)$ be a unitary representation

A map $\Delta : G \to B(H)$ is called a derivation.

if 1) $\Delta(gh) = \pi(g)\Delta(h) + \Delta(g)\pi(h)$.

2) $\exists\, C > 0$, $\|\Delta(g)\| < C$, $\forall g \in G$.

Δ is said to be inner if there exist $T \in B(H)$ s.t.

$\Delta(t) = \pi(g)T - T\pi(g)$

$H_b^1(G, B(H)) := \dfrac{\text{Derivation } \Delta : G \to B(H)}{\text{inner derivations}}$ - bounded homology coef in

Vector space

Observation if $\Delta : G \to B(H)$ is a bounded derivation

then $\tilde{\pi}(g) = \begin{pmatrix} \pi(g) & D(g) \\ 0 & \pi(g) \end{pmatrix} \in B(H \oplus H)$

uniformly bounded representation.

$\tilde{\pi}(g)\tilde{\pi}(h) = \tilde{\pi}(gh)$.

Lemma: Let $\pi : G \to \mathcal{U}(H)$ be unitary representation and $\Delta : G \to B(H)$ be bounded der.

$\tilde{\pi}(g)$ is unitarizable. iff Δ is inner.

Proof,

Δ is inner that π is unitarizable.

$S \in B(H)$ $\quad D(g) = \pi(g)S - S\pi(g)$

then defining $\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \in GL(H \oplus H)$.

It could be that G is amenable
$\iff H^1_b(G, B(\ell^2 G)) = 0$.

Dixmier $\Longrightarrow$

Theorem (Nikolas Monod, Geneva)

Let G be a group ~~(discrete)~~ and Λ be infinite abelian group.

G is amenable $\iff \Lambda \wr G$ is unitarizable.

$$\Lambda \wr G = \Big(\bigoplus_{g \in G} \Lambda\Big) \rtimes G \quad - \text{(wreath product of } \Lambda \text{ and } G\text{)}$$

"Sketch of proof that Burnside group are not unitarizable!"

Let $B(n,p)$ be $\mathbb{F}_n / \langle \mathbb{F}_n^{p} \rangle$ be non amenable Burnside.

$$\Downarrow \quad \Big(\bigoplus_{n \in \mathbb{Z}} \mathbb{Z}/q\mathbb{Z}\Big) \wr B(n,p) = W$$

Any element in W is of the form
$a \cdot g$ for
$$a \in \bigoplus_{g \in G} \Big(\bigoplus_{n \in \mathbb{N}} \mathbb{Z}/q\mathbb{Z}\Big), \quad g \in B(n,p).$$

$\Longrightarrow$ $a \cdot g$ has order $\mid p \cdot q$.

$$\begin{tikzcd}
\mathbb{F}_\infty \arrow[rr, two heads] \arrow[dr, "\pi"'] & & \Big(\bigoplus \mathbb{Z}/q\mathbb{Z}\Big) \wr B(n,p) \\
& \mathbb{F}_\infty / \langle \mathbb{F}_\infty^{pq} \rangle = B(\infty, p \cdot q) \arrow[ur, "\exists!"'] &
\end{tikzcd}$$

$\Longrightarrow$ $B(\infty, p\,q)$ cannot be unitarizable.

Claim $\exists$ groups that do not contain $\mathbb{F}_2$ and are not unitarizable.

Theorem Let G be a group. If G is unitarizable $\Rightarrow$ then G is deformation rigid.

Pf: Assume that G is not deformation rigid

$\exists \pi_n, \sigma_n : G \to \mathcal{U}(H_n)$ s.t.

$\|\pi_n(g) - \sigma_n(g)\| < \frac{1}{n}, \quad \forall g \in G \quad n \in \mathbb{N}$

But $\pi_n \not\sim \sigma_n \quad \forall n.$

We define ~~δ_n~~ $\Delta_n(g) = n(\pi_n(g) - \sigma_n(g))$

$\Rightarrow \mu_n(g) := \begin{pmatrix} \pi_n(g) & \Delta_n(g) \\ 0 & \pi_n(g) \end{pmatrix}$ – uniformly bounded.

Defining $\tilde{\Delta}_n(g) = \begin{pmatrix} 0 & \Delta_n(g) \\ 0 & 0 \end{pmatrix}$ is inner derivation.

with respect to $\pi_n \oplus \pi_n$ given by the vector $\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$

Consider $\varphi(g) := \bigoplus_{n\in\mathbb{N}} \pi_n \oplus \sigma_n \in \mathcal{U}(\bigoplus_{n\in\mathbb{N}} H_n)$

$\Delta(g) := \bigoplus_{n\in\mathbb{N}} \tilde{\Delta}_n(g).$

$\Rightarrow \Delta$ is uniformly bounded with respect to φ.

if G is unitarizable $\Rightarrow \Delta$ is inner.

" $\Rightarrow$ " $\exists T_n \in B(H_n \oplus H_n)$ $\quad \exists C > 0$

$\Delta_n = \pi_n(g) T_n - T_n \sigma_n(g) \quad \|T_n\| \leq C$

$n \pi_n(g) - n \cdot \sigma_n(g) = \Delta_n(g) = \pi_n(g) T_n - T_n \sigma_n(g).$

$$=_n \Pi_n(g)\left(1-\frac{T_n}{n}\right)=\left(1-\frac{T_n}{n}\right)\cdot G_n(y).$$

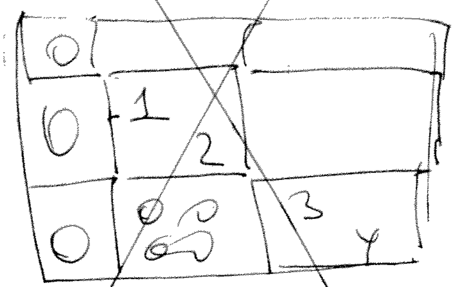
$\Rightarrow$ $n > C$ $\quad 1-\frac{T_n}{n}$ is invertible

and hence $G_n(y) \xrightarrow{} \Pi(y)$ - contradiction

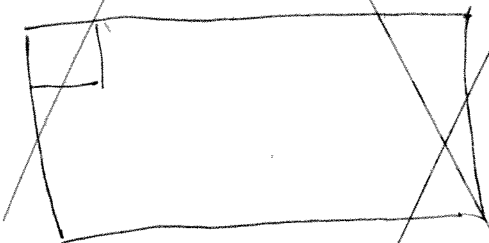
$\blacksquare$

$$B = \begin{bmatrix} \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} & & \\ & \begin{bmatrix} b_3 & 0 \\ 0 & b_4 \end{bmatrix} & \\ & & [\,\bullet\,] \end{bmatrix}$$

1, 2, 3, 4

b_2 

1

$C =$ 

$n=3$

$$B = \begin{bmatrix} b_1 & b_1+c_3-a_2 & b_1+c_3-a_2 \\ 0 & b_2 & b_2+c_2-a_2 \\ 0 & 0 & b_3 \end{bmatrix}$$

$$C = \begin{bmatrix} c_3 & a_1 & 0 \\ b_2+c_2-a_2 & c_2 & 0 \\ b_3+c_1-a_2 & b_3+c_1-a_2 & c_1 \end{bmatrix}$$

$$A = \begin{bmatrix} b_1+c_3 & b_1+c_3-a_2 & b_1+c_3-a_2 \\ b_2+c_2-a_2 & b_2+c_2 & b_2+c_2-a_2 \\ b_3+c_1-a_2 & b_3+c_1-a_2 & b_3+c_1 \end{bmatrix}$$

$(b_1+c_3-\lambda)(b_2+c_2-\lambda)(b_3+c_2-\lambda) + (b_1+c_3-a_2)(b_2+c_2-a_2)(b_3+c_1-a_2) +$

$+ \ldots - \qquad \lambda = a_2$ корень.