1. HIGHER-RANK GRAPHS, COLOURED GRAPHS AND SKELETONS

In these notes, a directed graph is a quadruple \((E_0, E_1, r, s)\) where \(E_0, E_1\) are countable (discrete) sets, and \(r, s\) are maps from \(E_1\) to \(E_0\). A path in \(E\) is a sequence \(a_1 \ldots a_n\) with each \(a_i \in E_1\) and with \(s(a_i) = r(a_{i+1})\) for all \(i\). We regard the set \(E^\ast\) of all paths as a category with objects \(E^n\) and composition given by concatenation of paths.

Definition 1.1. Let \(k \in \mathbb{N}\). A graph of rank \(k\) or a \(k\)-graph is a countable category \(\Lambda\) equipped with a functor \(d : \Lambda \to \mathbb{N}^k\), called the degree functor satisfying the following factorisation property:

- for all \(\lambda \in \Lambda\) and \(n, m \in \mathbb{N}\) such that \(d(\lambda) = m + n\) there are unique elements \(\mu \in d^{-1}(m)\) and \(v \in d^{-1}(n)\) such that \(\lambda = \mu v\).

Lemma 1.2. Let \(\Lambda\) be a \(k\)-graph. Then \(d^{-1}(0) = \{s(\lambda) : \lambda \in \text{Ob}(\Lambda)\}\).

Proof. If \(\alpha \in \text{Ob}(\Lambda)\), then

\[d(s(\alpha)) = d(\alpha) = 2d(\alpha),\]

forcing \(d(s(\alpha)) = 0\). Thus \(\{s(\lambda) : \lambda \in \text{Ob}(\Lambda)\} \subset d^{-1}(0)\).

For the reverse inclusion, fix \(\lambda\) with \(d(\lambda) = 0\). We have \(d(\lambda) = 0 + 0\), and

\[s(\text{id}_0(\lambda)) = \lambda = \lambda \text{id}_0(s(\lambda)).\]

Uniqueness of factorisations therefore forces \(\lambda = \text{id}_0(s(\lambda))\).

Note 1.3. We will adopt the following notation throughout these notes.

- \(\Lambda^n := d^{-1}(n)\).

Date: May 27, 2010.

2. SIMS

- \(r(\lambda) := \text{id}_{\text{dom}(\lambda)}(\lambda) \in \Lambda^n\) and \(s(\lambda) := \text{id}_{\text{dom}(\lambda)}(\lambda) \in \Lambda^n\).
- For \(E \subset \Lambda\) and \(\alpha \in \Lambda\), we write \(\alpha E := \{\lambda \in E : r(\lambda) = s(\alpha)\}\).
- \(E_{\alpha} := \{\lambda \in E : r(\lambda) = s(\lambda)\}\). So in particular, for \(r \in \Lambda^n\) and \(n \in \mathbb{N}\),

\[\Lambda^n = \{\lambda \in \Lambda : r(\lambda) = r\} \quad \text{and} \quad d(\lambda) = n\].

We describe \(k\)-graphs in terms of their \(k\)-coloured skeletons.

Definition 1.4. Let \(k \in \mathbb{N}\). A \(k\)-coloured graph is a directed graph \((E_0, E_1, r, s)\) together with a colour map \(c : E_0 \to \{1, \ldots, k\}\).

Given a \(k\)-coloured graph \(E\), we extend the colour map \(c\) to a functor \(c : E_1^\ast \to \mathcal{P}(\{1, \ldots, k\})\) so \(c(\alpha) = \{c(\alpha_1), c(\alpha_2), \ldots, c(\alpha_n)\}\) for \(\alpha \in E_1^\ast\).

Example 1.5. Fix \(k \in \mathbb{N}\) and \(m \in \mathbb{N}^k\). The coloured graph \(E_{\lambda,m}\) has vertices \(E_{\lambda,m}^0 = \{n \in \mathbb{N} : n \leq m\}\), and edges \(E_{\lambda,m}^1 = \{n + n_1 : n, n_1 \in \mathbb{N}^k, n + n_1 \in E_{\lambda,m}^0\}\) with structure maps

\[r(n + n_1) = n, \quad s(n + n_1) = n + c(n), \quad \text{and} \quad c(n + n_1) = i\].

For example, \(E_{3,3,2}\) could be drawn as follows:

![Graph Diagram]

A graph morphism \(\varphi\) from a graph \(E\) to a graph \(F\) is a pair of maps \(\varphi^0 : E^0 \to F^0\) and \(\varphi^1 : E_1 \to F_1\) such that \(r(\varphi^0(c)) = s(\varphi^1(r))\) and \(s(\varphi^0(c)) = s(\varphi^1(s))\) for all \(c \in E_1^\ast\).

We will often simply write \(\varphi\) for each of \(\varphi^0\) and \(\varphi^1\). A graph morphism between \(k\)-coloured graphs is then a graph morphism which preserves colour.

For disjoint \(i, j \leq k\), an \(ij\)-square in a \(k\)-coloured graph \(E\) is a \(k\)-coloured-graph morphism \(\varphi : E_{\lambda,m} \to E_{\mu,n}^i\) to \(E_{\lambda,m}^j\).

Definition 1.6. A complete and associative collection of squares for a \(k\)-coloured graph \(E\) is a set \(C\) of squares in \(E\) such that

1. For each \(ij\)-coloured path \(fg \in E^2\) there is a unique \(\varphi \in C\) such that \(\varphi(0 + n_1) = f\) and \(\varphi(n_1 + r) = g\); and
2. If we write \(fg \sim g' f'\) whenever there is a square \(\varphi\) such that

\(\varphi(0 + n_1) = f\), \(\varphi(n_1 + r) = g\), \(\varphi(0 + n_1) = f'\), \(\varphi(n_1 + r) = g\), and \(\varphi(0 + n_1) = f'\),

then if \(fg\) is a \(k\)-coloured path and

\(f_2 \sim g_1 f_1\), \(f_1 h \sim h_1 f_2\), \(g_1 h \sim h_2 g_2\), \(g h \sim h' g\), \(f_1 g_1 \sim h f_1\), and \(f_2 g_2 \sim h' g'\),

then \(f_2 = f_1, g_2 = g_1, h_2 = h_1, g_2 = g_1, h_2 = h_1, g_2 = g_1, h_2 = h_1\).
Given a $k$-coloured graph $E$ and a coloured-graph morphism $\varphi : E_{k,n} \to E$, we say that an $ij$-square $\psi$ in $E$ occurs in $\varphi$ if there exists $n \in \mathbb{N}^k$ such that $n + c_i + e_j \leq m$ and
\[
\varphi(n + n_i) = \psi(0 + n_i), \\
\varphi(n + c_i + n_j) = \psi(e_i + n_j), \\
\varphi(n + c_i + e_j) = \psi(0 + n_j), \\
\varphi(n + e_j) = \psi(n_j + n).
\]

If $E$ is a $k$-coloured graph and $C$ is a complete and associative collection of squares in $E$, we say that a coloured-graph morphism $\varphi : E_{k,m} \to E$ is $C$-compatible if every square which occurs in $\varphi$ belongs to $C$.

The next lemma is due to Robbie Backwood and is the key step in our construction of a $k$-graph from a coloured graph.

**Lemma 1.7.** Let $E$ be a $k$-coloured graph and let $C$ be a complete and associative collection of squares in $E$. Let $\pi : E_k^* \to \mathbb{N}^k$ be the homomorphism satisfying $\pi(i) = c_i$. Then for each path $\alpha = c_i \alpha_2 \ldots \alpha_n \in E$, there is a unique $C$-compatible coloured-graph morphism $\varphi : E_{k,n}(\alpha) \to E$ such that
\[
\varphi(\pi(c_i \alpha_2 \ldots \alpha_n)) = \alpha_{i+1} \quad \text{for all } i < |\alpha|.
\]

**Proof.** We proceed by induction on $|\alpha|$. If $|\alpha| = 0$ then the assertion is trivial.

Now fix $n \geq 1$ and suppose that there is a unique $\varphi$ satisfying (1.1) whenever $|\alpha| < n \in \mathbb{N}$, and fix $\alpha \in E^\alpha$. Let $i := \pi(c_i)$, and let $m := \pi(\alpha_i)$. By the inductive hypothesis, there is a unique $C$-compatible coloured-graph morphism $\varphi : E_{k,m} \to E$ which is traversed by $\alpha_1 \ldots \alpha_m$. For each $j \in \{1, \ldots, k\} \setminus \{i\}$ such that $m_j \neq 0$, that $C$ is a complete collection of squares ensures that there is a unique $\varphi_j \in \epsilon^*(j)$ and $\varphi' \in \psi(\alpha) \cap \psi(\alpha)$ such that $\varphi_j(m_j - e_j + n_j) = \varphi_j(n_j + m_j)$. 

For each $j$, the inductive hypothesis applied to $\lambda^j$ for any traversal $\xi$ of $\psi(E_{m,n-c-e})$, yields a unique $C$-compatible morphism $\lambda^j$ traversed by $\xi^j$.

We claim that for distinct $p, q$, the morphisms $\lambda^p$ and $\lambda^q$ agree on the intersection of their domains, namely $E_{k,m-n-e-p}$. To see this, let $\tau := \lambda^p(m - e_p - e_i + v_i)$. Then $\tau = \lambda^q((m - e_p - e_i) + v_i)$ because the two are the paths $b^2$ and $b_1$ obtained from Definition 1.6(2) with
\[
f = \psi((m - e_p - e_i) + v_i), \quad g = \psi((m - e_i) + v_i), \quad h = c_i.
\]

Hence each of $\lambda^p(E_{m-n-e-p})$ and $\lambda^q(E_{m-n-e-p})$ is traversed by $\xi^p$ for any traversal $\xi$ of $\psi(E_{m-n-e-p})$. The inductive hypothesis therefore gives
\[
\lambda^p(E_{m-n-e-p}) \subseteq \lambda^q(E_{m-n-e-p}).
\]

Since $E_{m,n} = \left( \bigcup_{n=p,q} E_{m,n} \right) \cup \{(m - e_p) + v_p : m_p \neq 0, p \neq q\}$, equation (1.3) implies that there is a well-defined coloured-graph morphism $\varphi : E_{k,m} \to E$ determined by
\[
\psi(E_{m,n}) = \lambda^p \quad \text{whenever } p \neq q \text{ and } m_p \neq 0, \text{ and}
\psi((m - e_p) + v_p) = \varphi \quad \text{for all } p \neq q \text{ and } m_p \neq 0.
\]

Every square which occurs in $\varphi$ either occurs in one of the $\lambda^p$ or occurs in the cube $\kappa^{(p)}$ traversed by the path $fgh$ of (1.2) for some $p, q$. Since the $\lambda^p$ and the $\kappa^{(p)}$ are all $C$-compatible, it follows that $\varphi$ is also. That the $\varphi$ and $\lambda^p$ were uniquely determined by requiring that all squares occurring in them belonged to $C$ implies that $\varphi$ is the unique $C$-compatible morphism traversed by $\alpha$. 

**Corollary 1.8.** Let $E$ be a $k$-coloured graph, and let $C$ be a complete and associative collection of squares for $E$. If
\[
\varphi : E_{k,m} \to E \quad \text{and} \quad \psi : E_{k,n} \to E
\]
are $C$-compatible coloured-graph morphisms such that $\varphi(m) = \psi(0)$, then there is a unique $C$-compatible morphism $(\varphi \psi) : E_{k,m+n} \to E$ such that
\[
\varphi(\psi)(p + v_i) = \varphi(p + v_i), \quad \text{whenever } p + c_i \leq m, \text{ and}
\varphi(\psi)(p + v_i) = \psi(0 - m) + v_i, \quad \text{whenever } m \leq p \leq m + n - c_i.
\]
Moreover, this defines an associative partial multiplication on the set
\[ \Lambda_{E,E} = \bigcup_{n \in \mathbb{N}} \{ \varphi : E_{k,n} \to E \mid \varphi \text{ is a C-compatible coloured-graph morphism} \} . \]

**Proof.** Fix paths \( \alpha^o \) and \( \alpha^o \) in \( E \) which traverse \( \varphi \) and \( \psi \). Then Lemma 1.7 implies that there is a unique C-compatible coloured-graph morphism \( \varphi \psi \) traversed by \( \alpha^o \alpha^o \).

The uniqueness assertion of Lemma 1.7 implies that \( \varphi \psi \) satisfies (1.4). Moreover, any coloured-graph morphism \( \pi \) satisfying (1.4) is traversed by \( \alpha^o \alpha^o \) and hence another application of uniqueness from Lemma 1.7 implies that \( \pi = \varphi \psi \).

Associativity follows from associativity of concatenation of paths in \( E \).

**Theorem 1.9.** Let \( E \) be a \( k \)-coloured graph, and let \( C \) be a complete and associative collection of squares for \( E \). Let \( \Lambda = \Lambda_{E,C} \) be as in Corollary 1.8, and define \( d : \Lambda \to \mathbb{N}^+ \) by \( d(\varphi) = m \) if \( \text{dom}(\varphi) = E_{k,m} \). Then \( \Lambda \) is the unique \( k \)-graph such that \( \Lambda^{\alpha} = c^{-1}(i) \) for each \( i \) and \( f g = g f' \) in \( \Lambda \) if and only if \( f \sim g \) in \( E \).

**Proof.** Corollary 1.8 shows that \( \Lambda \) is a category and it has \( \Lambda^{\alpha} = c^{-1}(i) \) and \( f g = g f' \) whenever \( f \sim g \) in \( E \) by definition. To see that \( \Lambda \) is a \( k \)-graph, we must verify the factorisation property. This follows from Lemma 1.7 and uniqueness of factorisations of paths in \( E \).

For uniqueness, observe that if \( \Gamma \) is a \( k \)-graph with the given properties, then each \( \gamma \in \Gamma \) determines a \( C \)-compatible coloured-graph morphism \( \varphi \), by \( \varphi(n + \nu) = \gamma \) where \( \nu \) is the unique path satisfying \( \gamma = i\gamma \psi \) with \( d(\gamma) = m \), \( d(\nu) = n \), and \( d(\psi) = d(\gamma) - m - n \).

**Example 1.10.** The associative condition is necessary in three or more dimensions as is demonstrated by the following three-coloured graph due to Jack Spielberg:

There is a unique complete collection of squares in this graph, but the collection is not associative as can be seen by chasing through the possible factorisations of the path \( fgh \).

2. **2-k-graph \( \Lambda \)-algebras and the gauge-invariant uniqueness theorem**

A \( k \)-graph is row-finite if \( |v|_k < \infty \) for all \( v \in \Lambda^k \) and \( n \in \mathbb{N}_0^k \). It is locally convex if whenever \( \mu \in \Lambda^k \) and \( \nu \in \Lambda^{k'} \) with \( i \neq j \) and \( r(\mu) = r(\nu) \), we have \( s(\mu) \Lambda^k \neq \emptyset \) and \( s(\nu) \Lambda^{k'} \neq \emptyset \).

Pictorially, the graph on the left is not allowed unless the two edges pictured extend to squares as on the right.

**Remark 2.1.** If \( \Lambda \) is locally convex, then a straightforward induction shows that if \( m \wedge n = 0 \) and \( \mu \in \Lambda^m \) and \( \nu \in \Lambda^n \) with \( r(\mu) = r(\nu) \), then \( s(\mu) \Lambda^m \) and \( s(\nu) \Lambda^n \) are nonempty.

We write \( \Lambda^{\leq} \) for the set
\[ \Lambda^{\leq} = \{ \lambda \in \Lambda : d(\lambda) \leq n \text{ and } d(\lambda) \leq n \implies s(\lambda) \Lambda^\leq = \emptyset \} . \]

**Lemma 2.2.** Let \( \Lambda \) be a locally convex k-graph. Fix \( m, n \in \mathbb{N}_0^k \). We have \( \Lambda^{\leq(m+n)} = \Lambda^{\leq m} \Lambda^{\leq n} \).

**Proof.** If \( \mu \in \Lambda^{\leq m} \) and \( \nu \in \Lambda^{\leq n} \), then certainly \( d(\mu) \leq m + n \), so suppose \( d(\mu) \leq m + n \). Suppose \( d(\nu) \leq m + n \). There are two cases to consider: \( d(\nu) < m \) or \( d(\mu) < m \). If \( d(\nu) < m \), then \( s(\mu) \Lambda^m = s(\mu) \Lambda^\leq = \emptyset \). On the other hand, if \( d(\mu) < m \), then \( s(\mu) \Lambda^m = s(\mu) \Lambda^\leq = \emptyset \). and then \( s(\mu) \Lambda^m \subset s(\nu) \Lambda^\leq \) by the factorisation property. So \( \Lambda^{\leq(m+n)} \subset \Lambda^{\leq m} \Lambda^{\leq n} \).

Now suppose that \( \Lambda \in \Lambda^{\leq(m+n)} \). Let \( m' := m \wedge d(\lambda) \), and let \( n' := n \wedge d(\lambda) - m' \). It is straightforward to check that \( m' + n' = (m + n) \wedge d(\lambda) \). Let \( \mu = \lambda(0, m', n') \). Clearly \( d(\mu) \leq m \) and \( d(\nu) \leq n \). If \( d(\mu) < m \), then \( s(\mu) \Lambda^m = s(\mu) \Lambda^\leq = \emptyset \). Given \( \nu \in \Lambda^{\leq n} \). Now suppose that \( d(\nu) < m \). Then \( d(\mu) = d(\lambda) \), so \( d(\nu) = 0 \). Moreover, \( d(\lambda) < m \), whence \( s(\lambda) \Lambda^\leq = \emptyset \). It then follows from Remark 2.1 that \( r(\nu) \Lambda^\leq = \emptyset \). So \( \mu \in \Lambda^{\leq n} \).

The following definition of a Cuntz-Krieger \( \Lambda \)-family, due originally to Yeeda, is the one suitable to locally convex row-finite \( k \)-graphs. However, it is very closely modelled on Kumjian and Pask's original definition for row-finite \( k \)-graphs with no sources. Likewise, our analysis in this section leading up to the gauge-invariant uniqueness theorem is largely due to Raeburn-S-Yeeda but is heavily based on Kumjian and Pask's seminal work.

**Definition 2.3.** Let \( \Lambda \) be a locally convex \( k \)-graph \( k \)-graph. A Cuntz-Krieger \( \Lambda \)-family is a \( C^* \)-algebra \( B \) for which \( \Lambda \to B \), \( \Lambda \to \Lambda_k \) such that
\[ (C_\Lambda) \{ \{v : v \in \Lambda^k \} \} \text{ is a set of mutually orthogonal projections} \]
We write $C^*\{\{\lambda : \lambda \in \Lambda\}\}$.

To give an example of a Cuntz-Krieger A-family, we introduce filters in k-graphs. The idea of using filters and ultrafilters to construct representations of combinatorial objects such as k-graphs is due to Exel in the context of inverse semigroups, though the procedure is greatly simplified in our setting. To introduce filters, we need the notion of a \textit{minimal common extension} of paths in $\Lambda$.

**Definition 2.4** Let $\Lambda$ be a k-graph, and let $\mu, \nu \in \Lambda$ We say that $\lambda$ is a \textit{minimal common extension} of $\mu$ and $\nu$ if $d(\lambda) = d(\mu) \lor d(\nu)$ and $\lambda = \mu\beta = \nu\beta'$ for some $\mu', \nu' \in \Lambda$. We write $\text{MCE}(\mu, \nu)$ for the set of all minimal common extensions of $\mu$ and $\nu$.

A filter of a k-graph $\Lambda$ is a set $x \subseteq \Lambda$ such that

(F1) if $x \in \Lambda$ and $\lambda \in \Lambda$, then $\mu \in x$ and

(F2) if $\mu, \nu \in x$, then $\text{MCE}(\mu, \nu) \subseteq x \neq \emptyset$.

It follows that if $x$ is a filter of $\Lambda$, then $\Lambda^n \cap x$ contains a unique element $\tau(x)$, and also that if $\mu, \nu \in x$ then there is a unique element $\mu \lor \nu$ of $\text{MCE}(\mu, \nu)$ which belongs to $x$.

An ultrafilter of $\Lambda$ is a filter which is maximal with respect to containment. A standard Zorn’s Lemma argument shows that for each $\Lambda \in \Lambda$ there exists an ultrafilter $x$ of $\Lambda$ such that $\lambda \in x$. We write $\Lambda$ for the set of filters of $\Lambda$, and $\Lambda_{\text{uf}}$ for the set of ultrafilters of $\Lambda$.

**Lemma 2.5** Let $\Lambda$ be a row-finite locally-convex k-graph. Let $x \in \Lambda$ and fix $\lambda \in x$ and $\mu, \nu \in \Lambda(\tau)$. Then

(1) $\lambda^* = \{ \lambda \alpha \in x \}$ and $\mu^* = \{ \beta : \beta \Lambda \cap \mu \neq \emptyset \}$ are filters;

(2) $\lambda^* \cap \mu = \mu^* \cap \lambda \neq \emptyset$.

Proof. (1) If $\alpha \in \lambda^* \cap \mu$ and $\alpha \beta \in x$, then (F1) forces $\lambda \beta \in x$ and hence $\beta \in \lambda^*$. If $\beta \in \lambda^* \cap \mu$, then $\beta \Lambda \cap \mu \neq \emptyset$, so $\beta \in \mu \cap \mu$. So $\lambda^* \cap \mu \neq \emptyset$.

(2) Suppose $\alpha, \beta \in \lambda \cap \mu$, then $\lambda \beta \in x$ and hence $\beta \in \lambda \cap \mu$. If $\beta \in \lambda \cap \mu$, then $\beta \Lambda \cap \mu \neq \emptyset$.

For (F2), suppose that $\alpha, \beta \in \lambda \cap \mu$, then $\lambda \beta \in x$ and hence $\beta \in \lambda \cap \mu$. If $\beta \in \lambda \cap \mu$, then $\beta \Lambda \cap \mu \neq \emptyset$.

We can now state the following.

**Lemma 2.6** Let $\Lambda$ be a locally convex row-finite k-graph. If $x \in \Lambda_{\text{uf}}$ and $\lambda \in \Lambda$, then $\tau(x) \Lambda^n \cap x \neq \emptyset$.

Proof. Fix an increasing cofinal sequence $(\mu_0, \mu_1, \ldots, \mu_n)$ of $x$ such that $\mu_0 = \tau(x)$. For each $i$, if $\mu_i \Lambda \cap \mu_{i+1} \neq \emptyset$, then $\tau(x) \Lambda^n \cap x \neq \emptyset$.

We can now state the following.

**Example 2.7** Let $\Lambda$ be a locally convex row-finite k-graph, and let $H := \ell^2(\Lambda_{\text{uf}})$.

Routine calculations show that $T_{\Lambda}x := \chi_x(s(\lambda))\chi_x$ yields an Cuntz-Krieger A-family $T$ in $B(H)$ (it satisfies (CK4) by Lemma 2.6). Moreover, for $x \in \Lambda$,

$T_{\Lambda}x = \chi_x(s(\lambda))x_{\lambda \Lambda}$.

Example 2.7 shows in particular that for any k-graph $\Lambda$ there exist Cuntz-Krieger A-families in which every $t_{\lambda^*}$ is nonzero.

**Lemma 2.8** Let $\Lambda$ be a locally convex row-finite k-graph. Then for $\mu, \nu \in \Lambda$, we have

(2.1) $\text{MCE}(\mu, \nu) = \mu(\Lambda^{(d(\mu) \lor d(\nu))} \cap \nu(\Lambda^{(d(\mu) \lor d(\nu))} \cap \nu)) = \text{MCE}(\nu(\Lambda^{(d(\nu) \lor d(\nu))} \cap \mu(\Lambda^{(d(\nu) \lor d(\nu))} \cap \mu))$}

and

(2.2) $t_{\mu^*} = \sum_{\mu^* \subseteq \text{MCE}(\mu, \nu)} t_{\mu^*}$.

Proof. To establish (2.1) first note that

$\text{MCE}(\mu, \nu) = \mu(\Lambda^{(d(\mu) \lor d(\nu))} \cap \nu(\Lambda^{(d(\mu) \lor d(\nu))} \cap \nu)) \subseteq \text{MCE}(\nu(\Lambda^{(d(\mu) \lor d(\nu))} \cap \mu(\Lambda^{(d(\mu) \lor d(\nu))} \cap \mu))$.

by definition. For the reverse inclusion, note that $\lambda \in \mu \Lambda \cap \nu \Lambda \Rightarrow d(\mu) \geq d(\mu) \lor d(\nu)$. To establish (2.2), let $\lambda := d(\mu), \nu := d(\nu)$ and use (CK4) to calculate

(2.3) $t_{\mu^*} = t_{\mu^*} t_{\mu^*} t_{\mu^*} t_{\mu^*} = \sum_{\mu^* \subseteq \text{MCE}(\mu, \nu)} t_{\mu^*} t_{\mu^*} t_{\mu^*} t_{\mu^*}$.
By Lemma 2.2, each $\mu \in \Lambda$ induces another application of (CK4) that ensures each $\Delta^*_{\mu\nu}\Lambda^*_{\mu\nu} = \Delta^*_{\mu\nu}\Lambda^*_{\mu\nu}$. Hence
\[
\Delta^*_{\mu\nu} = \sum_{\Lambda^*_{\mu\nu}} \Delta^*_{\mu\nu}\Lambda^*_{\mu\nu} = \sum_{\Lambda^*_{\mu\nu}} \Delta^*_{\mu\nu}\Lambda^*_{\mu\nu}.
\]
By (2.1)
\[
= \sum_{\Lambda^*_{\mu\nu}} \Delta^*_{\mu\nu}\Lambda^*_{\mu\nu}.
\]
By (CK3).

**Corollary 2.9.** Let $A$ be a locally convex row-finite $k$-graph and let $t$ be a Cuntz-Krieger $\Lambda$-family. Then $C^*(t) = \text{span} \{t_{\mu\nu}^*: \mu, \nu \in \Lambda\}$.

Our proof of the following result is taken more or less directly from Raeburn’s notes on graph algebras from the CBMS conference held at the University of Iowa in 2004.

**Proposition 2.10.** There is a $C^*$-algebra $C^*(\Lambda)$ generated by a Cuntz-Krieger $\Lambda$-family $s$ which is universal in the sense that each Cuntz-Krieger $\Lambda$-family $t$ induces a homomorphism $\tau_t: C^*(\Lambda) \to C^*(t)$ satisfying $\tau_t(s_{\mu\nu}) = t_{\mu\nu}$ for all $\Lambda \in \Lambda$. Moreover, each $s_{\mu\nu}$ is nonzero.

**Proof.** Let $A = \{\{\mu, \nu\} \in \Lambda \times \Lambda : s_{\mu\nu} = s(\mu) \neq 0\}$. Let $A_0 := c_0(A, \Lambda)$ and for each $\mu, \nu$, let $\delta_{\mu\nu} : A_0 \to \Lambda$ denote the indicator function. Define $f: \Lambda_0 \to A_0$ by $f(\mu, \nu) = (f(\mu, \nu))$, and define a multiplication on $A_0$ by extending the assignment
\[
\delta(\mu, \nu) \delta(\mu', \nu') \rightarrow \delta(\mu, \nu')
\]
to a bilinear map. For each Cuntz-Krieger $\Lambda$-family $t$ on Hilbert space, the partial isometries $t_{\mu\nu}$ satisfy the same relations as the $\delta_{\mu\nu}$, so each such family determines a representation $\tau_t$ of $A_0$ such that $\tau_t(t_{\mu\nu}) = t_{\mu\nu}$.

Each $s_{\mu\nu}$ is a partial isometry, so its norm is less than or equal to 1. Hence for $f \in A_0$,
\[
\|f(\mu, \nu)\| \leq \sum_{(\mu, \nu) \in \Lambda \times \Lambda} \|f(\mu, \nu)\| = \|f(\mu, \nu)\| = \|f(\mu, \nu)\|.
\]
Hence $\|f\|_0 := \sup_{\Lambda \in \tilde{A}} \|f(t)\|$ defines a seminorm on $A_0$. Let $I := \{f : \|f\|_0 = 0\}$, and let $A := A_0/I$. Let $C^*(A)$ be the completion of $A$ in the norm induced by $\| \cdot \|_0$, and let $\delta_{\mu\nu} : A \to \Lambda$ be the assignment $\delta(\mu, \nu) \rightarrow \delta(\mu, \nu')$. Since the Cuntz-Krieger family $T$ of Example 2.7 consists of nonzero partial isometries, the universal property of $C^*(\Lambda)$ ensures that the $s_{\mu\nu}$ are nonzero. □

**Remark 2.11.** Let $A$ be a locally convex row-finite $k$-graph and let $t$ be a Cuntz-Krieger $\Lambda$-family. Fix $\mu, \nu \in \Lambda$ with $s_{\mu\nu} = s(\mu) = \nu$, and suppose that $t_{\mu\nu} \neq 0$. Then
\[
\|s_{\mu\nu}t_{\mu\nu}\| = \|s_{\mu\nu}t_{\mu\nu}\| = \|s_{\mu\nu}t_{\mu\nu}\| = \|s_{\mu\nu}t_{\mu\nu}\| = \|t_{\mu\nu}\| = \|t_{\mu\nu}\| = \|t_{\mu\nu}\| = \|t_{\mu\nu}\|.
\]
In particular, each $s_{\mu\nu} \neq 0$ in $C^*(\Lambda)$. □

For each $\mu \in \tilde{A}$, the map $\lambda \mapsto \lambda^{\Lambda_\mu}\Lambda_{\mu\nu}$ is a Cuntz-Krieger $\Lambda$-family, so the universal property of $C^*(\Lambda)$ gives an endomorphism $\gamma: C^*(\Lambda) \to C^*(\Lambda)$ such that $\gamma_{\lambda}(s_{\mu\nu}) = \lambda^{\Lambda_\mu}\Lambda_{\mu\nu}$ for all $\lambda$. Since $\gamma_{\lambda} \circ \gamma_{\mu} = \gamma_{\lambda \mu}$, $\gamma_{\lambda}$ is an automorphism of $C^*(\Lambda)$ and $\lambda \mapsto \gamma_{\lambda}$ is an action of $A$. If $\lambda \mapsto z$, then $\gamma_{\lambda}(s_{\mu\nu}) = s_{\lambda\mu\lambda\nu}$ for all $\mu, \nu$, and then an argument shows that $\gamma$ is strongly continuous. It is then standard to show that $\Phi^t(s_{\mu\nu}) := \int_0^t \gamma_{\lambda}(s_{\mu\nu})d\lambda$ defines a conditional expectation from $C^*(\Lambda)$ to $C^*(\Lambda)$:
\[
\epsilon(s_{\mu\nu}) = \int_0^\infty \gamma_{\lambda}(s_{\mu\nu})d\lambda = s_{\mu\nu}.
\]

**Proposition 2.12.** Let $A$ be a locally convex row-finite $k$-graph. Then
\[
\begin{align*}
&1. C^*(\Lambda) = \text{span} \{s_{\mu\nu}^*: \mu, \nu \in \Lambda\} \\
&2. C^*(\Lambda)^e = \text{span} \{s_{\mu\nu}^*: \mu, \nu \in \Lambda\} \\
&3. \text{span} \{s_{\mu\nu}^*: \mu, \nu \in \Lambda\} \subset \text{span} \{s_{\mu\nu}^*: \mu, \nu \in \Lambda\}
\end{align*}
\]

**Proof.** (1) For $\mu, \nu \in \Lambda$, we have
\[
\Phi^t(s_{\mu\nu}^*) = \int_0^t s_{\mu\nu}(\lambda) d\lambda = \begin{cases} s_{\mu\nu} & \text{if } \mu = \nu, \\ 0 & \text{otherwise.} \end{cases}
\]
Hence $\Phi^t(s_{\mu\nu}^*) = s_{\mu\nu}$ for $\mu, \nu \in \Lambda$. Since $\Phi^t \circ \Phi^s = \Phi^{t+s}$, this proves (1).

(2) Since $\mu, \nu \in \Lambda$, implies $s_{\mu\nu}^* s_{\mu\nu} = \delta(\mu, \nu)$, we have
\[
\begin{align*}
\text{span} \{s_{\mu\nu}^*: \mu, \nu \in \Lambda\} &= \bigoplus_{n=m} \text{span} \{s_{\mu\nu}^*: \mu, \nu \in \Lambda\} \\
\text{span} \{s_{\mu\nu}^*: \mu, \nu \in \Lambda\} &= \bigoplus_{n=m} \text{span} \{s_{\mu\nu}^*: \mu, \nu \in \Lambda\}
\end{align*}
\]
for each $n, m$. If $n \leq m$, and $\mu, \nu \in \Lambda$ with $s_{\mu\nu} = s(\nu)$, then
\[
\text{span} \{s_{\mu\nu}^*: \mu, \nu \in \Lambda\} \subset \text{span} \{s_{\mu\nu}^*: \mu, \nu \in \Lambda\}
\]
by Lemma 2.2. Hence
\[
\begin{align*}
\text{span} \{s_{\mu\nu}^*: \mu, \nu \in \Lambda\} \subset \text{span} \{s_{\mu\nu}^*: \mu, \nu \in \Lambda\}
\end{align*}
\]
and (2) follows.

(3) Fix $n \in \tilde{A}$, $m \leq n$ and $\mu \in \Lambda$. For $\mu, \nu, \tau \in \Lambda$, we have $s_{\mu\nu} s_{\tau\lambda} = s_{\mu\nu}^* s_{\tau\lambda} = \delta(\mu, \nu) s_{\mu\nu}^* s_{\tau\lambda}$. Remark 2.11 therefore implies that the $s_{\mu\nu}$ form a family of nonzero matrix units indexed by $(\Lambda \cap \Lambda^\mu)^n$, and the result then follows from the uniqueness of $K(\tilde{A} \cap \Lambda^\mu)^n$.

**Proposition 2.13.** Let $A$ be a locally convex row-finite $k$-graph. Suppose that $t$ is a Cuntz-Krieger $\Lambda$-family such that each $\Lambda_\mu$ is nonzero, and suppose that there is a linear map $\Psi: C^*(t) \to C^*(t)$ such that $\Psi(t_{\mu\nu}^*) = \delta_{\mu\nu} t_{\mu\nu}^*$ for all $\mu, \nu$. Then $t_{\mu\nu}^* C^*(t) \to C^*(t)$ is injective.

**Proof.** By Remark 2.11, whenever $s(\mu) = s(\nu)$, we have $t_{\mu\nu}^* \neq 0$. Since $K(\tilde{A} \cap \Lambda^\mu)^n$ is simple, Proposition 2.12(3) implies that $t_{\mu\nu}^*$ is injective, hence isometric on
each $\Phi(\pi_v^* \mu) : \mu, v \in (\Lambda^\infty \cap \Lambda^m)$. So Proposition 2.12(2) implies that $\pi_v$ is isometric on $C^*(\Lambda^\infty)$. Since $\Phi \circ \pi_v = \pi_v \circ \Phi$, and since $\Phi$ is faithful on positive elements, we have $\pi_v(a) = 0 \implies \Phi(\pi_v(a)) = 0 \implies \pi_v(\Phi(a)) = 0 \implies \Phi(\pi_v(a)) = 0 \implies a = 0$.

The following is one of the many generalisations to date of an Hefu and Raeburn's gauge-invariant uniqueness theorem for unital Cuntz-Krieger algebras, and its proof like all the others is more or less identical to the one originally given by an Hefu and Raeburn. It will be the single most useful result in our repertoire later in the course, and plays a similar role in the theory of k-graph algebras in general.

**Corollary 2.14 (The gauge-invariant uniqueness theorem).** Let $\Lambda$ be a locally convex row-finite k-graph. Suppose that $t$ is a Cuntz-Krieger $\Lambda$-family such that $t_{\lambda}^* t = \delta_{\mu, \nu}$. Then for all $\mu, \nu \in (\Lambda^\infty \cap \Lambda^m)$, we have $\forall \lambda \in \Lambda^m$.

**Proof.** The map $\Phi : a \mapsto \int_{\mathbb{R}} \beta(\mu) a(z) dz$ from $C^*(\Lambda^\infty)$ to $C^*(\Lambda^\infty)$ satisfies the hypotheses of Proposition 2.13.

3. THE CUNTZ-KRIEGER UNIQUENESS THEOREM AND SIMPLICITY

The formulation of aperiodicity and cofinality used in this section are due to Lewin. The aperiodicity condition, in particular, is the latest refinement of a condition originally given by Kunjan and Pask which has since been re-cast and sharpened by many authors including Raeburn-Young, D. Robertson, and Shortt.

**Definition 3.1.** We say that a k-graph $\Lambda$ is aperiodic if, for all $\mu, \nu \in \Lambda$ with $s(\mu) = s(\nu)$, there exists $r \in s(\Lambda)$ such that $\text{MCE}(\mu r, \nu r) = 0$ for all $r$.

**Lemma 3.2.** Let $\Lambda$ be a k-graph, and fix $v \in \Lambda^0$ and a finite subset $H$ of $\Lambda v$. Then there exists $\tau \in \Lambda v$ such that $\text{MCE}(\mu r, \nu r) = 0$ for all $\tau$ such that $\pi_v(a) = 0$.

**Proof.** We proceed by induction on $H$. If $|H| = 1$ there is nothing to do.

Suppose there exists $\pi_v(a) = 0$ for all $\tau \in \Lambda v$ and fixed $H \subseteq \Lambda v$ with $|H| = n$. Let $\lambda$ be any element of $H$ and let $G := H \setminus \{\lambda\}$. By the inductive hypothesis there exists $\eta$ such that $\text{MCE}(\mu r, \nu r) = 0$ for all $\mu, \nu \in G$. Enumerate $G = \{1, \ldots, n-1\}$ and iteratively choose paths $\pi_v(a) = 0$ such that for each $i$, $\text{MCE}(\mu_i r_{\lambda_{i+1}}, \nu_i r_{\lambda_{i+1}}) = 0$. Then $\tau = \pi_v(a) = 0$.

The following theorem is a generalisation of Cuntz and Krieger’s original uniqueness result theorem for their $C^*$-algebras associated to $[0, 1]$-matrices. Indeed, the proof is, modulo the details, very much like the one given by Cuntz in his analysis of $\mathcal{O}_\lambda$.

**Theorem 3.3 (The Cuntz-Krieger uniqueness theorem).** Let $\Lambda$ be a locally convex row-finite k-graph. Suppose that $\Lambda$ is aperiodic. Let $t$ be a Cuntz-Krieger $\Lambda$-family such that $t_{\lambda}^* t = \delta_{\mu, \nu}$. Then $\pi_v$ is injective.

**Proof.** We aim to apply Proposition 2.13. It suffices to show that for any finite $F \subseteq \Lambda$ and any collection of scalars $\{a_{\mu, \nu} : \mu, \nu \in F\}$, we have

$$\sum_{\mu, \nu \in F} a_{\mu, \nu} t_{\mu}^* t_{\nu} = 0 \implies \sum_{\mu, \nu \in F} a_{\mu, \nu} t_{\mu}^* t_{\nu} = 0$$

for this implies that there is a well-defined linear map $\Phi$ satisfying $\Phi(t_{\mu}^* t_{\nu}) = \delta_{\mu, \nu} a_{\mu, \nu} t_{\mu}^* t_{\nu}$ for all $\mu, \nu$.

Let $\lambda := \lambda_{\mu, \nu}$. Then $\sum_{\mu, \nu \in F} a_{\mu, \nu} t_{\mu}^* t_{\nu} \in \mathbb{R}^n(F)$ by Proposition 2.12(2). Hence, by the same result there exist $v \in \Lambda^0$ and $\lambda \in \Lambda^m$ such that the strict-topology limit

$$\varphi_{\lambda, v} := \lim_{\lambda \in \Lambda^m, v \in \Lambda^0} t_{\lambda}^* t_{\nu}$$

satisfies

$$\sum_{\mu, \nu \in F} a_{\mu, \nu} t_{\mu}^* t_{\nu} = \left(\sum_{\mu, \nu \in F} a_{\mu, \nu} \varphi_{\lambda, v}^* \varphi_{\lambda, v} \right) \varphi_{\lambda, v}$$

or $\sum_{\mu, \nu \in F} a_{\mu, \nu} t_{\mu}^* t_{\nu} = 0$. Since $\Lambda$ is row finite, $F \cap \Lambda^\infty \cap \Lambda^m$ is finite, so Lemma 3.2 implies that there exists $\tau \in \Lambda^v$ such that $\forall \mu, \nu \in F$ with $d(\mu) \neq d(\nu)$, $\sum_{\mu, \nu \in F} a_{\mu, \nu} t_{\mu}^* t_{\nu} = 0$. Thus $\varphi_{\lambda, v}^* \varphi_{\lambda, v} = 0$.

By Proposition 2.12(3), $v_{\lambda}^* v_{\lambda} = \varphi_{\lambda, v}^* \varphi_{\lambda, v}$.

Moreover, if $\mu, \nu \in F$ with $d(\mu) = d(\nu)$, then $t_{\mu}^* t_{\nu} = 0$.

**Corollary 3.4.** Let $\Lambda$ be a locally convex row-finite k-graph. The following are equivalent:

1. $\Lambda$ is aperiodic.
(2) each nontrivial ideal of $C^*(\Lambda)$ contains $s_v$ for some $v \in \Lambda^0$.

**Proof.** (1) $\Rightarrow$ (2) is the Cuntz-Krieger uniqueness theorem.

For (2) $\Rightarrow$ (1) we prove the converse. Suppose that there exist distinct $\mu, \nu \in \Lambda$ such that $\text{MCE}(\mu, \nu) \neq \emptyset$ for all $\tau \in s(\mu)\Lambda$.

We claim that an ultrafilter $x$ of $\Lambda$ contains $\mu$ if and only if it contains $\nu$. To see this, it suffices by symmetry to show that $\mu \in x$ implies $\nu \in x$. Fix an ultrafilter $x$ such that $\mu \in x$. Fix a cofinal sequence $\{s_{\mu}^n\}$ of $x$ such that $s_{\mu}^n = \mu$. For each $i$, $s_{\mu}^i = \mu$, for some $\tau_i \in \Lambda$. By assumption, $\text{MCE}(s_{\mu}^i, s_{\nu}^i) \neq \emptyset$ for all $i$. For $j \leq i$, we have $\text{MCE}(s_{\mu}^j, s_{\nu}^j) \subset \text{MCE}(s_{\mu}^i, s_{\nu}^i) \Lambda$. Since each $\text{MCE}(s_{\mu}^i, s_{\nu}^i) \Lambda$ is finite, we may inductively choose $\gamma_i \in \text{MCE}(s_{\mu}^i, s_{\nu}^i)$ such that $\gamma_i \in \gamma_i \Lambda$ for all $j \leq i$, and such that $\gamma_i \Lambda \cap \text{MCE}(s_{\mu}^j, s_{\nu}^j) \neq \emptyset$ for infinitely many, and hence all, $j \geq i$. Let $y = \{o \in \Lambda : \tau_i \in \gamma_i \Lambda \text{ for some } j\}$. Then $y$ is a filter. We have $x \in y$ because the $\mu_i$ were cofinal. Hence $y = x$. Since $\nu \in y$ by definition, we conclude that $\nu \in x$. This proves the claim.

By the preceding paragraph, the Cuntz-Krieger $\Lambda$-family $T$ of Example 2.7 satisfies $\tau(T) \preceq T \tau$. Moreover, $T \not\preceq \text{ker}(\tau\gamma)$ for all $T$. So it suffices to show that $s_{\mu}^n \neq s_{\nu}^m$. We have $s_{\mu}^n \neq s_{\nu}^m$ by Remark 2.1. Since $\text{MCE}(\mu, \nu) \neq \emptyset$ and $\mu \neq \nu$, we have $d(\mu) \neq d(\nu)$. Hence there exists $z \in \Lambda^n$ such that $\mu(z) - \nu(z) = 1$. Now

$$\left(1 - \gamma_i\right)(s_{\mu}^n z - s_{\nu}^m z) = 2s_{\mu}^n z - s_{\nu}^m z \neq 0$$

and hence $s_{\mu}^n \neq s_{\nu}^m$, as required.

**Definition 3.5.** We say that a locally convex row-finite k-graph $\Lambda$ is cofinal if, for all $\mu, \nu \in \Lambda^0$, there exists $n \in \mathbb{N}$ such that $\text{MCE}(\mu, \nu) \neq \emptyset$ for all $\tau \in \Lambda^n \mu \Lambda$.

**Proposition 3.6.** Let $\Lambda$ be a locally convex row-finite k-graph. The following are equivalent:

1. $\Lambda$ is cofinal;
2. each ideal $I$ of $C^*(\Lambda)$ such that $s_\mu \in I$ for some $\mu \in \Lambda^0$ satisfies $I = C^*(\Lambda)$.

**Proof.** (1) $\Rightarrow$ (2). Fix an ideal $I$ and a vertex $\mu$ such that $s_\mu \in I$. Fix $\nu \in \Lambda^0$. Since $\Lambda$ is cofinal, there exists $n \in \mathbb{N}$ and paths $\{\mu_k : \Lambda \in \Lambda^n \mu \Lambda\}$ such that $\mu_k \in \text{MCE}(\mu, \nu)$ for each $\lambda \in \Lambda^n \mu \Lambda$.

Hence

$$s_\mu = \sum_{\lambda \in \Lambda^n \mu \Lambda} s_{\lambda} s_{\lambda}^* = \sum_{\lambda \in \Lambda^n \mu \Lambda} s_{\mu_k} s_{\mu_k}^* s_{\mu_k} s_{\mu_k}^* \in I.$$

(2) $\Rightarrow$ (1). We prove the contrapositive. Fix $\mu, \nu \in \Lambda^0$ and suppose that for each $n \in \mathbb{N}$ there exists $\lambda \in \Lambda^n \mu \Lambda$ such that $\text{MCE}(\lambda, \nu) = \emptyset$. As before, we may inductively choose paths $\mu_k : \Lambda \in \Lambda^n \mu \Lambda$ such that $\mu_k \in \text{MCE}(\mu, \nu)$ for all $m \leq n$ and such that for infinitely many (and hence all) $p > n$ there exists $\eta \in \Lambda^n \mu \Lambda \Lambda^p \mu \Lambda$ such that $\text{MCE}(\eta, \mu) = \emptyset$. The set $x = \{o : \mu \in oA \Lambda \text{ for some } n\}$ is a filter. It is an ultrafilter because $\Phi(x) = \emptyset$, and then since $\Phi(x) = \emptyset$, we have $\text{MCE}(\mu, \nu) = \emptyset$, and so there is no filter containing $x$ which also contains $\beta$. Let

$$\Lambda_\nu = \{x : \text{MCE}(\lambda, \nu) = \emptyset \text{ for all } x \in \lambda\}.$$

Then $\Phi(\Lambda_\nu) \subset \Phi(\Lambda_\nu)$ is invariant for the Cuntz-Krieger $\Lambda$-family $T$ of Example 2.7, so $S_\nu := T_T(\Lambda_\nu)$ is a Cuntz-Krieger $\Lambda$-family with $S_\nu = 0$ and $S_\mu = 0$. Hence $\text{ker}(\tau\gamma)$ is a proper ideal containing a vertex projection.

4. CONSTRUCTIONS OF k-GRAPHS

Constructions of $(k+1)$-graphs from $k$-graphs have appeared in many contexts beginning with the cartesian product construction of Kumjian and Pask, and including many authors since — we shall list them here, but we shall see a number of specific examples later in these notes. The notion of a k-morph was introduced by Kumjian-Pask-S as a unifying framework for these constructions.

**Definition 4.1.** A k-morph between k-graphs $\Lambda$ and $\Gamma$ (or a k $\Gamma$-morph for short) is a countable set $X$ equipped with maps $r : X \rightarrow \Lambda^0$ and $s : X \rightarrow \Lambda^0$ and a bijection $\theta : X \times X \rightarrow \Lambda^0 \times \Lambda^0$ such that whenever $\theta(x, y) = (\lambda, \mu)$ we have

1. $\theta(x) = \theta(y)$;
2. $\theta(x) = \theta(y)$;
3. $\theta(x) = \theta(y)$;
4. whenever, in addition, $\theta(x, \eta) = (\mu, z)$, we have $\Phi(\theta(x, \eta)) = \Phi(\mu, z)$.

if $\Lambda \subseteq \Gamma$, we call $X$ a $\Lambda$-endomorphism.

**Examples 4.2.** (1) Fix $k$-graphs $\Lambda, \Sigma$ and coverings $p : \Sigma \rightarrow \Lambda$ and $q : \Sigma \rightarrow \Gamma$; that is, degree-preserving functors which restrict to bijections on each $v \Sigma$ and $\Sigma v$. Let $X = \{x_n : n \in \mathbb{N}\}$, and define $x_n = \Phi(x_n)$, $r(x_n) = p(x_n)$, and $s(x_n) = q(x_n)$. Define $\theta : X \times X \rightarrow \Lambda^0 \times \Lambda^0$ by $\theta(x_n, x_m) = (p(x_n), x_{n+m})$. (To see that this makes sense observe that since $q$ is a covering, $\sigma$ can be recovered from $r(\sigma)$ and $q(\sigma)$.)

In the picture below, $\Lambda$ and $\Gamma$ are cycles of length 2 and 3 and $\Sigma$ is the common
covering cycle of length 6.

(2) Fix a k-graph Λ and an automorphism α of Λ. Let $X_α := \{ x_α : r ∈ \Lambda \}$ with $r = α, s = \text{id}, \theta(r|x_α, λ) = (α(λ), x_{α(λ)})$. Then $X_α$ is a $Λ$-endomorph. In fact, $X$ is precisely $X_α$ from (1).

Theorem 4.3. Let Λ and Γ be k-graphs.

(1) Let $X$ be a $Λ$-Γ-morph. There is a unique $(k+1)$-graph $Σ$, called the linking graph for $X$ admitting an isomorphism $ι_0 : X → Λ$ and a bijection $ι_0 : X → Σ^{k+1}$ such that $r(ι_0(x)) = ι_0(r(x))$ and $s(ι_0(x)) = ι_0(s(x))$ for all $x ∈ X$ and $ι_0(x)ι_0(y) = ι_0(x)ι_0(y)$ whenever $θ(x, y) = (λ, y)$.

(2) Let $Y$ be a $Λ$-endomorph. There is a unique $(k+1)$-graph $Λ × N$ admitting an isomorphism $ι_0 : \Lambda × N → Σ^{k+1}$ such that $r(ι_0(y)) = ι_0(r(y))$ and $s(ι_0(y)) = ι_0(s(y))$ for all $y ∈ Y$, and such that $ι_0(y)ι_0(y) = ι_0(r(y))ι_0(y)$ whenever $θ(y, y) = (y, y)$.

Proof. (1) Let $E = E_{Γ, Γ^0}$ be the k-coloured graph with colour map $c$ associated to $Λ$ and $Γ$. Define $F$ by $F_0 := E_0$ and $F_1 := E_1$, and $ι_0 : F_1 → F_0$ inherited from $Λ$ and $Γ$, and $ι_0$ agrees with $c$ on $E_1$. Define $E$ by $E_0 := Λ$ and $E_1 := \{ (x, y) : x, y ∈ X \}$, and $ι_0 : E_1 → F_0$ gives an isomorphism $ι_0 : Λ → Λ_0Σ_0\Lambda^0$. The unique assertion of Theorem 1.9 gives an isomorphism $ι_0 : Λ → Λ_0Σ_0\Lambda^0$, and similarly for $Γ$. The $(k+1)$-graph $Σ$ satisfies the desired factorisation regime by definition. Uniqueness of $Σ$ follows from another application of the uniqueness assertion of Theorem 1.9.

(2) The proof is basically the same as that of (1), except that $ι_0 : Λ → Σ$ maps onto $\{ σ : σ(σ) = 0 \}$ rather than $Λ^0Σ_0\Lambda^0$.

Examples 4.4. (1) The common covering of the 2-cycle and the 3-cycle by the 6-cycle above gives the following linking graph.

(2) Let Λ be the complete directed binary tree described with the vertices at level $n$ indexed by $2^nZ$ and an edge from the vertex $i$ at level $n$ to the vertex $j$ at level $n+1$ if $i$ is congruent to $j$ mod $2^{n-1}$. There is a unique automorphism $α$ which acts on the vertices at level $n$ by addition of 1 modulo $2^n$. The resulting endomorphism crossed product $Λ ×_{α} N$ has the following coloured graph.

Theorem 4.5. Let Λ and Γ be locally convex row-finite k-graphs, and let $X$ be a $Λ$-Γ-morph in which both $r$ and $s$ are surjective, and $r$ is finite-to-one. Let $Σ$ be the linking graph. Then $\sum_{ι_0(α)} s_{ι_0(α)}$ and $\sum_{ι_0(α)} s_{ι_0(α)}$ converge to full projections $P_Λ$ and $P_Γ$ in the multiplier algebra $M(C^*(Σ))$. The map $ι_0 : Λ → Σ^{k+1}$ determines an injective homomorphism $ι_0^* : C^*(Λ) → \mathcal{P}_Λ C^*(Σ) P_Λ$, and the map $ι_0^* : Λ → Σ^{k+1}$ determines an isomorphism $ι_0^* : C^*(Γ) → \mathcal{P}_Γ C^*(Σ) P_Γ$. 

*
Corollary 4.8. Let $\Lambda \to \Lambda$ be an automorphism. Then there is an automorphism $\tilde{\alpha}$ of $C^*(\Lambda)$ satisfying $\tilde{\alpha}(s_\omega) = s_{\alpha(\omega)}$ for all $\omega$, and $C^*(\Lambda \times \mathbb{R}) \cong C^*(\Lambda) \times \mathbb{R}$.

Proof sketch. Let $X = X_\sigma$. Since $\alpha$ is a bijection, $H_X$ is isomorphic to a vector-space to $C^*(\Lambda)$. This isomorphism carries the inner product on $H_X$ to the standard right inner-product $(\cdot, \cdot) = (\cdot, \cdot)_0$ on $C^*(\Lambda)$, so $H_X \cong C^*(\Lambda)(\times \mathbb{R})$ as a right Hilbert module.

By definition of $\tilde{\sigma}: X \to \Lambda \times \mathbb{R}$, the left action on $H_X$ is given by $(x, \omega) \in H_X \mapsto \tilde{\sigma}(x)(\cdot, \omega)_0 = (x, \omega)_0$, so $H_X$ is isomorphic as a right Hilbert bimodule to $C^*(\Lambda)$. Pimsner's theorem therefore shows that $H_{X_\tilde{\sigma}} \cong C^*(\Lambda) \times \mathbb{R}$; combined with Corollary 4.6, this proves the result. \(\square\)

Hookeandale. In fact, the assignments $X \to H_X$ and $\Lambda \to C^*(\Lambda)$ determine a contravariant functor from a category $M_k$ whose objects are $k$-graphs and whose morphisms are isomorphism classes of $k$-morphs (the fibered product of $k$-morphs determines a composition) to the category $C^*$ with $C^*$-algebras as objects and isomorphism classes of $C^*$-correspondences as morphisms. We can thus construct graphs of $k$-morphs: these are functors from 1-graphs to $M_k$. Indeed, this can be made to work for $k$-graphs of $k$-morphs, though in that instance more information is required than just the functor. We won't go into this as we are only interested in a special case.

Proposition 4.9. Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ be locally convex von-finite k-graphs, and let $X_i$ be a $k$-graph, $\Lambda_i$ with $r_i:\Lambda_i$ surjective and $r_i$ finite-to-one for each $1 \leq i \leq n$. Then for any $k$-graph $\Sigma$, there is a unique $k$-graph $\Sigma \to \Lambda_1 \times \cdots \times \Lambda_n$ such that the factorization property in $\Sigma$ is inherited from the $\Lambda_i$, and the bijections $\theta_i: X_i \times \Lambda_i \to X_i \times X_{i+1}$.

The maps $\theta_i$ determine injective homomorphisms $\tilde{\theta}_i: C^*(\Lambda_i) \to C^*(\Sigma)$, and $\tilde{\theta}: \bigcap_{i=1}^n \tilde{\theta}_i(s_\omega) = \tilde{\theta}(C^*(\Lambda))$.

Proof. The proof is almost identical to those of Theorems 4.3 and 4.5.

Remark 4.10. If $n = \infty$ in Proposition 4.9, it is still straightforward to establish the existence of the enveloping $(k + 1)$-graph $\Sigma$ and that $\tilde{\theta}(s_\omega) = \tilde{\theta}(C^*(\Lambda))$ for any $\omega$.

5. Rank-2 Bratteli diagrams and AT-algebras.

The results in this section are due to Paauw-Rachman-Rosam-S, though we have proved them in a very different manner to streamline arguments and highlight demonstrate how the $k$-morph construction can be used.

We will write $e_i$ for the 1-graph with vertices $\{v_i: v_i \in \mathbb{Z}/n\mathbb{Z}\}$ and edges $\{e_i: v_i \in \mathbb{Z}/n\mathbb{Z}\}$ with $s(e_i) = i$ and $t(e_i) = i+1$.

Fix, for the section, a sequence $\{\Lambda_n\}_{n=1}^\infty$ of $1$-graphs such that each $\Lambda_n$ is a graph such that each $\Lambda_n \cong \overset{\text{End}}{\text{auto}}(\Lambda_n)$ for each $n$. For each $\Lambda_n \geq 1$, and each pair $i$ with $i \leq m_n$ and $j \leq m_n$.
fix $c_{ij} \in \mathbb{N}$; we assume that for each $i$ there exists $j$ such that $c_{ij} \neq 0$ and that for each $j$ there exists $i$ such that $c_{ij} \neq 0$.

Whenever $c_{ij} \neq 0$, let $X_{ij} := \pi_i X_{ij}$ be the $k$-morphism of Example 4.2.1(1) for the canonical
coverings

\[ p : C_{ij} \rightarrow X_{ij} \text{ and } q : C_{ij} \rightarrow X_{i1} \text{ and } 1_{X_{ij}} \rightarrow \Lambda_{ij}. \]

For each $m, X^m := \coprod_{i \in I, j \in J} X^m_{ij}$ is a $\Lambda_{ij}$-morph.

As in Remark 4.10, there is a unique 2-graph $\Gamma$ such that $P^2 = \bigcup X^2_{ij}$, $P^3 = \bigcup X^3_{ij}$ and $P^4 = \bigcup X^4_{ij}$, where $\lambda$ are the factorisation rules determined by the bijections

\[ \theta_{ij} : X^2_{ij} \rightarrow \Lambda_{ij} \quad \text{and} \quad \psi_{ij} : X^3_{ij} \rightarrow \Lambda_{ij}. \]

We call $\Gamma$ a rank-2 Bratteli diagram.

To analyse $C^*(\Gamma)$, we first study its building blocks.

**Lemma 5.1.** For $n \geq 1$, $C_n(\Lambda) \cong M_n \otimes C(T)$.

**Proof.** For $t \in \mathbb{T} \setminus \{0\}$, define $t_n := \theta_{n+1, n} \otimes 1 \in M_n \otimes C(T)$, and for $t_n := \theta_{n, n} \otimes 1$ for each $n \in \mathbb{Z}$. Hence the gauge-invariant uniqueness theorem implies that $\psi$ is an isomorphism.

**Corollary 5.2.** For each $N \in \mathbb{N}$, let $\Gamma(0, N) := \bigcup_{i=1}^{N} \Lambda_i \mathbin{\bigcap} \bigcup_{i=1}^{N} \Lambda_i$. Then $C^*(\Gamma) \cong \bigoplus_{i=1}^{\infty} M_i \otimes C(T)$.

**Proof.** Proposition 4.9 and

\[ C^*(\Gamma(0, N)) \cong M_{N+1} \otimes C(T) \cong C^*(\Lambda_N). \]

**Lemma 5.1 then implies that $C^*(\Gamma(0, N)) \cong \bigoplus_{i=1}^{\infty} M_i \otimes C(T)$.

Since $C^*_{*}(\Lambda_N)$ is unital (with identity $\Sigma_{i=1}^{N} \Lambda_i$), the result follows because of the deep result due to一幕 Lesseppe.

**Proposition 5.3.** For each $N \in \mathbb{N}$, let $P_N := \sum_{m=1}^{\infty} \sum_{n=1}^{N} \psi_{ij} \in C^*(\Gamma)$. Then $P_N C^*(\Gamma) P_N \cong C^*(\Lambda_N)$.

**Proof.** Similar to previous proofs.

In particular, $C^*(\Gamma)$ is an AT-algebra.

**Theorem 5.4.** Suppose that

1. for all $w \in \Lambda_0$ there exists $n \in \mathbb{N}$ such that $w^N_0 \neq \emptyset$ for all $n \in \Lambda_0$; and
2. for all $l \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $|\Lambda_{ij}| \geq l$ whenever $c_{ij} \neq 0$.

Then $C^*(\Gamma)$ is simple and has real rank $0$.

**Proof.** For simplicity, we just need to show that $\Gamma$ is cofinal and aperiodic. — Corollary 3.7 does the rest.

For cofinality, fix $v, w \in \Gamma^0$. By (1), there exists $n \in \mathbb{N}$ such that $w^N_0 \neq \emptyset$ for all $u \in \Lambda_0^n_\infty$. Since the $X_{ij}$ are $k$-morphs, their source maps are surjective, we then have $w^N_0 \neq \emptyset$ for all $n \in \Lambda_0^n_\infty_\infty$. Let $m \in \mathbb{N}$ be the integer such that $w^m \in \Gamma^0$, and let $N := \max(m, n)$. Then $s^m_0 \Gamma^m \cap \Lambda_0^N \neq \emptyset$, and hence $\Gamma_0(s^m_0 \Gamma^m \cap \Lambda_0^N \neq \emptyset)$ and $\Lambda_0^N \neq \emptyset$. For aperiodicity, fix distinct paths $\mu, \nu \in \Gamma$. If $d(\mu) = d(\nu)$, then $s(\mu) \neq s(\nu)$ or $s(\mu) \neq s(\nu)$, and $\tau := \mu$ satisfies $MCE(\mu, \tau) = 0$. Moreover, if $d(\mu) \neq d(\nu)$, then either $\mu(\mu) \neq \nu(\nu)$ or $s(\mu) \neq s(\nu)$ since one path or the other must be in different levels of $\Gamma^0$. So suppose that $\mu(\mu) = \nu(\nu)$, and $\Lambda_0^N \neq \emptyset$ and hence $d(\mu) = d(\nu)$ and that $d(\mu) \neq d(\nu)$, so $d(\mu) \neq d(\nu)$. Factorise $\mu = \mu' \circ \nu$ and $\nu = \nu' \circ \nu$ where $d(\mu') = d(\nu') = 0 = d(\mu')_2 = d(\nu')_2$. Using (2), fix $m \geq n + 1$ such that $|\Lambda_{ij}| > |\Lambda_{ij}| - |\Lambda_{ij}|$ for all $i, j, k \in \mathbb{N}$ such that $\mu_0 \neq 0$. Let $\tau$ be any element of $s(\mu(\mu))$. By definition of the $X_{ij}$, we may identify each with $Z / \Lambda_0^N \otimes Z$ so we can identify $\tau$ with a sequence $\beta_{\mu, \nu} \in \Lambda_0$, where each $\beta_{\mu, \nu} \in \mathbb{Z} / \Lambda_0^N \otimes \mathbb{Z}$, for each $i, j$.

In particular, $\beta_{\mu, \nu} \in \mathbb{Z} / \Lambda_0^N \otimes \mathbb{Z}$ for each $i, j$ and by choice of $m$ it follows that $\mu_0 = \mu \circ \nu$, $\nu_0 = \nu_0 \circ \nu$, $\nu_0 \neq 0$, $\mu_0 \neq 0$, and $\mu(\mu) \neq \nu(\nu)$.

For some $\mu', \nu'$. In particular,

\[
\beta(\mu') = c_\beta(\mu', \nu') = c_\beta(\mu_0 \nu, \mu \nu_0) \neq \mu \circ \nu = \nu \circ \nu,
\]

so $MCE(\mu, \nu) = 0$.

It remains to show that $C^*(\Gamma)$ has real rank 0. To do this we apply a powerful result of Blackadar-Bratteli-Elliott-Kirjanov which says that a simple AT algebra has real rank 0 only if and only if projections separate tracial states. For this, fix a trace $\tau$ on $C^*(\Gamma)$. For paths $\alpha, \beta \in \Gamma$ with $d(\alpha) = d(\beta) = 0$ and a path $\mu$ such that $d(\mu) = 0$, and suppose that $\tau(s_0 \mu) = 0$. Then $s(\tau(s_0 \mu) = 0$, forcing $s_0 \mu = 0$. Since $d(\mu) = 0$, each of $\mu$ and $s(\mu)$ belong to the same $\Lambda_0^N$, and it follows that $d(\mu) = d(\beta)$, and then $s_0 \mu = 0$ forces $\mu = \beta$ and $\mu = r$ is a cycle. Choose $m > n$ such that $\Lambda_{ij}^m > |\mu|$. Then $\tau(s_0 \mu) = 0$.

\[ 0 = \tau(s_0 \mu) = \sum_{\nu \in \Lambda_0^m} \tau(s_0 \nu) \tau(s_0 \nu) \]

where each $\nu = \tau(\mu' \circ \nu)$ with $d(\mu') = d(\nu)$, and this forces $\nu = \tau$ for all $\nu$. By choice of $m$, this forces $d(\mu) = 0$. A similar argument applies to show that $\tau(s_0 \nu) = 0$, forces $\mu = \alpha$ and $d(\mu) = 0$. It follows that $C^*(\Gamma)$ is spanned by elements of the form $s_0 \mu_0 \nu_0 \nu_0 \nu_0 \nu_0 \nu_0 \nu_0$ and that it follows if $\tau_1, \tau_2$ agree on all elements of the form $s_0 \mu_0 \nu_0 \nu_0 \nu_0 \nu_0 \nu_0 \nu_0$ then they are equal. In particular, projections separate tracial states as required.
Theorem 5.5. Let $E_0$ be the directed graph with one vertex $v_{0,j}$ for each $\lambda_0,j$ and $[m_0]E_0[m_{0,1}] = [X^0_0]/[A^0_0]$, and let $E_1$ be the directed graph with one vertex $v_{1,j}$ for each $\lambda_1,j$ and $[m_1]E_1[m_{1,1}] = [X^1_1]/[A^1_1]$. Then $K_0(C^*(\Gamma))$ is the dimension group associated to the Bratteli diagram $E_0$, and $K_1(C^*(\Gamma))$ is group-isomorphic to the dimension group associated to the Bratteli diagram $E_1$.

Proof sketch. Let $A := \{(\lambda_1 : d(\lambda_1) = 0)\}$ regarded as a 1-graph. So $A^1 = \bigsqcup_{n=0}^\infty X^n$. The map $v_0 \mapsto v_{1,n}$ is a bijection of each $X^n_0$ and determines an automorphism of $A$. It is straightforward to see that $\Gamma \cong A \times_{X^0_0} N_0$, so Corollary 4.8 implies that $C^*(\Gamma) \cong C^*(A) \times_\pi Z$.

A theorem of Drinen shows that $C^*(A)$ is Morita equivalent to the AF algebra with Bratteli diagram $A$. The Pasnau-Voirolnes exact sequence in $K$-theory then implies that $K_0(C^*(\Gamma)) = \text{coker}(1 - \tilde{\alpha})$ and $K_1(C^*(\Gamma)) = \text{ker}(1 - \tilde{\alpha})$.

To describe the $K$-theory of $C^*(A)$, recall that $K_i(M_n(A)) = (\mathbb{Z},(0))$ with generator $[p]$ for any minimal projection $p$. Hence

$$K_i(C^*(A)) = 0 \quad \text{and} \quad K_0(C^*(A)) = \varprojlim \bigoplus_{n \in \mathbb{N}} Z[s_n],$$

with linking maps determined by

$$[s_n] = \sum_{a_0 \in A_0} [a_0 s_n a_0] = \sum_{a_0 \in A_0} [a_0 A^0 a_0] \sum_{a_1 \in A_1} [a_1 A^1 a_0] [s_{n+1}].$$

The automorphism $\tilde{\alpha}$ permutes the $s_n$ for $n \in a$ in a given $A^0_0$. So ker$(1 - \tilde{\alpha})$ consists of functions which are constant on cycles. That is

$$\ker(1 - \tilde{\alpha} + \text{proj}) \cap \bigoplus_{n \in \mathbb{N}} Z[s_n] = \bigoplus_{n \in \mathbb{N}} Z[P_n],$$

where $P_n = \sum_{a \in A^0_0} s_a$. Relation (CK) gives

$$[P_n] = \sum_{a \in A^0_0} \sum_{b \in A^0_0} [s_{n+1}] = \sum_{j=1}^{n+1} |A^0_{n+1} A^1_{n+1} A^0_{n+1}| [P_{n+1,j}],$$

Continuity of $K$-theory then establishes the formula for $K_1(C^*(\Gamma))$.

Similarly, for each $n, i$, the classes $[a_n] \in C^*(A)$ where $a_i \in A^0_0$ are all equivalent modulo the image of $(1 - \tilde{\alpha})$. Hence coker$(1 - \tilde{\alpha}) \cap \bigoplus_{n \in \mathbb{N}} Z[s_n] = \bigoplus_{n \in \mathbb{N}} Z[s_{n+1}]$ where $(n, i) \mapsto s_{n,i}$ is a fixed choice of representative for each $A^0_{n,i}$. The Cuntz-Krieger relations for $A$ show that in coker$(1 - \tilde{\alpha})$,

$$[A^0_{n,i} s_{n,i}] = [P_{n+1,j}] = \sum_{j=1}^{n+1} [A^0_{n+1} A^1_{n+1} A^0_{n+1}] [s_{n+1,j}].$$

Continuity of $K$-theory once again establishes the formula for $K_0(C^*(\Gamma))$.

Examples 5.6. (1) For the rank-2 Bratteli diagram $\Gamma$ with coloured graph

the graphs $E_1$ and $E_2$ are

where $E_0$ and $E_1$ are isomorphic to the 1-graph obtained by deleting the loops. Hence $K_0(C^*(\Gamma)) = \mathbb{Z} \oplus \mathbb{Z}$.

so we have $K_*(C^*(\Gamma)) = (\mathbb{Z},(0))$ and hence $C^*(\Gamma)$ is stably isomorphic to the $2^\infty$ Bunce-Deddens algebra.

(2) For the rank-2 Bratteli diagram $\Gamma$ with coloured graph

both $E_0$ and $E_1$ are isomorphic to the 1-graph obtained by deleting the loops. Hence $K_0(C^*(\Gamma)) = \mathbb{Z} \oplus \mathbb{Z}$.

Hence $K_*(C^*(\Gamma)) = \mathbb{Z} \oplus \mathbb{Z}$.
(Z + 0Z, Z²), and it follows that C*(Λ) is Morita equivalent to the irrational rotation algebra for rotation θ.

6. COACTIONS, CROSS-PRODUCTS AND COVERINGS

The connection between skew products and coaction crossed-products was first established for graph C*-algebras by Kaliszewski-Quigg-Raeburn and was extended to k-graphs by Pask-Quigg-Raeburn.

**Definition 6.1.** Let Λ be a k-graph, and let c : Λ → G be a functor into a discrete group G. The skew-product k-graph Λ ×c G is given by (Λ ×c G)n = Λn × G with
\[ r(λ, g) = (r(λ), c(λ)g), \quad s(λ, g) = (s(λ), y) \quad \text{and} \quad (λ, c(μ)g)(μ, y) = (λμ, y). \]
It is straightforward to see that (Λ ×c G)G = ΛG × G.

**Example 6.2.** Let E be the l-graph with a single vertex v and a single edge e. The following are, from left to right, the skew-product graphs for the functors determined by c₁(e) = 1 ∈ Z, c₂(e) = 2 ∈ Z and c₂(e) = [1] ∈ Z/2Z.

**Theorem 6.3.** Let Λ be a locally convex row-finite k-graph and let c : Λ → G be a functor into a discrete group. Then there is a coaction δ of G on C*(Λ) determined by δ(λ) = s(λ) ⊗ c(λ) for each λ ∈ Λ. Furthermore, C*(Λ ×c G) ≅ C*(Λ) × G via an isomorphism which carries e(λ, y) to (s(λ), y), where e : C*(Λ) → C*(Λ ×c G) is the canonical inclusion, and the e y are the images of the indicator functions χx ∈ cG(G).

**Proof.** Define t : Λ → C*(Λ) ⊗ C*(G) by tₖ = sₖ ⊗ c(λ). Since c is a cocycle, t is a cocycle. So t satisfies (CK1). If s(μ, g) = r(μ, v), then g = c(μ)h, and
\[ tₖₗₙ = (sₖ ⊗ c(μ))(sₗ ⊗ c(ν)) = sₖₗₙ c(μ) c(ν) = tₗₙ, \]
since s satisfies (CK2) and c is a cocycle. So t satisfies (CK2). For λ ∈ Λ, we have
\[ tₖₗₙ = sₖₗₚ ⊗ (c(μ))(c(λ))r(λ) = sₖₗₚ ⊗ 1 = tₖₗₚ, \]
because s satisfies (CK3) and the c(μ) are unitaries. Similarly, for v ∈ Λ and n ∈ N₀,
\[ \sum_{λ ∈ Λⁿ} tₖₚₗₙ = \sum_{λ ∈ Λⁿ} sₖₜₖₚₗₙ ⊗ (c(μ))(c(λ))r(λ) = \left( \sum_{λ ∈ Λⁿ} sₕₕₜₖₚₗₙ \right) ⊗ 1 = sₖₚₗₙ, \]
The universal property of C*(Λ) yields a homomorphism \[ δ : C*(Λ) → C*(Λ) ⊗ C*(G) \]
such that \[ δ(sₖₜₖₚₗₙ) = sₖₚₗₙ ⊗ c(μ) \]
and \[ (λ, c(μ)g)(μ, y) = (λμ, y). \]

**Lemma 6.5.** Let Λ be a locally convex row-finite k-graph. Then C*(Λ ×c Z) is AF.

**Proof.** Fix a finite subset F of Λ ×c Z. Let DF := \{m ∈ NF : (μ, m) ∈ F \text{ for some } μ ∈ Λ\}. Let N ⊆ VDF ∈ NF, and let \[ F := \bigcup_{(μ, m) ∈ F} (μ, m) : m ∈ \mathbb{N}, μ ∈ Λ(n)^{AP^*} \]
be the set of elements of F. It is straightforward to check that \( F \) is a finite-dimensional subalgebra of \( C^*(Λ ×c Z²) \) which contains \( C^*(sₕₙ, n, μ, n) \). Since \( C^*(Λ ×c Z²) \) is the increasing union of the subalgebras \( sₕₙ, n, μ, n \), the result follows.
Corollary 6.6. Let $\Lambda$ be a locally convex row-finite $k$-graph. Then $C^*(\Lambda)$ is stably isomorphic to a crossed product of an AF algebra by $\mathbb{Z}^\Lambda$.

Proof. Let $\hat{e}$ be the dual action of $\mathbb{Z}^\Lambda$ on $C^*(\Lambda) \rtimes_\Lambda \mathbb{Z}^\Lambda$ given by $\hat{e}(n)(x_g) = e(n)(x_{g+e})$. By Takai duality, $C^*(\Lambda) \rtimes_\Lambda \mathbb{Z}^\Lambda \cong C^*(\Lambda \times_{\Lambda} \mathbb{Z}^\Lambda)$ is AF, and the result follows.

We finish with a third take on the Bunce-DeRidder algebra of type $\mathfrak{A}$. We have seen it as an AF-algebra and as a crossed product of an AF-algebra by $\mathbb{Z}$. Now we will see it as a coaction crossed product by the profinite group of $2$-adic numbers.

Fix a discrete group $\mathbb{G}$ and a sequence $G = H_0 \triangleright H_1 \triangleright H_2 \triangleright \cdots$ of finitely index normal subgroups of $\mathbb{G}$. For each $n$, let $G_n := G/H_n$. We obtain a projective system

$$\cdots \rightarrow G_n \rightarrow G_{n-1} \rightarrow G_{n-2} \rightarrow \cdots$$

of finite groups. Fix a locally convex row-finite $k$-graph $\Lambda$ and a sequence of cocycles $c_n : \Lambda \rightarrow G_n$ such that $c_n(\gamma(\alpha)) = c_{n-1}(\alpha)$ for all $\alpha, n$. We will use $[g]_n$ for the class of $g$ in $G_n$.

Each $\Gamma_n := \Lambda \times_{c_n} G_n$ is a $k$-graph, and the map $\phi_n : \Gamma_n \rightarrow \Gamma_{n-1}$ given by $\phi_n(\lambda, [g]_n) = (\lambda, [g]_{n-1})$ is a cocovering. Let $\Sigma$ be the infinite $(k+1)$-graph of Remark 4.10 obtained from the tower of $k$-morphs $X_{\Gamma_n}$. For each $n$, let $P_n := \sum_{\gamma \in \Pi_n} \delta_\gamma$ be the sum of the counit $\delta_\gamma$ over $\Pi_n$. We have $P_n \in C^*(\Sigma)$.

Definition. We have $P_n C^*(\Sigma) P_n \cong \lim_n C^*(\Gamma_n)$ under coactions satisfying $S_{\alpha(\beta)} \rightarrow S_{\alpha(\beta)} \rightarrow S_{\alpha(\beta)} \rightarrow \cdots$.

Lemma 6.7. We have $P_n C^*(\Sigma) P_n \cong \lim_n C^*(\Gamma_n)$ under coactions satisfying $S_{\alpha(\beta)} \rightarrow S_{\alpha(\beta)} \rightarrow S_{\alpha(\beta)} \rightarrow \cdots$.

Lemma 6.8. For each $n$, let $\delta_n$ be the coaction of $G_n$ on $C^*(\Lambda)$ determined by $\delta_n(s) = s_n \otimes c_n(\alpha)$. Then $P_n C^*(\Sigma) P_n \cong \lim_n C^*(\Lambda) \rtimes_\Lambda G_n$ under coactions satisfying $s_{\alpha(\beta)} \rightarrow s_{\alpha(\beta)} \rightarrow s_{\alpha(\beta)} \rightarrow \cdots$.

Proof. Combine Lemma 6.7 and Theorem 6.3.

Theorem 6.9. There is a coaction $\delta_n$ of $G_n$ on $C^*(\Lambda)$ satisfying $\delta_n(s) = s_n \otimes c_n(\alpha)$ for all $\alpha, n$. Moreover,

$$C^*(\Lambda) \times_{G_n} G_n \cong \lim_n C^*(\Lambda) \times_{\Lambda} G_n,$$

and in particular is Morita equivalent to $C^*(\Sigma)$.

Proof. More or less the same argument as in Theorem 6.3 shows that there is a coaction $\delta_n : C^*(\Lambda) \rightarrow C^*(\Lambda) \rtimes C^*(\Gamma_n)$ satisfying $\delta_n(s) = s_n \otimes c_n(\alpha)$ — nondegeneracy as a coaction follows from nondegeneracy as a homomorphism by a result of Laca and Raeburn because $G_n$ is amenable. Since $C^*(\Gamma_n) = \lim_n C^*(\Gamma_n)$, for each $n$ the map $\delta_n : s_{\alpha(\beta)} \rightarrow s_{\alpha(\beta)} \otimes c_n(\delta(\beta))$ determines a homomorphism of $G_n C^*(\Gamma_n)$ into $C^*(\Gamma_n)$. Thus Theorem 6.3 implies that $\Psi_{\alpha(\beta)} : \lim_n C^*(\Lambda) \rtimes C^*(\Gamma_n)$ determines a Cuntz-Krieger algebra. The universal property of $C^*(\Lambda) \rtimes C^*(\Lambda)$ implies that there is an action $\beta : T^\Lambda$ on $C^*(\Lambda) \rtimes C^*(\Lambda)$ which fixes the copy of $C^*(\Gamma_n)$ and satisfies $\beta_\Lambda \circ \alpha = \alpha \circ \gamma$, and it follows that $\beta_\Lambda \circ \alpha = \alpha \circ \gamma$, for each $n$. The gauge-invariant uniqueness theorem therefore implies that the $\alpha_n$ are injective. The universal property of $\lim_n C^*(\Lambda) \rtimes C^*(\Lambda)$ gives $r_n : \lim_n C^*(\Lambda) \rtimes C^*(\Lambda) \rightarrow C^*(\Lambda) \times_{G_n} G_n$ for each $n \in \mathbb{N}$. The universal property of $C^*(\Lambda) \times_{G_n} G_n$ implies that there is an action $\beta : \Gamma^\Lambda \times_{G_n} \Gamma^\Lambda$ on $C^*(\Lambda) \times_{G_n} G_n$, which fixes the copy of $C^*(\Lambda)$ and satisfies $\beta_n \circ \alpha_n = \alpha_n \circ \gamma_n$, for each $n$. The gauge-invariant uniqueness theorem therefore implies that the $\alpha_n$ are injective. The universal property of $\lim_n C^*(\Lambda) \rtimes C^*(\Lambda)$ gives $r_n : \lim_n C^*(\Lambda) \rtimes C^*(\Lambda) \rightarrow C^*(\Lambda) \times_{G_n} G_n$, and $r_n$ is injective because the $\alpha_n$ are all injective. It is surjective because the $\chi_{\alpha_n(\beta)} \otimes \chi_{\alpha_n(\beta)}$ span a dense subalgebra of $C^*(\Lambda)$, so the image of $r_n$ contains all the generators of $C^*(\Lambda) \times_{G_n} G_n$. Remark 4.10 implies that $P_n$ is full, so the Morita equivalence of $C^*(\Lambda) \rtimes C^*(\Lambda)$ follows from Corollary 6.8.

Remark 6.10. In fact the continuity of coaction crossed products by projective systems of finite discrete groups is a general phenomenon, but the proof is more involved.

Example 6.11. Let $\Lambda$ be the 1-graph with one edge $e$ and one vertex $v$. Let $\mathbb{G} := \mathbb{Z}$ and $H_n = \mathbb{Z}^n$ for all $n$, so $G_n = \mathbb{G}/H_n$ is the finite cyclic group of order $2^n$ for all $n$. 

\[ \text{Not } \text{AI}^*(\mathbb{G}) \text{ because the projective limit is compact.} \]
Hence $G_{\infty} = \mathbb{Z}_2$ the group of 2-adic numbers. Then the 2-graph $\Sigma$ is

which is precisely the rank-2 Bratteli diagram corresponding to the $2^\infty$ Bunce-Deddens algebra as described in Example 5.6(1). By Theorem 6.9 and Lemma 5.1, we have $\mathcal{P}_0 C_0(\Sigma) \mathcal{P}_0 \cong C(T) \times_{\mathbb{Z}_2} \mathbb{Z}_2$.