A COLLECTION OF DEFINITIONS AND NOTATION

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This summary of definitions and notation is to accompany my lectures at the XV ELAM in Cordoba, 23-27 May 2011. These are not all assumed to be familiar; the majority will be introduced in the lectures.

1. Rings

All rings are assumed to be associative but not necessarily commutative or unital.

The standard *unitification* of a ring R is the unital ring $\widetilde{R}$ based on the abelian group $\mathbb{Z} \oplus R$, with multiplication rule $(m, a)(n, b) = (mn, mb + na + ab)$.

An *idempotent* in a ring is any element $e = e^2$.

A ring R *has local units* iff for each finite set $X \subseteq R$, there exists an idempotent $e \in R$ such that $X \subseteq eRe$. Note: eRe is a unital ring with identity element e.

$M_n(R)$ denotes the ring of all $n \times n$ matrices over a ring R. For all n, identify $M_n(R)$ with the non-unital subring of $M_{n+1}(R)$ consisting of those matrices whose $(n+1)$st row and $(n+1)$st column are zero, and set $M_\infty(R) = \bigcup_{n=1}^\infty M_n(R)$. This is a ring with local units.

A ring R is (*von Neumann*) *regular* iff for all $x \in R$, there exists $y \in R$ such that $xyx = x$. Further, R is *unit-regular* iff for all $x \in R$, there exists a unit (= invertible element) $u \in R$ such that $xux = x$.

Idempotents e and f in a ring R are *orthogonal* iff $ef = fe = 0$. The idempotent e is *dominated by* f, written $e \leq f$, iff $e = ef = fe$. Next, e and f are (*Murray-von Neumann*) *equivalent*, denoted $e \sim f$, iff there exist $x, y \in R$ such that $xy = e$ and $yx = f$. Lemma: $\sim$ is an equivalence relation. Finally, e is *subequivalent to* f, written $e \lesssim f$, iff there exists an idempotent $g \in R$ such that $e \sim g \leq f$. Lemma: $e \lesssim f$ iff there exist $x, y \in R$ such that $xfy = e$.

An idempotent $e \in R$ is *infinite* iff $e = e_1 + e_2$ for some orthogonal idempotents e_1 and e_2 such that $e_1 \sim e$ and $e_2 \neq 0$; otherwise, e is called *finite* (or *Murray-von Neumann finite*). It is common to call e *properly infinite* (respectively, *purely infinite*) iff there exist orthogonal idempotents $e_1, e_2 \in R$ such that each $e_i \sim e$ and $e_1, e_2 \leq e$ (respectively, $e_1 + e_2 = e$). Warning: In some literature, "properly infinite" is defined by the second set of conditions.

A unital ring R is *directly finite* (or *von Neumann finite*, or *Dedekind finite*) iff all one-sided inverses in R are two-sided: $xy = 1$ implies $yx = 1$. Observation: R is directly finite iff the idempotent 1_R is finite. The ring R is *stably finite* iff all the matrix rings $M_n(R)$ are directly finite.

An element z in a ring R is a *zero-divisor* iff there is some nonzero element $a \in R$ such that $za = 0$ or $az = 0$. Thus, z is a *non-zero-divisor* in R iff ($za = 0 \implies a = 0$) and ($az = 0 \implies a = 0$) for all $a \in R$.

A *classical right ring of fractions* (or *classical right quotient ring*) for a ring R is a unital ring S such that

(1) R is a subring of S.
(2) Every non-zero-divisor in R is invertible in S.
(3) Every element of S can be written in the form ab^{-1} with $a, b \in R$ and b a non-zero-divisor.

If S exists, it is unique up to isomorphism. A *classical ring of fractions* for R is a ring which is both a right and a left classical ring of fractions.

A ring R is *right self-injective* iff the right R-module R_R is injective.

The *right singular ideal* of a ring R is $Z_r(R) = Z(R_R)$ (the singular submodule of the module R_R); it is a two-sided ideal of R. The ring R is a *right nonsingular ring* iff $Z_r(R) = 0$.

Suppose R is a right nonsingular ring. A *maximal right quotient ring of* R is a ring Q such that

(1) R is a subring of Q.
(2) R is an essential right R-submodule of Q, that is, R_R is essential in Q_R.
(3) Whenever R is a subring of a ring S such that R_R is essential in S_R, the identity map on R extends to a ring homomorphism $S \to Q$.

Theorem: Every unital right nonsingular ring has a maximal right quotient ring, and it is unique up to isomorphism.

A *unital exchange ring* is a unital ring R such that the module R_R has the finite exchange property. Theorem: R is an exchange ring iff ${}_RR$ has the finite exchange property. Theorem: R is an exchange ring iff for all right (or left) ideals I and J of R with $I + J = R$, there exists an idempotent $e \in R$ such that $e \in I$ and $1 - e \in J$. Theorem: If R is an exchange ring, then all finitely generated projective R-modules have the finite exchange property.

A non-unital ring R is an *exchange ring* iff for each $x \in R$, there exist an idempotent $e \in R$ and elements $r, s \in R$ such that $e = xr = x + s - xs$.

A *matricial algebra* over a field F is any F-algebra isomorphic to $\bigoplus_{i=1}^{k} M_{n_i}(F)$, for some $n_i \in \mathbb{N}$. An *ultramatricial F-algebra* is any F-algebra $A = \bigcup_{n=1}^{\infty} A_n$ where $A_1 \subseteq A_2 \subseteq \cdots$ is a countable increasing sequence of matricial F-subalgebras; equivalently, A is the direct limit of a countable sequence $A_1 \to A_2 \to \cdots$ of matricial F-algebras and F-algebra homomorphisms.

A *simple ring* is a nonzero ring which has no proper nonzero ideals. A *purely infinite simple* ring (in the algebraic sense) is a simple ring in which each nonzero right (equivalently, left) ideal contains an infinite idempotent. Theorem: A unital simple ring R is purely infinite (algebraically) iff R is not a division ring and for each nonzero $a \in R$, there exist $x, y \in R$ such that $xay = 1$. In general, a ring R is *purely infinite* (algebraically) iff

(1) No quotient R/I is a division ring.
(2) Whenever $a \in R$ and $b \in RaR$, there exist $x, y \in R$ such that $xay = b$.

2. *-ALGEBRAS

An *involution* on a ring R is an additive map $* : R \to R$ such that $(ab)^* = b^*a^*$ and $(a^*)^* = a$ for all $a, b \in R$. (If R is unital, automatically $1^* = 1$.) A **-ring* is a ring equipped with a particular involution *. An element a in a *-ring is *self-adjoint* iff $a^* = a$.

A *projection* in a *-ring is any self-adjoint idempotent: $p = p^* = p^2$.

A *real *-algebra* is a *-ring A which is also an $\mathbb{R}$-algebra such that $(\alpha a)^* = \alpha a^*$ for all $\alpha \in \mathbb{R}$ and $a \in A$. A *complex *-algebra* is a *-ring A which is also a $\mathbb{C}$-algebra such that $(\alpha a)^* = \overline{\alpha} a^*$ for all $\alpha \in \mathbb{C}$ and $a \in A$, where $\overline{\alpha}$ denotes the complex conjugate of α.

When A is a (real or complex) *-algebra, it is usual to equip the matrix algebras $M_n(A)$ with the *conjugate transpose involution*, given by the rule $(a_{ij})^* = (a^*_{ji})$.

3. Algebras of operators

Let H be a real or complex Hilbert space, with inner product $(-,-)$. A linear map $T : H \to H$ is *bounded* iff there exists a real number $r \geq 0$ such that $\|T(x)\| \leq r\|x\|$ for all $x \in H$. Lemma: There is a least such r. It is called the *operator norm* of T, denoted $\|T\|$. Lemma: A linear map on H is bounded iff it is continuous. The set of all bounded linear maps on H is denoted $B(H)$ (or $\mathcal{L}(H)$); it is a real or complex algebra, and a Banach space with respect to the operator norm. Lemma: For any $T \in B(H)$, there is a unique $T^* \in B(H)$ such that $(T(x), y) = (x, T^*(y))$ for all $x, y \in H$. It is called the *adjoint* of T. The adjoint map $* : B(H) \to B(H)$ is an involution, so $B(H)$ is a real or complex *-algebra. The *strong operator topology* on $B(H)$ is the *topology of pointwise convergence*: a net (T_α) converges to an operator T in this topology iff $T_\alpha(x) \to T(x)$ for all $x \in H$.

A *W*-algebra* (or *von Neumann algebra*) *on H* is any *-subalgebra of $B(H)$ which is closed in the strong operator topology. A *C*-algebra on H* is any *-subalgebra of $B(H)$ which is closed in the norm topology. Observation: Every W*-algebra on H is also a C*-algebra on H.

An (*abstract*) *C*-algebra* is any real or complex *-algebra A equipped with a norm $\|\cdot\|$ such that

1. A is complete with respect to $\|\cdot\|$ (i.e., A is a Banach space).
2. $\|ab\| \leq \|a\| \cdot \|b\|$ for all $a, b \in A$.
3. $\|aa^*\| = \|a\|^2$ for all $a \in A$.

Gelfand-Naimark-Segal Theorem: Every C*-algebra is isomorphic (as a normed *-algebra) to a C*-algebra on some Hilbert space.

An *AW*-algebra* is a C*-algebra A such that for any subset $X \subseteq A$, there is a projection $p \in A$ with $pA = \{a \in A \mid xa = 0 \text{ for all } x \in X\}$. Theorem: Every W*-algebra is an AW*-algebra.

An *AF* (= *approximately finite dimensional*) C*-algebra is any C*-algebra A containing finite dimensional *-subalgebras $A_1 \subseteq A_2 \subseteq \cdots$ such that $\bigcup_{n=1}^\infty A_n$ is dense in A; equivalently, A is the C*-direct limit of a countable sequence $A_1 \to A_2 \to \cdots$ of finite dimensional C*-algebras and C*-algebra homomorphisms.

A complex C*-algebra A has *real rank zero* iff the set of $\mathbb{R}$-linear combinations of orthogonal projections is dense in the set of self-adjoint elements of A.

The *positive cone* of a C*-algebra A is the set $A_+ = \{aa^* \mid a \in A\}$. Theorem: A_+ is closed under addition. *Cuntz subequivalence* for elements $a, b \in A_+$ is defined by $b \precsim a$ iff there exists a sequence (x_n) in A such that $x_n a x_n^* \to b$. A complex C*-algebra A is *purely infinite* (in the C* sense) iff

1. There are no nonzero C*-algebra homomorphisms $A \to \mathbb{C}$.
2. $b \in \overline{AaA}$ implies $b \precsim a$, for all $a, b \in A_+$.

Theorem: A unital simple complex C*-algebra is purely infinite algebraically iff it is purely infinite in the C* sense.

4. Ordered sets

A *pre-order* on a set X is a relation $\leq$ on X which is

(1) reflexive: $x \leq x$ for all $x \in X$.
(2) transitive: $x \leq y \leq z$ implies $x \leq z$, for all $x, y, z \in X$.

It is a *partial order* if it is also

(3) antisymmetric: $x \leq y \leq x$ implies $x = y$, for all $x, y \in X$.

A *poset* (= *partially ordered set*) is a set X equipped with a particular partial order.

An *upper bound* for a subset Y of a poset X is any element $b \in X$ such that $y \leq b$ for all $y \in Y$. A *supremum* for Y is a *least upper bound*: an upper bound s for Y such that $s \leq b$ for all upper bounds b of Y. A supremum is unique if it exists, and is then denoted $\bigvee Y$ or $\sup Y$. The supremum of a two-element set $\{y, z\}$ is written $y \vee z$. *Lower bounds* and *infima* (= *greatest lower bounds*) are defined dually, and are denoted $\bigwedge Y$ or $\inf Y$ or $y \wedge z$. A *greatest element* for X is $\bigvee X$. If it exists, it is denoted 1 or $\top$. A *least element* for X is $\bigwedge X$. If it exists, it is denoted 0 or $\perp$. The poset X is *bounded* iff it has both a greatest element and a least element.

A *lattice* (in the context of ordered sets) is a poset in which every nonempty finite subset has a supremum and an infimum. A *complete lattice* is a poset in which every subset has a supremum and an infimum.

Suppose L is a bounded lattice with least element 0 and greatest element 1. A *complement* for an element $x \in L$ is any element $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$. The lattice L is *complemented* iff every element of L has a complement.

A lattice L is *modular* iff $a \wedge (b \vee c) = (a \wedge b) \vee c$ for all $a, b, c \in L$ such that $a \geq c$.

5. Modules

The *endomorphism ring* of a module M, denoted $\mathrm{End}(M)$, is the set of all endomorphisms of M (= module homomorphisms $M \to M$), with pointwise addition and composition of functions as operations. We also write $\mathrm{End}_R(M)$ if M is an R-module.

A submodule N of a module M is a *direct summand of* M provided there exists a submodule N' of M such that $M = N \oplus N'$, that is, $N + N' = M$ and $N \cap N' = 0$. Such an N' is called a *complement of* N.

The *left regular module* over a ring R is just R itself, viewed as a left R-module using the ring multiplication. This module is denoted ${}_RR$. The *right regular module* over R, denoted R_R, is R viewed as a right R-module.

Over a unital ring R, a *free* left (respectively, right) R-module is any direct sum of copies of ${}_RR$ (respectively, of R_R). Proposition: An R-module is free iff it has a basis. A *projective* R-module is any direct summand of a free R-module.

A module M is *directly finite* (or *von Neumann finite*, or *Dedekind finite*) iff M is not isomorphic to any proper direct summand of itself; equivalently, $M \ncong M \oplus N$ for any nonzero module N. Lemma: M is a directly finite module iff $\mathrm{End}(M)$ is a directly finite ring.

A submodule N of a module M is *essential in* M iff $N \cap K \neq 0$ for all nonzero submodules $K \subseteq M$.

The *singular submodule* of a right R-module M is the set

$$Z(M) = \{x \in M \mid \exists \text{ an essential right ideal } I \subseteq R \text{ with } xI = 0\},$$

which is a submodule of M. The module M is *nonsingular* iff $Z(M) = 0$, and it is *singular* iff $Z(M) = M$.

A module M has the *finite exchange property* iff for every module K and all finite direct sum decompositions $K = M' \oplus N = \bigoplus_{i=1}^n K_i$ with $M' \cong M$, there exist submodules $K_i' \subseteq K_i$ such that $K = M' \oplus \bigoplus_{i=1}^n K_i'$.

6. Abelian monoids

A *monoid* is a semigroup (i.e., a set with an associative operation) which has an identity element. A monoid is *abelian* iff its operation is commutative. Abelian monoids will be written additively, with operation $+$ and identity element 0.

The *units* of an abelian monoid M are those elements which have additive inverses. M is *conical* iff 0 is the only unit in M, that is, $x+y=0$ implies $x=y=0$, for all $x, y \in M$.

The *algebraic pre-order* on M is the pre-order $\leq$ given by the existence of subtraction: $x, y \in M$ satisfy $x \leq y$ iff there exists $x' \in M$ such that $x + x' = y$. An *ideal* (or *o-ideal*) of M is any submonoid I which is *hereditary* with respect to $\leq$, that is, $x \leq y \in I$ implies $x \in I$, for all $x, y \in M$.

An *order-unit* in M is any element $u \in M$ such that all elements of M are bounded above by multiples of u: for each $x \in M$, there exists $n \in \mathbb{N}$ such that $x \leq nu$.

M has *cancellation* iff $x + z = y + z$ implies $x = y$, for all $x, y, z \in M$. A weaker property is that M is *separative* (or: M has *separative cancellation*) iff $2x = x + y = 2y$ implies $x = y$, for all $x, y \in M$.

An element $x \in M$ is *infinite* iff $x + y = x$ for some non-unit $y \in M$; otherwise, x is *finite*. Observation: If M has cancellation, all elements of M are finite.

M has the *Riesz decomposition property* iff whenever $x, y_1, y_2 \in M$ with $x \leq y_1 + y_2$, there exist $x_1, x_2 \in M$ such that $x = x_1 + x_2$ and each $x_i \leq y_i$.

M satisfies the *Riesz refinement property* iff whenever $x_1, x_2, y_1, y_2 \in M$ with $x_1+x_2 = y_1+y_2$, there exist elements $z_{ij} \in M$, for $i, j = 1, 2$, such that $x_i = z_{i1}+z_{i2}$ for $i = 1, 2$ and $y_j = z_{1j} + z_{2j}$ for $j = 1, 2$. A mnemonic for these equations is the *refinement matrix*

$$\begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & z_{11} & z_{12} \\ x_2 & z_{21} & z_{22} \end{array}.$$

M is *strongly periodic* iff each $x \in M$ satisfies $(m + 1)x = x$ for some $m \in \mathbb{N}$.

As a monoid, a *semilattice* is any abelian monoid in which $2x = x$ for all x. A *distributive semilattice* is any semilattice which satisfies the Riesz decomposition property. Lemma: A semilattice is distributive iff it satisfies the Riesz refinement property.

If I is an ideal of an abelian monoid M, there is a congruence relation $\sim_I$ on M defined by $x \sim_I y$ iff $x + a = y + b$ for some $a, b \in I$. The quotient $M/{\sim_I}$ is called the *quotient of M modulo I*, and is denoted M/I.

7. Pre-ordered abelian groups

A *pre-ordered abelian group* is an abelian group G equipped with a pre-order $\leq$ which is *translation-invariant*: $x \leq y$ implies $x + z \leq y + z$, for all $x, y, z \in G$. If $\leq$ is antisymmetric, then G is called a *partially ordered abelian group*.

The *positive cone* of a pre-ordered abelian group G is the set

$$G^+ = \{x \in G \mid x \geq 0\};$$

it is a submonoid of G.

G is *directed* iff each pair of elements of G has an upper bound in G. Lemma: G is directed iff G is generated (as a group) by G^+.

G is *unperforated* iff $nx \geq 0$ implies $x \geq 0$, for all $n \in \mathbb{N}$ and $x \in G$.

An *order-unit* in G is an element $u \in G^+$ such that for each $x \in G$, there exists $n \in \mathbb{N}$ with $-nu \leq x \leq nu$.

An *interpolation group* is a partially ordered abelian group G satisfying the *Riesz interpolation property*: whenever $x_1, x_2, y_1, y_2 \in G$ with $x_i \leq y_j$ for all $i, j = 1, 2$, there exists $z \in G$ such that $x_i \leq z \leq y_j$ for all $i, j = 1, 2$. Lemma: A partially ordered abelian group G has the Riesz interpolation property if and only if G^+ has the Riesz decomposition property, if and only if G^+ has the Riesz refinement property.

A *dimension group* is any directed, unperforated, interpolation group.

An *ideal* of a pre-ordered abelian group G is any subgroup H which is directed (with respect to the pre-order inherited from G) and *convex*: for any $x, z \in H$ and $y \in G$, the relation $x \leq y \leq z$ implies $y \in H$.

8. V and K_0

Let R be any ring, and write $\mathrm{Idem}_\infty(R)$ for the set of all idempotent matrices over R, that is, the set of all idempotents in $M_\infty(R)$. The *orthogonal sum* of $e, f \in \mathrm{Idem}_\infty(R)$ is the block matrix $e \oplus f = \left[\begin{smallmatrix} e & 0 \\ 0 & f \end{smallmatrix}\right]$. Write $[e]$ for the $\sim$-equivalence class of any $e \in \mathrm{Idem}_\infty(R)$, set

$$V(R) = \mathrm{Idem}_\infty(R)/\!\sim\; = \{[e] \mid e \in \mathrm{Idem}_\infty(R)\},$$

and define an addition operation in $V(R)$ by the rule $[e]+[f] = [e \oplus f]$. Observation: This operation is well-defined, and $V(R)$ is a conical abelian monoid. It is called the *V-monoid of R*. Observation: If R is unital, $[1_R]$ is an order-unit in $V(R)$.

If R is unital, there is the following *projective module picture* of $V(R)$. Let $\mathrm{FP}(R)$ denote the class of all finitely generated projective right R-modules, and write $[P]$ for the isomorphism class of any $P \in \mathrm{FP}(R)$. The set $\mathrm{FP}(R)/\!\cong$, that is, $\{[P] \mid P \in \mathrm{FP}(R)\}$, is an abelian monoid with respect to the addition operation induced from direct sum: $[P] + [Q] = [P \oplus Q]$. Proposition: There is a natural monoid isomorphism $V(R) \to \mathrm{FP}(R)/\!\cong$ such that $[e] \mapsto [eR^n]$ for any idempotent matrix $e \in M_n(R)$.

Continue to assume R is unital. Idempotents $e, f \in \mathrm{Idem}_\infty(R)$ are *stably equivalent*, written $e \approx f$, iff there is some $g \in \mathrm{Idem}_\infty(R)$ such that $e \oplus g \sim f \oplus g$. Write $[e]_0$ for the stable equivalence class of $e \in \mathrm{Idem}_\infty(R)$. Then the set

$$K_0(R)^+ = \mathrm{Idem}_\infty(R)/\!\approx\; = \{[e]_0 \mid e \in \mathrm{Idem}_\infty(R)\}$$

is an abelian monoid with respect to the addition operation induced from orthogonal sum, and this monoid has cancellation. Now define $K_0(R)$ to be the *group of differences of $K_0(R)^+$*:

(1) As a set, $K_0(R) = \{[e]_0 - [f]_0 \mid e, f \in \mathrm{Idem}_\infty(R)\}$.
(2) Equality in $K_0(R)$ is given by $[e]_0 - [f]_0 = [e']_0 - [f']_0$ in $K_0(R)$ iff $[e]_0 + [f']_0 = [e']_0 + [f]_0$ in $K_0(R)^+$, iff $e \oplus f' \approx e' \oplus f$.

(3) Addition in $K_0(R)$ is given by

$$([e]_0 - [f]_0) + ([e']_0 - [f']_0) = [e \oplus e']_0 - [f \oplus f']_0 \,.$$

(4) $K_0(R)^+$ is embedded as a submonoid of $K_0(R)$ by identifying $[e]_0$ with $[e_0] - [0]_0$ for all $e \in \mathrm{Idem}_\infty(R)$.

Further, define a relation $\leq$ on $K_0(R)$ by the rule $x \leq y$ iff $y - x \in K_0(R)^+$. Lemma: $(K_0(R), \leq)$ is a pre-ordered abelian group with positive cone $K_0(R)^+$, and $[1_R]$ is an order-unit in $K_0(R)$.

Any unital ring homomorphism $\phi : R \to S$ induces a ring homomorphism $\phi : M_\infty(R) \to M_\infty(S)$, which in turn induces an order-preserving group homomorphism $K_0(\phi) : K_0(R) \to K_0(S)$ by the rule

$$K_0(\phi)\big([e]_0 - [f]_0\big) = [\phi(e)]_0 - [\phi(f)]_0 \,.$$

In this way, K_0 becomes a functor from the category of unital rings to the category of pre-ordered abelian groups with order-unit.

For a non-unital ring R, the natural projection map $\phi : \widetilde{R} \to \mathbb{Z}$, given by $\phi(m, a) = m$, is a unital ring homomorphism, and $K_0(R)$ is defined as the abelian group

$$K_0(R) = \ker K_0(\phi) \subseteq K_0(\widetilde{R}),$$

equipped with the pre-order inherited from $K_0(\widetilde{R})$. The *scale* of $K_0(R)$ is the set $\{[e]_0 \mid e = e^2 \in R\}$. Theorem: If R has local units, $K_0(R)$ is the group of differences of $K_0(R)^+$, and $K_0(R)^+$ is naturally isomorphic to $\mathrm{Idem}_\infty(R)/\!\approx$.