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1

Leavitt path algebras. An Introduction.

1.1 Path algebras. Local units.

Definition 1.1.1 A *(directed) graph* $E = (E^0, E^1, r, s)$ consists of two sets E^0, E^1 and functions $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. (We place no restriction on the cardinalities of E^0 and E^1.)

If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called *row-finite*. A vertex v for which $s^{-1}(v) = \emptyset$ is called a *sink*, while a vertex v for which $r^{-1}(v) = \emptyset$ is called a *source*. (In other words, v is a sink (resp., source) if v is not the source (resp., range) of any edge of E.) The expressions $\mathrm{Sink}(E), \mathrm{Source}(E)$ will be used to denote, respectively, the sets of sinks and sources of E.

A *path* μ in a graph E is a sequence of edges $\mu = e_1 \dots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. In this case, $s(\mu) = s(e_1)$ is the *source* of μ, $r(\mu) = r(e_n)$ is the *range* of μ, and $n = \ell(\mu)$ is the *length* of μ. (We view the vertices of E as paths of length 0; to streamline notation, we will sometimes extend the functions s and r to E^0 by defining $s(v) = r(v) = v$ for $v \in E^0$.) If $\mu = e_1 \dots e_n$ is a path then we denote by μ^0 the set of its vertices, that is, $\mu^0 = \{s(e_1), r(e_i) \mid 1 \le i \le n\}$. For $n \ge 2$ we define E^n to be the set of paths in E of length n, and $\mathrm{Path}(E) = \bigcup_{n\geq 0} E^n$ the set of all paths in E.

Definition 1.1.2 Given a directed graph E and a field K, the *path algebra* KE is the vector space with basis $\mathrm{Path}(E)$, the set of all paths in E, with the product defined by

$$\lambda\mu = \begin{cases} \lambda\mu & \text{if } r(\lambda) = s(\mu) \\ 0 & \text{if } r(\lambda) \neq s(\mu) \end{cases}$$

Definition 1.1.3 We say that a ring R has *local units* in case for any $x_1, \dots, x_n \in R$ there exists an idempotent e in R such that $x_i \in eRe$ for $i = 1, \dots, n$. In other words R is a directed union of its "corners" eRe, $e = e^2 \in R$.

For a finite subset X of E^0, write $e_X = \sum_{v\in X} v$. Then KE is the directed union $\bigcup_X e_X KE e_X$, so that KE is a ring with local units.

1.2 Leavitt path algebras.

Definition 1.2.1 Let E be any directed graph, and let K be any field. We define a set $(E^1)^*$ consisting of symbols of the form $\{e^* \mid e \in E^1\}$. The **Leavitt path algebra of E with coefficients in** K, denoted $L_K(E)$, is the free associative K-algebra generated by the set $E^0 \cup E^1 \cup (E^1)^*$, subject to the following relations:

Condition V: $vv' = \delta_{v,v'}v$ for all $v, v' \in E^0$,

Condition E1: $s(e)e = er(e) = e$ for all $e \in E^1$,

Condition E2: $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$,

Condition CK1: $e^*e' = \delta_{e,e'}r(e)$ for all $e, e' \in E^1$, and

Condition CK2: $v = \sum_{\{e\in E^1 | s(e)=v\}} ee^*$ for every regular vertex $v \in E^0$.

Phrased another way, $L_K(E)$ is the free associative K-algebra on the symbols $E^0 \cup E^1 \cup (E^1)^*$, modulo the ideal generated by the five types of relations indicated in the previous list.

We can express the Leavitt path algebra $L_K(E)$ as a certain quotient of an ordinary path algebra, as follows. Let $\hat{E}$ be the *double* of E, that is, the graph obtained from E by adjoining, for each $e \in E^1$, an edge e^* going in the reverse direction of e, that is $s(e^*) = r(e)$ and $r(e^*) = s(e)$. The path algebra $K(\hat{E})$ is then obviously the algebra with the same set of generators as $L_K(E)$, but subject only to the relations (V), (E1) and (E2), and $L_K(E) = K(\hat{E})/I$, where I is the ideal of $K(\hat{E})$ generated by the relations (CK1) and (CK2).

Observe that $L_K(E)$ has also local units. The same set of local units e_X, where X is a finite subset of E^0, will work here.

1.3 Examples: Classical Leavitt algebras. Finite-dimensional LPA's. The Toeplitz algebra.

Definition 1.3.1 A unital ring R has the *Invariant Basis Number* property in case the following holds:

If m and m' are positive integers with the property that the free right modules R^m and $R^{m'}$ are isomorphic, then $m = m'$.

Less formally, a ring has the IBN property in case any two bases (i.e., linearly independent spanning sets) of any free right R-module have the same number of elements. It turns out that many general classes of rings have this property (e.g., noetherian rings and commutative rings), classes of rings which include all of the basic examples to which the student was first made acquainted.

Unfortunately, since all of the examples the student first encounters have the IBN property, the student more then likely comes away with the wrong impression. Indeed, there are many classes of rings which do not have the IBN property. Maybe the most common such example is the ring $B = \mathrm{End}_K(V)$, where V is an infinite dimensional vector space over the field K. Then B is not IBN: it is not hard to show that $B^m \cong B^{m'}$ for *all* positive integers m, m'.

Definition 1.3.2 Suppose R is not IBN. Let $m \in \mathbb{N}$ be minimal with the property that $R_R^m \cong R_R^{m'}$ for some $m' > m$. For this m, let n denote the minimal such m'. In this case we say that R has *module type* (m, n).

For example, $B = \text{End}_K(V)$ thus has module type $(1, 2)$.

As we shall see, there is a perhaps surprising amount of structure inherent in non-IBN rings. To start with, in the groundbreaking article [20], Leavitt proves the following fundamental result.

Theorem 1.3.3 *For each pair of positive integers $n > m$ and field K there exists a unital K-algebra $L_K(m, n)$, unique up to K-algebra isomorphism, such that:*

1. *$L_K(m, n)$ has module type (m, n), and*
2. *For each unital K-algebra A having module type (m, n) there exists a unit-preserving K-algebra homomorphism $\phi : L_K(m, n) \to A$ which satisfies certain (natural) compatibility conditions.*

The algebra $L_K(m, n)$ is defined as the K-algebra generated by the entries of matrices x_{ij} and y_{ji} of matrices $X = (x_{ij})$ and $Y = (y_{ji})$ of sizes $m \times n$ and $n \times m$ respectively, subject to the basic relations:

$$XY = I_m, \quad YX = I_n\,.$$

Our motivational focus here is on non-IBN rings of module type $(1, n)$ for some $n > 1$. (The Leavitt algebras $L_K(m, n)$ with $1 < m < n$ appear as Leavitt path algebras of *separated graphs*, see Chapter 5.) In particular, such a ring then has the property that there exist isomorphisms of free modules

$$\phi \in \text{Hom}_R(R^1, R^n) \ \text{ and } \ \psi \in \text{Hom}_R(R^n, R^1)$$

having

$$\psi \circ \phi = \iota_R \ \text{ and } \ \phi \circ \psi = \iota_{R^n}$$

where ι denotes the identity map on the appropriate module. Using the usual interpretation of homomorphisms between free modules as matrix multiplications (a description which the student encounters for the real numbers in an undergraduate linear algebra course, and which is easily shown to be valid for any unital ring), we see that such isomorphisms exist if and only if there exist $1 \times n$ and $n \times 1$ R-vectors

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

for which

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = (1_R)$$

and

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} = \begin{pmatrix} 1_R & 0 & \cdots & 0 \\ 0 & 1_R & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1_R \end{pmatrix}.$$

Rephrased,

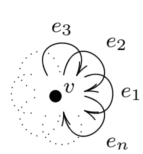
$$R^1 \cong R^n \quad \text{for some} \quad n > 1$$

if and only if there exist $2n$ elements $x_1, ..., x_n, y_1, ..., y_n$ of R for which

$$\sum_{i=1}^{n} x_i y_i = 1_R \quad \text{and} \quad y_i x_j = \delta_{ij} 1_R \text{ (for all } 1 \leq i, j \leq n). \tag{1.1}$$

Part of the beauty of the Leavitt path algebras is that they include many well-known, but seemingly disparate, classes of algebras. To make these connections clear, we introduce some notation which will be used throughout.

Notation 1.3.4 *We let R_n denote the "rose with n petals" graph having one vertex and n loops:*

$R_n \quad =$ 

In particular, a special role in the theory is played by the graph R_1:

$$R_1 \quad = \quad \bullet^{v} \circlearrowright e$$

For any $n \in \mathbb{N}$ we let A_n denote the "oriented n-line" graph having n vertices and $n-1$ edges:

$$A_n \quad = \quad \bullet^{v_1} \xrightarrow{e_1} \bullet^{v_2} \xrightarrow{e_2} \bullet^{v_3} \cdots\cdots \bullet^{v_{n-1}} \xrightarrow{e_{n-1}} \bullet^{v_n}$$

The examples presented in the following three propositions may be viewed as the three primary colors of Leavitt path algebras. Making good now on a promise offered earlier, we here validate our claim that the Leavitt algebras $L_K(1,n)$ are truly motivating examples for the more general notion of a Leavitt path algebra.

Proposition 1.3.5 *Let $n \geq 2$ be any positive integer, and K any field. Let $L_K(1,n)$ be the Leavitt K-algebra of type $(1,n)$ presented above, and let R_n be the rose with n petals. Then*

$$L_K(1,n) \cong L_K(R_n).$$

That these two algebras are isomorphic follows directly from the definition of $L_K(1,n)$ as a quotient of the free associative algebra on $2n$ variables, modulo the relations given in display (1.1). Specifically, we map $x_i \mapsto e_i$ and $y_i \mapsto e_i^*$. Then the relations given in (1.1) are precisely the relations provided by the (CK1) and (CK2) conditions of Definition 1.2.1.

The rose with one petal produces a more-familiar (certainly a less-exotic) algebra. Prior to the description of $L_K(R_1)$, the following remark is very much in order.

Remark 1.3.6 If E is a graph and $e \in E^1$, then the expression ee^* of $L_K(E)$ is always an idempotent, since using (CK1) we have $(ee^*)(ee^*) = e(e^*e)e^* = er(e)e^* = ee^*$. However, ee^* does *not* equal $s(e)$ unless e is the only edge emitted by $s(e)$ (since in that case the (CK2) condition reduces to the equation $s(e) = ee^*$).

For any field K, the "Laurent polynomial K-algebra" is the associative K-algebra generated by the two symbols x and y, with relations $xy = yx = 1$. For obvious reasons this algebra is denoted by $K[x, x^{-1}]$. The elements of $K[x, x^{-1}]$ may be written as $\sum_{i=m}^{n} k_i x^i$ (where $k_i \in K$ and $m \leq n \in \mathbb{Z}$); note in particular that the exponents are allowed to include negative integers. Viewed another way, $K[x, x^{-1}]$ is the group algebra of $\mathbb{Z}$ over K.

Proposition 1.3.7 *Let K be any field. Then*

$$K[x, x^{-1}] \cong L_K(R_1).$$

By the (CK1) relation we have $x^*x = v = 1$ in $L_K(R_1)$. But since v emits only the edge x, then Remark 1.3.6 yields $xx^* = v = 1$ in $L_K(R_1)$ as well, and the result now follows.

The third of the three primary colors of Leavitt path algebras moves us from the less-exotic $K[x, x^{-1}]$ to the almost-mundane matrix algebras $\mathrm{M}_n(K)$.

Proposition 1.3.8 *Let K be any field, and $n \geq 1$ any positive integer. Then*

$$\mathrm{M}_n(K) \cong L_K(A_n).$$

Let $\{f_{i,j} \mid 1 \leq i, j \leq n\}$ denote the standard matrix units in $\mathrm{M}_n(K)$. We define the map $\varphi : L_K(A_n) \to \mathrm{M}_n(K)$ by setting $\varphi(v_i) = f_{i,i},\ \varphi(e_i) = f_{i,i+1}$, and $\varphi(e_i^*) = f_{i+1,i}$. Using Remark 1.3.6, it is then easy to check that φ is an isomorphism of K-algebras as desired.

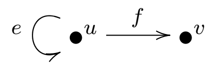
The structure of finite-dimensional LPAs is described in the following:

Proposition 1.3.9 *The Leavitt path algebra $L_K(E)$ is finite-dimensional if and only if E is finite and acyclic (no oriented closed paths). In this case, we have*

$$L_K(E) \cong \prod_{v \in \mathrm{Sink}(E)} M_{n_v}(K),$$

where, for a sink v, n_v denotes the number of paths λ in E such that $r(\lambda) = v$.

We provide a fourth example of a well-known classical algebra which arises as a specific example of a Leavitt path algebra. Let E_T denote the graph

$E_T \quad =$ 

and let K be any field. We denote by $\mathcal{T}_K$ (or more simply by $\mathcal{T}$ when the field is understood) the *algebraic Toeplitz K-algebra*

$$\mathcal{T}_K = L_K(E_T).$$

Proposition 1.3.10 *For any field K, the Leavitt path algebra $L_K(E_T)$ is the free associative K-algebra $K[x, y]$, with relations $yx = 1$ and $xy \neq 1$.*

We begin by noting that in $L_K(E_T)$ we have the relations $ee^* + ff^* = u$ and $u + v = 1$. We consider the elements $X = e + f$ and $Y = e^* + f^*$ of $L_K(E_T)$. Then by (CK1) we have $YX = u + v = 1$, while $XY = ee^* + ff^* = u \neq 1$ by (CK1) and (CK2). The subalgebra of $\mathcal{T}_K = L_K(E_T)$ generated by X and Y then contains $1 - u = v$, which in turn gives that this subalgebra contains $e = Xu, f = Xv, e^* = Yu$, and $f^* = Yv$. These observations establish that the map $\varphi : K[U, V] \to L_K(E_T)$ given by the extension of $\varphi(U) = e + f, \varphi(V) = e^* + f^*$ is a surjective K-algebra homomorphism.

The injectivity of φ will follow from Exercise 1.3.11(iii) below.

Exercise 1.3.11 *Let $\mathcal{T} = K[x, y \mid yx = 1]$ be the Toeplitz algebra.*

(i) *For $i, j \geq 0$, set $e_{ij} = x^i(1 - xy)y^j$. Show that e_{ij} is a set of matrix units, and that the linear span $\mathcal{K}$ of this set is isomorphic to the algebra $M_\infty(K)$ of infinite matrices (indexed by $\mathbb{Z}^+$) with only finitely many nonzero entries.*

(ii) *Show that $\mathcal{K}$ is precisely the (two-sided) ideal generated by $1 - xy$, and that there is an exact sequence of K-algebras:*

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \longrightarrow K[x, x^{-1}] \longrightarrow 0$$

(iii) *Let B be any K-algebra containing elements z, t such that $tz = 1$. Show that there is a unique K-algebra homomorphism $\varphi \colon \mathcal{T} \to B$ such that $\varphi(x) = z$ and $\varphi(y) = t$. Show that if $zt \neq 1$ then φ is injective.*

2

Basic structure

To simplify the exposition, we will work only with row-finite graphs.

2.1 The basic lemma.

Definition 2.1.1 Let E be a graph, let $\mu = e_1 \dots e_n$ be a path in E, with $n \geq 1$, and let $e \in E^1$.

(i) We say that μ is a *cycle* in case $r(e_n) = s(e_1)$ and $r(e_i) \neq r(e_j)$ for $1 \leq i, j \leq n$, $i \neq j$.

(ii) We say that e is an *exit* for μ if there exists i ($1 \leq i \leq n$) such that $s(e) = s(e_i)$ and $e \neq e_i$.

(iii) We say that E satisfies *Condition* (L) if every cycle in E has an exit.

Notation 2.1.2 *For a cycle without exits c in a graph E, being $v = s(c)$, we will use the following notation:*

$$c^0 = v, \ c^{-n} = (c^*)^n, \ \textit{for any} \ \ n \geq 1.$$

The following reduction lemma is a basic tool:

Lemma 2.1.3 (Reduction Lemma). *Let K be a field and E be a graph. For any nonzero element $\alpha \in L_K(E)$ there exist $\mu, \nu \in \mathrm{Path}(E)$ such that:*

(i) $0 \neq \mu^* \alpha \nu = kv$, *for some $k \in K$ and $v \in E^0$, or*

(ii) $0 \neq \mu^* \alpha \nu = p(c, c^*)$, *where c is a cycle without exits and $p(x, x^{-1})$ is a nonzero polynomial in $K[x, x^{-1}]$.*

The first step to prove this is to show that for $0 \neq \alpha \in L_K(E)$ there exists $\nu \in \mathrm{Path}(E)$ such that $0 \neq \alpha\nu \in KE$. Then the result is proved for this kind of elements.

Corollary 2.1.4 *Let K be a field and E be any graph that satisfies Condition* (L). *Then every nonzero ideal of $L_K(E)$ contains a vertex.*

Proof Let I be a nonzero ideal of $L_K(E)$, and let α be a nonzero element in I. Since the graph satisfies Condition (L) then, by Theorem 2.1.3, there exist $\mu, \nu \in \mathrm{Path}(E)$ such that $0 \neq \mu^* \alpha \nu = kv$ with $v \in E^0$ and $k \in K^\times$. This implies $0 \neq v = k^{-1} \mu^* \alpha \nu \in L_K(E) I L_K(E) \subseteq I$.

2.2 Simple LPAs. Purely infinite LPAs.

A graph is said to be *cofinal* in case every vertex connects to any infinite path and to any sink. (In a cofinal graph, if there are sinks then there is only one sink.)

Theorem 2.2.1 *Let K be a field and E a graph. Then the Leavitt path algebra $L_K(E)$ is simple if and only if the graph E satisfies the following conditions:*

(i) *The graph E is cofinal.*

(ii) *E satisfies Condition (L).*

To give an idea of the proof we show that $L(E)$ is simple when E satisfies both conditions. Indeed, let I be a nonzero (two-sided) ideal of $L(E)$. By condition (L) I contains a vertex v. Note that if $v_0 \in E^0$ is in the tree of v then $v_0 \in I$ by $(CK1)$. Moreover by $(CK2)$, if $r(e) \in I$ for all $e \in s^{-1}(v_0)$ then $v_0 \in I$ too.

Now given a vertex w in E^0 with $w \notin I$, then either w is a sink or $r(e_1) \notin I$ for some $e_1 \in s^{-1}(w)$. Applying the same argument to $r(e_1)$, we obtain $e_2 \in s^{-1}(r(e_1))$ such that $r(e_2) \notin I$ (or $r(e_1)$ is a sink). Continuing in this way we obtain an infinite path γ (or a path ending in a sink) such that all the vertices in the path do not belong to I. So v cannot connect to the path γ, a contradiction.

When trying to classify a class of algebras, one of the first steps is to try to classify the simple ones. Purely infinite simple Leavitt path algebras appear when studing simple Leavitt path algebras. As we will see, every simple Leavitt path algebra is locally matricial of purely infinite simple.

Definition 2.2.2 Let R be a ring and e and f two idempotents in R. We say that e and f are *(Murray-von Neumann) equivalent* provided there exist elements $x \in eRf$ and $y \in fRe$ such that $xy = e$ and $yx = f$, and we will write $e \sim f$ to denote this fact.

Note that $e \sim f$ if and only if $eR \cong fR$ as right R-modules.

Definition 2.2.3 Let R be a ring. An idempotent e in R is said to be *infinite* if there exist orthogonal idempotents $f, g \in R$ such that $e = f + g$, with $g \neq 0$ and $e \sim f$. Observe that the idempotent e is infinite if and only if eR is isomorphic to a proper direct summand of itself, that is, eR is a *directly infinite module*.

For idempotents e and f, we will write $e \leq f$ in case $e = ef = fe$ and $e < f$ in case $e \leq f$ and $e \neq f$. Note that an idempotent e is infinite if and only if $e \sim f$ for some idempotent f such that $f < e$.

We now provide a characterization of the infinite vertices of a Leavitt path algebra.

Lemma 2.2.4 *Let E be a graph and $v \in E^0$. Then v is an infinite idempotent in $L_K(E)$ if and only if v connects to a cycle with exits.*

Proof. Observe that for every finite family $e_1, \ldots, e_n \in s^{-1}(w)$, $w \in E^0$, we have

$$\bigoplus_{i=1}^{n} r(e_i) \precsim w.$$

This follows from the facts that $r(e_i) = e_i^* e_i \sim e_i e_i^*$ for all i, and that $e_1e_1^*, e_2e_2^*, \ldots, e_ne_n^*$ is a family of orthogonal idempotents with $\sum_{i=1}^n e_ie_i^* \leq w$. Using this observation it is clear that $w' \precsim w$ if $w \geq w'$, where $w \geq w'$ means that w' belongs to the tree of w.

Now assume that $v \in E^0$ connects to a cycle with exits c. Let w be a vertex in c^0. Then $w \precsim v$ so that it is enough to show that w is infinite. If u is a bifurcation in c, let f be the edge in c such that $s(f) = u$, and let e be an exit of c at u (so that $e \neq f$). Then $u \precsim w$, indeed $u' \precsim u''$ for all $u', u'' \in c^0$, and

$$u \oplus r(e) \precsim r(f) \oplus r(e) \precsim u,$$

which shows that u is infinite. Since $u \precsim w$, we see that w is infinite too.

The converse uses some structure theory. We give an outline. Assume that the tree of v does not contain any cycle with exits. First one reduces by a direct limit argument to the case where E is a finite graph. In that case, one shows that the ideal generated by v is of the form

$$M_{r_1}(K) \oplus \cdots \oplus M_{r_k}(K) \oplus M_{s_1}(K[x, x^{-1}]) \oplus \cdots \oplus M_{s_\ell}(K[x, x^{-1}]),$$

and it is clear that all the idempotents in this algebra are finite.

Definition 2.2.5 Let R be a simple ring. We say that R is *purely infinite* if any nonzero right ideal of R contains an infinite idempotent. Equivalently if for any $0 \neq a \in R$ there are $s, t \in R$ such that sat is an infinite idempotent.

Theorem 2.2.6 *Let K be a field and E a graph. The Leavitt path algebra $L_K(E)$ is purely infinite simple if and only if the graph E satisfies the following conditions:*

(i) *E is cofinal.*

(ii) *E satisfies Condition* (L).

(iii) *Every vertex in* E^0 *connects to a cycle.*

Note that condition (iii) is equivalent to the existence of a cycle in the presence of cofinality.

Proof. Suppose first that conditions (i), (ii) and (iii) are satisfied. By Theorem 2.2.1 we know that $L_K(E)$ is a simple ring. Note that (ii) and (iii) together give that every vertex connects to a cycle with exits. So by lemma 2.2.4 we get that all vertices in E are infinite idempotents.

Given a nonzero α in $L_K(E)$, and since E satisfies Condition (L), by Theorem 2.1.3 there exist $x, y \in L_K(E)$ such that $x\alpha y = v$ for some vertex v. Since v is an infinite idempotent by the previous paragraph, we see from Definition 2.2.5 that $L_K(E)$ is purely infinite.

Now, suppose that $L_K(E)$ is simple purely infinite. By Theorem 2.2.1, the graph E satisfies conditions (i) and (ii) in the statement. Now we see that condition (iii) holds too. Since every vertex in E^0 must be an infinite idempotent in $L_K(E)$ by Definition 2.2.5, it follows from Lemma 2.2.4 that every vertex connects to a cycle with exits, which indeed gives (ii) and (iii) at once.

2.3 Graded ideals.

Among the nice properties of Leavitt path algebras we find that of being a $\mathbb{Z}$-graded algebra.

Let A be an algebra over a field K. We say that the algebra A is $\mathbb{Z}$-*graded* if there exists a family $\{A_n\}_{n\in\mathbb{Z}}$ of vector subspaces of A such that

$$A = \bigoplus_{n\in\mathbb{Z}} A_n \quad \text{and} \quad A_n A_m \subseteq A_{n+m} \text{ for every } n, m \in \mathbb{Z}.$$

For a graph E, the grading in $L(E)$ is defined by taking $L(E)_n$ the linear span of the monomials $\lambda\mu^*$ with $|\lambda| - |\mu| = n$.

An ideal I of a $\mathbb{Z}$-graded algebra A is said to be a *graded ideal* if

$$I = \bigoplus_{n \in \mathbb{Z}} I \cap A_n, \quad \text{equivalently,} \quad y = \sum_{n \in \mathbb{Z}} y_n \in I,\, y_n \in A_n, \quad \text{implies} \quad y_n \in I.$$

We can now describe the *graded ideals* of $L(E)$ in terms of certain subsets of E^0:

Definition 2.3.1 Let E be a graph.

We say that a subset $H \subseteq E^0$ is *hereditary* if $w \in H$ and $w \geq v$ imply $v \in H$.

We say that H is *saturated* if whenever v is any non-sink vertex for which

$$\{r(e) \mid e \in s^{-1}(v)\} \subseteq H,$$

then $v \in H$. (In other words, H is saturated if, for any non-sink vertex v such that *all* of the range vertices $r(e)$ for those edges e having $s(e) = v$ are in H, then v must be in H as well.)

Theorem 2.3.2 *There is a bijection between graded ideals of $L(E)$ and hereditary saturated subsets of E^0, namely*

$$I \mapsto H = \{v \in E^0 : v \in I\}, \ \textit{for}\ I \lhd_{\text{graded}} L(E),$$

$$H \mapsto I(H) := \langle v : v \in H \rangle.$$

Moreover, if H is a hereditary saturated subset of E^0, then

$$L(E)/I(H) \cong L(E \setminus H),$$

where $E \setminus H$ is the graph with vertices $E^0 \setminus H$ and edges those $e \in E^1$ such that $r(e) \notin H$ (and thus $s(e) \notin H$).

3

The monoid $\mathcal{V}(L_K(E))$.

To simplify the exposition, we will only work with row-finite graphs.

3.1 The monoid $M(E)$ associated to a quiver.

Let $M(E)$ be the abelian monoid given by the generators $\{a_v \mid v \in E^0\}$, with the relations:

$$a_v = \sum_{\{e \in E^1 \mid s(e)=v\}} a_{r(e)} \qquad \text{for every } v \in E^0 \text{ that emits edges.} \tag{M}$$

Theorem 3.1.1 *Let E be a row-finite graph. Then there is a natural monoid isomorphism $\mathcal{V}(L(E)) \cong M(E)$.*

For each row-finite graph E, there is a unique monoid homomorphism

$$\gamma_E \colon M(E) \to \mathcal{V}(L(E))$$

such that $\gamma_E(a_v) = [p_v]$. This is the map we want to show to be an isomorphism. To do that we employ (part of) Bergman's machinery, as follows:

Bergman constructs in [11] a unital K-algebra

$$S := R\langle i, i^{-1} \colon \overline{P} \cong \overline{Q} \rangle,$$

together with an algebra homomorphism $R \to S$, such that there is a universal isomorphism $i\colon \overline{P} \to \overline{Q}$, where $\overline{X} = S \otimes_R X$ for a left R-module X. The universal property of i is expressed as follows. If $R \to T$ is an algebra homomorphism and $\phi\colon T \otimes_R P \to T \otimes_R Q$ is an isomorphism of T-modules, then there is a unique algebra homomorphism $\psi\colon S \to T$ such that $\mathrm{id}_T \otimes_S i = \phi$, where T is an S-module via ψ.

We shall refer to the algebra S described above as *the Bergman algebra obtained from R by adjoining a universal isomorphism between P and Q*. By [11, Theorem 5.2], the monoid $\mathcal{V}(S)$ is exactly the quotient monoid of $\mathcal{V}(R)$ modulo the congruence generated by $([P],[Q])$, so that we modify $\mathcal{V}(R)$ by just introducing a single new relation $[P] = [Q]$.

Let E be a finite graph and assume that $\{v_1, \dots, v_m\} \subseteq E^0$ is the set of vertices which emit edges. We start with an algebra

$$A_0 = \prod_{v \in E^0} K.$$

In A_0 we have a family $\{p_v : v \in E^0\}$ of orthogonal idempotents such that $\sum_{v \in E^0} p_v = 1$. Let us consider the two finitely generated projective left A_0-modules $P = A_0 p_{v_1}$ and $Q = \oplus_{\{e \in E^1 | s(e) = v_1\}} A_0 p_{r(e)}$. There exists an algebra $A_1 := A_0\langle i, i^{-1}\colon \overline{P} \cong \overline{Q}\rangle$ with a universal isomorphism $i\colon \overline{P} := A_1 \otimes_{A_0} P \to \overline{Q} := A_1 \otimes_{A_0} Q$, see [11, page 38]. Note that this algebra is precisely the algebra $L(X_1)$, where X_1 is the graph having $X_1^0 = E^0$, and where v_1 emits the same edges as it does in E, but all other vertices do not emit any edge. Namely the row $(e : s(e) = v_1)$ implements an isomomorphism $\overline{P} = A_1 p_{v_1} \to \overline{Q} = \oplus_{\{e \in E^1 | s(e) = v_1\}} A_1 p_{r(e)}$ with inverse given by the column $(e^* : s(e) = v_1)^T$, which is clearly universal. By [11, Theorem 5.2], the monoid $\mathcal{V}(A_1)$ is obtained from $\mathcal{V}(A_0)$ by adjoining the relation $[P] = [Q]$. In our case we have that $\mathcal{V}(A_0)$ is the free abelian group on generators $\{a_v \mid v \in E^0\}$, where $a_v = [p_v]$, and so $\mathrm{V}(A_1)$ is given by generators $\{a_v \mid v \in E^0\}$ and a single relation

$$a_{v_1} = \sum_{\{e \in E^1 | s(e) = v_1\}} a_{r(e)}.$$

Now we proceed inductively. For $k \geq 1$, let A_k be the graph algebra $A_k = L(X_k)$, where X_k is the graph with the same vertices as E, but where only the first k vertices $v_1, \dots, v_k$ emit edges, and these vertices emit the same edges as they do in E. Then we assume by induction that $\mathcal{V}(A_k)$ is the abelian group given by generators $\{a_v \mid v \in E^0\}$ and relations

$$a_{v_i} = \sum_{\{e \in E^1 | s(e) = v_i\}} a_{r(e)},$$

for $i = 1, \ldots, k$. Let A_{k+1} be the similar graph, corresponding to vertices $v_1, \ldots, v_k, v_{k+1}$. Then we have $A_{k+1} = A_k\langle i, i^{-1} \colon \overline{P} \cong \overline{Q}\rangle$ for $P = A_k p_{v_{k+1}}$ and $Q = \oplus_{\{e \in E^1 | s(e) = v_{k+1}\}} A_k p_{r(e)}$, and so we can apply again Bergman's Theorem [11, Theorem 5.2] to deduce that $\mathrm{V}(A_{k+1})$ is the monoid with the same generators as before and the relations corresponding to $v_1, \ldots, v_k, v_{k+1}$.

The following properties of $M(E)$ were shown by Ara, Moreno and Pardo [10].

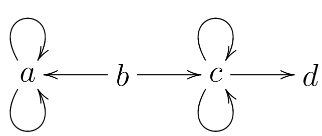
Theorem 3.1.2 *The monoid $M(E)$ associated with any row-finite graph E has the following properties:*

1. *$M(E)$ is a refinement monoid.*
2. *$M(E)$ is separative. (that is, $2x = x + y = 2y \implies x = y$)*
3. *$M(E)$ is unperforated, that is, $nx \leq ny \implies x \leq y$ for $n \in \mathbb{N}$.*

3.2 The Grothendieck group $K_0(L_K(E))$.

The group $K_0(R)$ of a unital ring R is the universal group of $\mathcal{V}(R)$. Recall that, as any universal group of an abelian monoid, the group $K_0(R)$ has a standard structure of partially pre-ordered abelian group. The set of positive elements in $K_0(R)$ is the image of $V(R)$ under the natural monoid homomorphism $\mathcal{V}(R) \to K_0(R)$.

Example 3.2.1 Consider the following graph E:



Then $M(E)$ is the monoid generated by a, b, c, d with defining relations $a = 2a$, $b = a + c$, $c = 2c + d$. The Grothendieck group of $M(E)$ is infinite cyclic generated by the class of c. It follows that $K_0(L_K(E))$ is infinite cyclic generated by $[p_c]$, and $K_0(L_K(E)) = K_0(L_K(E))^+$.

For a non-unital ring R, one defines $K_0(R)$ as the kernel of the homomorphism

$$\pi_*\colon K_0(\widetilde{R}) \to K_0(\mathbb{Z}),$$

being $\pi\colon \widetilde{R} \to \mathbb{Z}$ the natural projection.

If R is a ring with local units, then it can be seen that $K_0(R)$ is also the Grothendieck group of $\mathcal{V}(R)$. In particular this applies to $L(E)$.

We obtain the following description of K_0 of a Leavitt path algebra $L(E)$.

Proposition 3.2.2 *Let E be a finite graph with no sinks, with $E^0 = \{v_1, \dots, v_n\}$, and let $A = (a_{ij})$ be the adjacency matrix of E, so that a_{ij} is the number of arrows from vertex v_i to vertex v_j. Then we have:*

$$K_0(L(E)) \cong \mathit{Coker}(I - A^t\colon \mathbb{Z}^n \to \mathbb{Z}^n)$$

The result follows immediately from the structure of the $\mathcal{V}$-monoid. There are versions for infinite graphs, even with sinks.

3.3 The realization problem

Let us denote by $FP(R)$ the class of all finitely generated projective right modules over a unital ring R, and by $\mathcal{V}(R)$ the monoid of isomorphism classes of modules from $FP(R)$. In general the monoid $\mathcal{V}(R)$ is a conical abelian monoid, where conical means that $x + y = 0$ implies $x = y = 0$. It was proved by Bergman [11] and Bergman and Dicks [12] that every conical monoid with an order unit can be realized as $\mathcal{V}(R)$ for some unital ring R, even more one can take R to be a hereditary ring.

Now if R is an exchange ring there is an additional condition that $\mathcal{V}(R)$ must satisfy. It follows easily from the module theoretic characterization that $\mathcal{V}(R)$ must be a refinement monoid.

Definition Let M be an abelian monoid. Then M is a *refinement monoid* in case whenever $a+b=c+d$ in M there exist $x,y,z,t \in M$ such that $a=x+y$, $b=z+t$, $c=x+z$ and $d=y+t$.

Now a natural question in view of Bergman and Dicks results is:

R1. Realization Problem for Exchange Rings Is every refinement conical abelian monoid realizable by an exchange ring?

We can ask also the same question for particular classes of exchange rings. Most important for us is the following:

R2. Realization Problem for von Neumann Regular Rings Is every refinement conical abelian monoid realizable by a von Neumann regular ring?

A related problem was posed by K.R. Goodearl in [18]:

FUNDAMENTAL OPEN PROBLEM Which abelian monoids arise as $\mathcal{V}(R)$'s for a von Neumann regular ring R?

The striking result here is that Question R2 has a negative answer! Fred Wehrung [24] proved that there are (even cancellative) refinement cones of size $\aleph_2$ such that cannot be realized as $\mathcal{V}(R)$ for any von Neumann regular ring R.

So a reformulation of R2 is in order. The following is still an open problem:

R3. Realization Problem for small von Neumann Regular Rings Is every *countable* refinement conical abelian monoid realizable by a von Neumann regular ring?

It turns out that very little is known about R3. A *dimension monoid* is a cancellative, refinement, unperforated cone. These are the positive cones of the dimension groups [17, Chapter 15]. If M is a countable *dimension monoid* and F is any field, then there exists an ultramatricial F-algebra R (=direct limit of a sequence of finite direct products of matrix algebras over F) such that $\mathcal{V}(R) \cong M$, see [17, Theorem 15.24(b)].

Apart from this there seems to be no systematic constructions realizing large classes of countable refinement cones, except for:

Theorem 3.3.1 *[9, Theorem 8.4]. Let G be a countable abelian group and K any field. Then there is a purely infinite simple regular K-algebra R such that $K_0(R) \cong G$.*

Since $\mathcal{V}(R) = K_0(R) \sqcup \{0\}$ for a purely infinite simple regular ring [9, Corollary 2.2], we get that all cones of the form $G \sqcup \{0\}$, where G is a countable abelian group can be realized. Observe that the main point to get that result is to realize every cyclic group as K_0 of a purely infinite simple ring in a sort of functorial way.

Problems R1, R2 and R3 are related to the separativity problem.

A class $\mathcal{C}$ of modules is called *separative* if for all $A, B \in \mathcal{C}$ we have

$$A \oplus A \cong A \oplus B \cong B \oplus B \implies A \cong B.$$

A ring R is *separative* if $FP(R)$ is a separative class. Separativity is an old concept in semigroup theory, see [15]. A semigroup S is called *separative* if for all $a, b \in S$ we have $a + a = a + b = b + b \implies a = b$. Clearly a ring R is separative if and only if $\mathcal{V}(R)$ is a separative semigroup. Separativity provides a key to a number of outstanding cancellation problems for finitely generated projective modules over exchange rings, see [8].

Separativity can be tested in various different ways:

Theorem [8, Section 2] For a ring R the following conditions are equivalent:

(i) R is separative.

(ii) For $A, B \in FP(R)$, if $2A \cong 2B$ and $3A \cong 3B$, then $A \cong B$.

(iii) For $A, B \in FP(R)$, if there exists $n \in \mathbb{N}$ such that $nA \cong nB$ and $(n+1)A \cong (n+1)B$, then $A \cong B$.

(iv) For $A, B, C \in FP(R)$, if $A \oplus C \cong B \oplus C$ and C is isomorphic to direct summands of both mA and nB for some $m, n \in \mathbb{N}$, then $A \cong B$.

In case R is an exchange ring, separativity is also equivalent to the condition

(v) For $A, B, C \in FP(R)$, if $A \oplus 2C \cong B \oplus 2C$, then $A \oplus C \cong B \oplus C$.

Outside the class of exchange rings, separativity can easily fail. In fact it is easy to see that a commutative ring R is separative if and only if $\mathcal{V}(R)$ is cancellative. Among exchange rings, however, separativity seems to be the rule. It is not known whether there are non-separative exchange rings. This is one of the major open problems in this area. See [3] for some classes of exchange rings which are known to be separative. We single out the problem for von Neumann regular rings.

SP Is every von Neumann regular ring separative?

We have ($R3$ has positive answer) $\Longrightarrow$ (SP has a negative answer). To explain why we have to recall results of Bergman and Wehrung concerning *existence* of countable non-separative refinement cones.

Proposition(cf. [23]) Let M be a countable cone. Then there is an order-embedding of M into a countable refinement cone.

So let us apply the above Proposition to the cone M generated by a with the only relation $2a = 3a$. Then

$$a + a = a + (2a) = (2a) + (2a)$$

but $a \neq 2a$ in M. By the above Proposition there exists an order-embedding $M \to M'$, where M' is a countable refinement cone and M' cannot be separative.

Thus if R3 is true we can represent M' as $\mathcal{V}(R)$ for some von Neumann regular ring and R will be non-separative.

4

The regular algebra of a quiver.

4.1 Construction of the regular algebra of a quiver.

The results in this section come from a joint paper with Miquel Brustenga [5]. In the following, K will denote a fixed field and $E = (E^0, E^1, r, s)$ a finite quiver with $E^0 = \{1, \dots, d\}$, and we will denote by $p_1, \dots, p_d$ the corresponding idempotents in KE. We will denote the length of a path α by $|\alpha|$.

Observe that $A = \oplus_{i \in E^0} Kp_i \subseteq P(E)$ is a subring isomorphic to K^d. In general we will identify $A \subseteq KE$ with K^d. An element in KE can be written in a unique way as a finite sum $\sum_{\gamma \in \mathrm{Path}(E)} \lambda_\gamma \gamma$ with $\lambda_\gamma \in K$. We will denote by ε the augmentation homomorphism, which is the ring homomorphism $\varepsilon \colon KE \to K^d \subseteq KE$ defined by $\varepsilon(\sum_{\gamma \in \mathrm{Path}(E)} \lambda_\gamma \gamma) = \sum_{\gamma \in E^0} \lambda_\gamma \gamma$.

Definition 4.1.1 *Let $I = \ker(\varepsilon)$ be the augmentation ideal of KE. Then the K-algebra of formal power series over the graph E, denoted by $K\langle\langle E \rangle\rangle$, is the I-adic completion of KE, that is $K\langle\langle E \rangle\rangle \cong \varprojlim KE/I^n$.*

The elements of $K\langle\langle E \rangle\rangle$ can be written in a unique way as a possibly infinite sum $\sum_{\gamma \in E^*} \lambda_\gamma \gamma$ with $\lambda_\gamma \in K$. We will also denote by ε the augmentation homomorphism on $K\langle\langle E \rangle\rangle$.

There is an important subalgebra of $K\langle\langle E\rangle\rangle$, namely the algebra of rational series $K_{\mathrm{rat}}\langle E\rangle$. It is defined as the division closure of KE in $K\langle\langle E\rangle\rangle$, that is the smallest subalgebra of $K\langle\langle E\rangle\rangle$ containing KE and closed under inversion, that is, for any element a in $K_{\mathrm{rat}}\langle E\rangle$ which is invertible over $K\langle\langle E\rangle\rangle$ we have $a^{-1}\in K_{\mathrm{rat}}\langle E\rangle$. Observe that for any square matrix A over $K\langle\langle E\rangle\rangle$, we have

$$A \text{ is invertible over } K\langle\langle E\rangle\rangle \iff \varepsilon(A) \text{ is invertible over } K^d.$$

By using this, one can see that indeed $K_{\mathrm{rat}}\langle E\rangle$ is *rationally closed* in $K\langle\langle E\rangle\rangle$, that is, every square matrix over $K_{\mathrm{rat}}\langle E\rangle$ which is invertible over $K\langle\langle E\rangle\rangle$ is already invertible over $K_{\mathrm{rat}}\langle E\rangle$. Indeed, assume that A is invertible over $K\langle\langle E\rangle\rangle$. Then $\varepsilon(A)$ is invertible over K^d, and so replacing A with $\varepsilon(A)^{-1}A$ we can assume that $\varepsilon(A)=1_n$. Now this implies that all the diagonal entries of A are invertible over $K\langle\langle E\rangle\rangle$ and so they are invertible over $K_{\mathrm{rat}}\langle E\rangle$, so by performing elementary transformations to the rows (say) of A we get a diagonal invertible matrix over $K_{\mathrm{rat}}\langle E\rangle$. It follows that A is invertible over $K_{\mathrm{rat}}\langle E\rangle$.

Let us recall some basic facts about (Cohn) universal localization. Let R be a ring and let Σ be a set (or even a class) of maps between finitely generated projective R-modules. A ring homomorphism $\varphi\colon R\to S$ is Σ-*inverting* in case

$$f\otimes 1\colon P\otimes_R S\to Q\otimes_R S$$

is invertible for every $f\colon P\to Q$ in Σ. Cohn proved that there exists a universal Σ-inverting ring homomorphism $\iota_\Sigma\colon R\to\Sigma^{-1}R$, that is , every Σ-inverting homomorphism $\varphi\colon R\to S$ factors as $R\to\Sigma^{-1}R\to S$. Bergman and Dicks [12] proved that if R is a hereditary ring then $\Sigma^{-1}R$ is also hereditary.

Cohn and Dicks proved that the division closure $K_{\mathrm{rat}}\langle X\rangle$ of $K\langle X\rangle$ in $K\langle\langle X\rangle\rangle$ coincides with the universal localization of $K\langle X\rangle$ with respect to the set Σ of all square matrices A over $K\langle X\rangle$ such that $\varepsilon(A)$ is invertible over K (or equivalently A is invertible over $K\langle\langle X\rangle\rangle$). This can be generalized to path algebras

Theorem 4.1.2 *[5] Let Σ be the set of matrices over KE which are invertible over $K\langle\langle E\rangle\rangle$. Then $K_{\mathrm{rat}}\langle E\rangle$ coincides with the universal localization of KE with respect to Σ.*

For every non-sink vertex $i\in E^0$, consider the homomorphism:

$$\mu_i\colon KEp_i\longrightarrow\bigoplus_{e\in s^{-1}(i)} KEp_{r(e)}$$

given by $\mu_i(rp_i) = (re)_{e\in s^{-1}(i)}$. Let $\Sigma_1 = \{\mu_i\}$ be the set of all these maps. Observe that $L(E) = \Sigma_1^{-1}KE$.

Theorem 4.1.3 *We have a commutative diagram of K-algebra inclusions:*

$$\begin{array}{ccccccc} K^d & \longrightarrow & KE & \longrightarrow & K_{\mathrm{rat}}\langle E\rangle & \longrightarrow & K\langle\langle E\rangle\rangle \\ \big\downarrow & & \big\downarrow{\scriptstyle \iota_{\Sigma_1}} & & \big\downarrow{\scriptstyle \iota_{\Sigma_1}} & & \big\downarrow{\scriptstyle \iota_{\Sigma_1}} \\ KE^* & \longrightarrow & L_K(E) & \longrightarrow & Q(E) & \longrightarrow & U(E) \end{array}$$

The algebras $Q(E)$ and $U(E)$ are von Neumann regular ,and we have

$$M(E) \cong \mathcal{V}(L(E)) \cong \mathcal{V}(Q(E)) \cong \mathcal{V}(U(E))$$

We call the algebra $Q(E)$ the *regular ring* of the quiver E.

Let us have a look at the standard examples. In the first place we consider the case of $E = R_1$. Thus $KE = K[x]$, $K_{\mathrm{rat}}\langle E\rangle = K_{\mathrm{rat}}[x]$, the algebra of rational series, and $K\langle\langle E\rangle\rangle = K[[x]]$. We have

$$\begin{array}{ccccccc} K & \longrightarrow & K[x] & \longrightarrow & K_{\mathrm{rat}}[x] & \longrightarrow & K[[x]] \\ \big\downarrow & & \big\downarrow{\scriptstyle \iota_x} & & \big\downarrow{\scriptstyle \iota_x} & & \big\downarrow{\scriptstyle \iota_x} \\ K[x^{-1}] & \longrightarrow & K[x,x^{-1}] & \longrightarrow & K(x) & \longrightarrow & K((x)) \end{array}$$

For the graph R_n, the situation is similar, but now the free non-commutative algebra $K\langle x_1,\dots,x_n\rangle$ shows up. Set $X = \{x_1,\dots,x_n\}$. Then

$$\begin{array}{ccccccc} K & \longrightarrow & K\langle X\rangle & \longrightarrow & K_{\mathrm{rat}}\langle X\rangle & \longrightarrow & K\langle\langle X\rangle\rangle \\ \big\downarrow & & \big\downarrow{\scriptstyle \iota_{\sigma_1}} & & \big\downarrow{\scriptstyle \iota_{\sigma_1}} & & \big\downarrow{\scriptstyle \iota_{\sigma_1}} \\ K\langle X^*\rangle & \longrightarrow & L_K(1,n) & \longrightarrow & Q(R_n) & \longrightarrow & U(R_n) \end{array}$$

where $\sigma_1\colon K\langle X\rangle \to K\langle X\rangle^n$ is the map given by right multiplication by the row $(x_1,\dots,x_n)$.

4.2 Example: The Toeplitz algebra.

In this section we consider the specific example of the algebraic Toeplitz algebra $\mathcal{T}$. In this case we can describe quite explicitly what is the regular ring $Q(E_T)$. Recall that the graph E_T is the graph described just before Proposition 1.3.10, and that $L_K(E_T) \cong K[x, y \mid yx = 1] = \mathcal{T}$ by that proposition. The ideal $I = \langle 1 - xy \rangle$ is isomorphic to $M_\infty(K)$, (using the matrix units $e_{ij} = x^i(1-xy)y^j$, $i, j \geq 0$), and the quotient $\mathcal{T}/I$ is isomorphic to $L_K(R_1) \cong K[x, x^{-1}]$.

For R_1, we have $Q(R_1) = K(x) = \Sigma^{-1}K[x, x^{-1}]$, where Σ is the set of all the Witt polynomials $1 + a_1x + \cdots + a_nx^n$, $n \geq 1$.

There is a well-known representation of the Toeplitz algebra $\mathcal{T}$ on a countable dimensional vector space. Namely, let $v_0, v_1, v_2, \ldots$ be a basis of a countably dimensional (left) K-vector space V. Consider the map $\rho\colon \mathcal{T} \to \mathrm{End}_K(V)$, given by

$$(v_i)\rho(y) = v_{i+1}, \qquad (v_i)\rho(x) = \begin{cases} v_{i-1} & \text{if } i \geq 1 \\ 0 & \text{if } i = 0 \end{cases}.$$

The representation ρ is faithful. With Σ the set of Witt polynomials considered before we have:

Proposition 4.2.1 *$Q := Q(E_T) \cong \Sigma^{-1}\mathcal{T}$. Moreover there is a unique non-trivial ideal in Q, the ideal J generated by $e_{00} = 1 - xy$, and $Q/J \cong K(x)$. Moreover ρ extends to a faithful representation of Q on V.*

Sketch of the proof: It is easy to show that, in this case, $Q(E_T) = \Sigma^{-1}L(E_T) = \Sigma^{-1}\mathcal{T}$, because any square matrix A over KE_T such that $\epsilon(A)$ is an invertible matrix over K^2, will be invertible over $\Sigma^{-1}\mathcal{T}$.

We observe that Σ is a left Ore set in $\mathcal{T}$, so that the universal localization $\Sigma^{-1}\mathcal{T}$ is a classical localization in this case. For, we show that given $f = 1 + a_1x + \cdots + a_nx^n \in \Sigma$, and $\beta \in \mathcal{T}$, there is $\alpha \in \mathcal{T}$ such that $\beta f^{-1} = f^{-1}\alpha$ in $\Sigma^{-1}\mathcal{T}$. Observe that it is enough to show this for $\beta = y$.

Now observe that

$$yf = y + \sum_{i=1}^{n} a_i x^{i-1},$$

so that

$$yf^{-1} = y - \sum_{i=1}^{n} a_i f^{-1} x^{i-1} = f^{-1}(fy - \sum_{i=1}^{n} a_i x^{i-1}),$$

showing the desired relation. It is easy to show that Σ *is not* a right Ore set in $\mathcal{T}$ (see below).

We can now provide an alternate (and direct) proof of the regularity of Q in this case, using the structure of this algebra. First note that the map $\mathcal{T} \to K(x)$ factors through $\Sigma^{-1}\mathcal{T}$, so we obtain a surjective homomorphism

$$\pi\colon Q \to K(x).$$

The kernel of this map is the ideal J of Q generated by $e_{00} = 1 - xy$, and so it suffices to show that this ideal is von Neumann regular. But it is easy to show that the ideal J is linearly spanned by elements of the form $f^{-1}x^i(1 - xy)y^j$, where $f \in \Sigma$, and is thus simple as an algebra. Moreover the corner $e_{00}Qe_{00}$ is easily seen to be isomorphic to K. It follows from Litoff's Theorem that J must be locally matricial, and in particular von Neumann regular.

Now consider the representation $\rho\colon \mathcal{T} \to \mathrm{End}_K(V)$ described above. The image of a Witt polynomial $f = 1+a_1x+\cdots+a_nx^n$ under ρ is a lower triangular matrix, with the n diagonals down to the main diagonal corresponding to elements $a_1, a_2, \dots, a_n$. This matrix is invertible in $\mathrm{End}_K(V)$. Indeed, if we consider the power series $1 + b_1x + b_2x^2 + \cdots$ giving the formal inverse of f, then this series is convergent in $\mathrm{End}_K(V)$ and so it defines an inverse for $\rho(f)$. It follows that there is a unique homomorphism $\tilde{\rho}\colon Q = \Sigma^{-1}\mathcal{T} \to \mathrm{End}_K(V)$ extending ρ. Since the ideal J is essential in Q, we see that $\tilde{\rho}$ must be injective.

We can now see easily that Σ is not a right Ore set in $\mathcal{T}$. Indeed observe that the element $\tilde{\rho}((1-x)^{-1}(1-xy))$ is the infinite matrix having the column $(1, 1, 1, \dots)^t$ as a first column, and whose other entries are all 0. It is a simple matter to show that, for any $f \in \Sigma$ and $\alpha \in \mathcal{T}$, the matrix $\alpha\tilde{\rho}(f^{-1})$ cannot be of this form.

5

Separated graphs and dynamical systems.

This chapter is based on work by Ara and Goodearl [AG1], [AG2] and on work on progress by Ara, Exel, and Katsura.

To simplify the exposition we will assume in this section that all graphs are finite. We will focus mainly in examples.

5.1 The Leavitt path algebra of a separated graph

Definition 5.1.1 A *separated graph* is a pair (E, C) where E is a graph, $C = \bigsqcup_{v \in E^0} C_v$, and C_v is a partition of $s^{-1}(v)$ (into pairwise disjoint nonempty subsets) for every vertex v. (In case v is a sink, we take C_v to be the empty family of subsets of $s^{-1}(v)$.)

Definition 5.1.2 The *Leavitt path algebra of the separated graph* (E, C) with coefficients in the field K is the K-algebra $L_K(E, C)$ with generators $\{v, e, e^* \mid v \in E^0,\ e \in E^1\}$, subject to the following relations:

(V) $vv' = \delta_{v,v'} v$ for all $v, v' \in E^0$, (E1) $s(e)e = er(e) = e$ for all $e \in E^1$,

(E2) $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$,

(SCK1) $e^*e' = \delta_{e,e'}r(e)$ for all $e, e' \in X$, $X \in C$, and

(SCK2) $v = \sum_{e\in X} ee^*$ for every finite set $X \in C_v$, $v \in E^0$.

The Leavitt path algebra $L_K(E)$ is just $L_K(E, C)$ where $C_v = \{s^{-1}(v)\}$ if $s^{-1}(v) \neq \emptyset$ and $C_v = \emptyset$ if $s^{-1}(v) = \emptyset$.

5.2 Examples: Leavitt's algebras $L(m, n)$.

Example 5.2.1 *We consider now a key class of examples, the separated graphs which correspond to the Leavitt algebras $L_K(m, n)$ for $1 \leq m \leq n$. Indeed we can think of these Leavitt path algebras as versions of $L_K(m, n)$ which are generated by "partial isometries". Let us consider the separated graph $(E(m, n), C(m, n))$, where*

1. $E(m, n)^0 := \{v, w\}$ *(with $v \neq w$).*
2. $E(m, n)^1 := \{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m\}$ *(with $n + m$ distinct edges).*
3. $s(\alpha_i) = s(\beta_j) = v$ *and* $r(\alpha_i) = r(\beta_j) = w$ *for all* i, j.
4. $C(m, n) = C(m, n)_v := \big\{\{\alpha_1, \dots, \alpha_n\}, \{\beta_1, \dots, \beta_m\}\big\}$.

We will show that the structure of $A_{m,n} := L_K(E(m, n), C(m, n))$ is closely related to the structure of the classical Leavitt algebra $L_K(m, n)$.

Observe that the corner algebras $vA_{m,n}v$ and $wA_{m,n}w$ are full corners of $A_{m,n}$, that is, $A_{m,n}vA_{m,n} = A_{m,n}wA_{m,n} = A_{m,n}$. In particular, $vA_{m,n}v$, $wA_{m,n}w$, and $A_{m,n}$ are Morita equivalent to each other. We will describe their structure below.

Recall that $L_K(m, n)$ is generated by elements X_{ij} and X^*_{ij}, for $i = 1, \dots, m$ and $j = 1, \dots, n$, such that $XX^* = I_m$ and $X^*X = I_n$, where X denotes the $m \times n$ matrix (X_{ij}) and X^* denotes the *-transpose of X, i.e., the $n \times m$ matrix with entries $(X^*)_{ji} = X^*_{ij}$.

Proposition 5.2.2 *Let $m \leq n$ be positive integers, and define $E(m,n)$, $C(m,n)$, $A_{m,n}$ as above.*

1. *There are K-algebra isomorphisms*
$$A_{m,n} \cong M_{m+1}(L_K(m,n)) \cong M_{n+1}(L_K(m,n))$$
$$vA_{m,n}v \cong M_m(L_K(m,n)) \cong M_n(L_K(m,n)) \qquad wA_{m,n}w \cong L_K(m,n).$$

2. *The monoids $\mathcal{V}(L_K(m,n))$, $\mathcal{V}(A_{m,n})$, $\mathcal{V}(vA_{m,n}v)$, and $\mathcal{V}(wA_{m,n}w)$ are all of the form $\langle x \mid mx = nx \rangle$, where the generator x corresponds to the classes $[1]$, $[w]$, $[\alpha_1\alpha_1^*]$, $[w]$ in the four respective cases.*

3. *Moreover, $vA_{m,n}v$ is the Bergman algebra obtained from $R := M_n(K) * M_m(K)$ by adjoining a universal isomorphism between the left R-modules Rf_{11} and Rg_{11}, where $(f_{ij})_{i,j=1}^n$ and $(g_{ij})_{i,j=1}^m$ are sets of matrix units corresponding to the factors $M_n(K)$ and $M_m(K)$ respectively.*

4. *There is a surjective unital K-algebra homomorphism $\rho : L_K(m,n) \to vA_{m,n}v$. If $\mathcal{V}(L_K(m,n))$ and $\mathcal{V}(vA_{m,n}v)$ are identified with $\langle x \mid mx = nx \rangle$ as in (2), then $\mathcal{V}(\rho)$ is given by multiplication by m (equivalently, multiplication by n).*

Proof. Set $A := A_{m,n}$ and $L := L_K(m,n)$. We construct various K-algebra homomorphisms between algebras presented by generators and relations. In all cases, it is routine to check that the appropriate relations are satisfied by the proposed images for the generators, and we omit these details.

(1) We identify L with the diagonal copies of itself in the various matrix algebras $M_d(L)$. Let $(e_{ij})_{i,j=1}^{m+1}$ be the standard family of matrix units in $M_{m+1}(L)$, and observe that $M_{m+1}(L)$ is presented by the generators e_{ij}, X_{ij}, X_{ij}^* together with the following three types of relations:

(a) The defining relations for the X_{ij} and X_{ij}^* in L.

(b) The matrix unit relations for the e_{ij}.

(c) The commutation relations $e_{kl}X_{ij} = X_{ij}e_{kl}$ and $e_{kl}X_{ij}^* = X_{ij}^*e_{kl}$ for all i, j, k, l.

Moreover, $M_{m+1}(L)$ is a free left (or right) L-module with basis $\{e_{ij} \mid 1 \le i,j \le m+1\}$. Analogous statements hold for $M_m(L)$, and we identify $M_m(L)$ with the corner $eM_{m+1}(L)e$ where $e := e_{11} + \cdots + e_{mm}$.

There exist a K-algebra homomorphism $\psi : A \to M_{m+1}(L)$ such that

$$\begin{aligned}
&\psi(v) = e && \psi(w) = e_{m+1,m+1} && \\
&\psi(\alpha_i) = \sum_{l=1}^{m} X_{li} e_{l,m+1} && \psi(\alpha_i^*) = \sum_{l=1}^{m} X_{li}^* e_{m+1,l} && (i = 1, \dots, n) \\
&\psi(\beta_j) = e_{j,m+1} && \psi(\beta_j^*) = e_{m+1,j} && (j = 1, \dots, m),
\end{aligned}$$

and a K-algebra homomorphism $\phi : M_{m+1}(L) \to A$ such that

$$\begin{aligned}
\phi(e_{ij}) &= \beta_i \beta_j^* && (i,j = 1, \dots, m) \\
\phi(e_{i,m+1}) &= \beta_i && (i = 1, \dots, m) \\
\phi(e_{m+1,j}) &= \beta_j^* && (j = 1, \dots, n) \\
\phi(e_{m+1,m+1}) &= w && \\
\phi(X_{ij}) &= \beta_i^* \alpha_j + \sum_{l=1}^{m} \beta_l \beta_i^* \alpha_j \beta_l^* && (i = 1, \dots, m;\ j = 1, \dots, n) \\
\phi(X_{ij}^*) &= \alpha_j^* \beta_i + \sum_{l=1}^{m} \beta_l \alpha_j^* \beta_i \beta_l^* && (i = 1, \dots, m;\ j = 1, \dots, n).
\end{aligned}$$

Moreover, ϕ and ψ are mutual inverses. Thus, $A \cong M_{m+1}(L)$.

Isomorphisms between A and $M_{n+1}(L)$ are obtained in a similar fashion, by interchanging the roles of the α_i and β_j in the constructions of ψ and ϕ above.

Since ψ maps v to e, it restricts to an isomorphism of vAv onto $eM_{m+1}(L)e \equiv M_m(L)$. Similarly, $vAv \cong M_n(L)$. Since ψ maps w to $e_{m+1,m+1}$, it restricts to an isomorphism of wAw onto $e_{m+1,m+1}M_{m+1}(L)e_{m+1,m+1}$, and the latter algebra is isomorphic to L.

(2) By [11, Theorem 6.1], $\mathcal{V}(L) = \langle x \mid mx = nx \rangle$ with x corresponding to the class of the free module ${}_LL$, that is, to the class $[1_L]$. Applying the last isomorphism of part (1) immediately yields $\mathcal{V}(wAw) = \langle x \mid mx = nx \rangle$ with x corresponding to $[w]$. In view of the equivalence

$$Aw \otimes_{wAw} (-) : wAw - Mod \longrightarrow A - Mod,$$

it follows that $\mathcal{V}(A) = \langle x \mid mx = nx \rangle$ with x corresponding to $[w]$. Note that $[w] = [\alpha_1\alpha_1^*]$ in $\mathcal{V}(A)$ and that $\alpha_1\alpha_1^* \in vAv$. In view of the equivalence

$$vA \otimes_A (-) : A - Mod \longrightarrow vAv - Mod,$$

it thus follows that $\mathcal{V}(vAv) = \langle x \mid mx = nx \rangle$ with x corresponding to $[\alpha_1\alpha_1^*]$.

(3) Let $\hat{R}$ be the Bergman algebra obtained from R by adjoining a universal isomorphism between the modules Rf_{11} and Rg_{11}. Thus, $\hat{R}$ is presented by generators f_{ij}, g_{ij}, u, u^* where

(a) The f_{ij} satisfy the relations for a complete set of $n \times n$ matrix units.

(b) The g_{ij} satisfy the relations for a complete set of $m \times m$ matrix units.

(c) $u = f_{11}ug_{11}$, $u^* = g_{11}u^*f_{11}$, $uu^* = f_{11}$, and $u^*u = g_{11}$.

There is a K-algebra homomorphism $\theta : R \to vAv$ such that

$$\theta(f_{ij}) = \alpha_i\alpha_j^* \qquad\qquad \theta(g_{ij}) = \beta_i\beta_j^*$$

for all i, j. The universal property of the Bergman construction implies that θ extends uniquely to a K-algebra homomorphism $\hat{\theta} : \hat{R} \to vAv$ such that

$$\theta(u) = \alpha_1\beta_1^* \qquad\qquad \theta(u^*) = \beta_1\alpha_1^* .$$

There is a K-algebra homomorphism $\xi : M_m(L) \to \hat{R}$ such that

$$\begin{aligned} \xi(e_{ij}) &= g_{ij} && (i, j = 1, \dots, m) \\ \xi(X_{ij}) &= \sum_{l=1}^{m} g_{li}f_{j1}ug_{1l} && (i = 1, \dots, n;\ j = 1, \dots, m) \\ \xi(X_{ij}^*) &= \sum_{l=1}^{m} g_{l1}u^*f_{1j}g_{il} && (i = 1, \dots, n;\ j = 1, \dots, m). \end{aligned}$$

Let $\psi' : vAv \to M_m(L)$ be the isomorphism obtained by restricting ψ to vAv. Then $\hat{\theta}$ and $\xi\psi'$ are mutual inverses, proving that $\hat{R} \cong vAv$.

(4) There is a K-algebra homomorphism $\rho : L \to vAv$ such that

$$\rho(X_{ij}) = \alpha_j\beta_i^* \qquad\qquad \rho(X_{ij}^*) = \beta_i\alpha_j^*$$

for $i = 1, \dots, m$ and $j = 1, \dots, n$. Since vAv is generated by the elements $\alpha_j\alpha_i^*$, $\alpha_j\beta_i^*$, $\beta_i\alpha_j^*$, $\beta_i\beta_j^*$, we see that ρ is surjective. As in (2), $\mathcal{V}(L) = \langle x \mid mx = nx\rangle$ with x corresponding to the class [1], and $\mathcal{V}(vAv) = \langle x \mid mx = nx\rangle$ with x corresponding to $[\alpha_1\alpha_1^*]$. Since $\mathcal{V}(\rho)$ maps [1] to $[v] = \sum_{l=1}^m [\alpha_l\alpha_l^*] = m[\alpha_1\alpha_1^*]$, we conclude that $\mathcal{V}(\rho)$ is given by multiplication by m. In the given monoid, this is the same as multiplication by n.

5.3 Dynamical systems. An easy example: the algebra generated by a partial isometry

The C^*-algebra generated by a partial isometry has raised some recent interest in the literature. We will provide a purely algebraic version of it. Although this algebraic version is a $*$-algebra in a natural way, it may be defined without any reference to the involution (indeed this also happens with many other algebras, such as group algebras). So, given any field K, we define the (universal) K-algebra generated by a partial isometry as

$$A = K\langle s, s^* \mid ss^*s = s,\ s^*ss^* = s^*\rangle.$$

Note that indeed this is the K-algebra generated by a (universal) von Neumann regular element. This algebra is related to the Leavitt path algebra of the following separated graph (E, C):

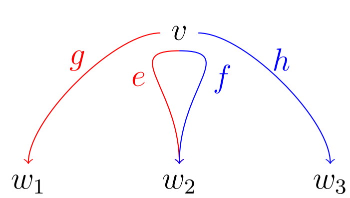


Figura 5.1: The separated graph (E, C)

As indicated in the picture, we have $C_v = \{X, Y\}$, with $X = \{g, e\}$, and $Y = \{f, h\}$. We have $vL(E, C)v \cong A$, with s corresponding to ef^*. We can compute the $\mathcal{V}$-monoid of this algebra (indeed of any Leavitt path algebra of a separated graph) by using the Bergman's results, obtaining that

$$\mathcal{V}(A) = \langle p, q, r \mid p + q = p + r \rangle.$$

A shortcoming of Leavitt path algebras of separated graphs is that, although they are generated by partial isometries, the product of the generating partial isometries is not in general a partial isometry. This is the case in particular with the algebra A. To circumvent this problem, we consider the multiplicative semigroup U generated by $\{s, s^*\}$, and we construct the following algebra:

$$\mathcal{A} = A/\langle x - xx^*x \mid x \in U \rangle.$$

The algebra $\mathcal{A}$ is much nicer than A, and it admits a representation

$$\mathcal{A} = C_K(\Omega) \rtimes_\theta \mathbb{Z}\,,$$

where $C_K(\Omega)$ is the space of locally constant functions on a certain totally disconnected Hausdorff compact space Ω, and θ is a partial automorphism of $C(\Omega)$.

Indeed the space Ω admits a decomposition:

$$\Omega = X_1 \sqcup X_2 = Y_1 \sqcup Y_2,$$

where X_1, X_2, Y_1, Y_2 are clopen subsets of Ω, and there is a homeomorphism α from X_1 onto Y_1 such that $\theta = \alpha^*$. Moreover given any other such dynamical system $(\Omega', X_1', Y_1', X_2', Y_2', \beta)$, there is a unique continuous equivariant map $\psi \colon \Omega' \to \Omega$. (Here equivariant of course means that $\psi(Y_i') \subseteq Y_i$, $\psi(X_i') \subseteq X_i$ and $\psi \circ \alpha' = \alpha \circ \psi_{|X_1'}$.)

We are going to describe here this system, and the structure of $\mathcal{A}$. The points of Ω are all the closed subintervals $[a, b]$ containing 0 of $\{-\infty\} \cup \mathbb{Z} \cup \{+\infty\}$, and

$$X_1 = \{[a, b] \mid b > 0\}, \quad Y_1 = \{[a, b], a < 0\}.$$

Then $\alpha([a, b]) = [a - 1, b - 1]$ for $[a, b] \in X_1$ is the desired partial homeomorphism of Ω.

For $i, j \in \mathbb{Z}^+$, set

$$q_{-i,j} = (s^i(s^*)^i - s^{i+1}(s^*)^{i+1})((s^*)^j s^j - (s^*)^{j+1} s^{j+1}).$$

We have

Proposition 5.3.1 *(i) The elements $q_{-i,j}$ are a family of pairwise orthogonal, minimal idempotents of $\mathcal{A}$. Moreover $q_{-i,j} \sim q_{-k,l}$ if and only if $j+i=k+l$.*

(ii) $\mathrm{Soc}(\mathcal{A})$ *is the ideal generated by the idempotents $q_{-i,j}$. Thus*

$$\mathrm{Soc}(\mathcal{A}) \cong \bigoplus_{t\in\mathbb{N}} M_t(K).$$

(iii) $\mathcal{A}/\mathrm{Soc}(\mathcal{A}) = I \oplus J$, *where* $I = \langle 1 - ss^* \rangle$ *and* $J = \langle 1 - s^*s \rangle$.

(iv) $\mathcal{A}/\mathrm{Soc}_2(\mathcal{A}) = K[s, s^{-1}]$.

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