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SHORT COMMUNICATIONS

Resolvent Convergence of Sturm–Liouville Operators with Singular Potentials

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1. MAIN RESULT

Suppose that the formal differential expression

$$l(y) = -y''(t) + q'(t)y(t), \qquad q(\,\cdot\,) \in L_2([a,b],\mathbb{C}) =: L_2, \tag{1.1}$$

is given on a compact interval $[a,b]$. In rigorous terms, it can be defined as *quasidifferential*, $l(y) := D^{[2]}y$, by introducing the *quasiderivatives* [1]:

$$D^{[0]}y = y, \qquad D^{[1]}y = y' - qy, \qquad D^{[2]}y = -(D^{[1]}y)' - qD^{[1]}y - q^2y.$$

Consider the family of quasidifferential expressions $l_\varepsilon(\,\cdot\,)$ of the form (1.1) with potentials $q_\varepsilon(\,\cdot\,) \in L_2$, $\varepsilon \in [0,\varepsilon_0]$. In the Hilbert space L_2 with norm $\|\cdot\|_2$, with each of these expressions we can associate a closed densely given quasidifferential operator $L_\varepsilon y := l_\varepsilon(y)$,

$$\operatorname{Dom}(L_\varepsilon) := \{y \in L_2 : \exists D_\varepsilon^{[2]}y \in L_2,\ \alpha(\varepsilon)\mathscr{Y}_a(\varepsilon) + \beta(\varepsilon)\mathscr{Y}_b(\varepsilon) = 0\},$$

where the matrices $\alpha(\varepsilon), \beta(\varepsilon) \in \mathbb{C}^{2\times 2}$ and the vectors

$$\mathscr{Y}_a(\varepsilon) := \{y(a), D_\varepsilon^{[1]}y(a)\},\ \mathscr{Y}_b(\varepsilon) := \{y(b), D_\varepsilon^{[1]}y(b)\} \in \mathbb{C}^2.$$

Let us recall that the family of operators L_ε converges to L_0 in the sense of norm resolvent convergence, $L_\varepsilon \xrightarrow{R} L_0$, if there exists a number $\mu \in \mathbb{C}$ belonging to the resolvent sets $\rho(L_0)$ and $\rho(L_\varepsilon)$ (for all sufficiently small ε) and

$$\|(L_\varepsilon - \mu)^{-1} - (L_0 - \mu)^{-1}\| \to 0, \qquad \varepsilon \to +0.$$

This definition is independent of the choice of the point $\mu \in \rho(L_0)$ [2].

For the case in which the matrices $\alpha(\varepsilon)$, $\beta(\varepsilon)$ are independent of ε, the following important theorem was established in [1].

Theorem 1. *Suppose that $\|q_\varepsilon - q_0\|_2 \to 0$ as $\varepsilon \to +0$ and the resolvent set of the operator L_0 is not empty. Then $L_\varepsilon \xrightarrow{R} L_0$.*

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The goal in the present paper is to generalize Theorem 1 to the case of boundary conditions that depend on ε, and to use the results of [3], [4] to weaken the conditions on the L_2-convergence of the potentials.

We introduce the following notation:

$$c^{\vee}(t) := \int_a^t c(x)\,dx$$

and $\|\cdot\|_C$ for the sup-norm.

Theorem 2. *Suppose that the resolvent set of the operator L_0 is not empty and, as $\varepsilon \to +0$, the following conditions hold*:

1) $\|q_\varepsilon\|_2 = O(1)$;

2) $\|(q_\varepsilon - q_0)^{\vee}\|_C \to 0$;

3) $\|(q_\varepsilon^2 - q_0^2)^{\vee}\|_C \to 0$;

4) $\alpha(\varepsilon) \to \alpha(0)$, $\beta(\varepsilon) \to \beta(0)$.

Then $L_\varepsilon \xrightarrow{R} L_0$.

Note that condition 3) is not additive. Condition 1) (in view of 2), 3)) can be weakened in several directions.

Actually, we shall prove a stronger assertion on the convergence of the Green function of the operators under consideration in the norm $\|\cdot\|_\infty$ of the space L_∞ on the square $[a,b]\times[a,b]$.

2. COMPARISON OF THEOREMS 1 AND 2

Let us show that if $\|q_\varepsilon - q_0\|_2 \to 0$, $\varepsilon \to +0$, then conditions 1), 2), 3) of Theorem 2 hold.

Indeed,

$$\|q_\varepsilon\|_2 \le \|q_\varepsilon - q_0\|_2 + \|q_0\|_2 = O(1).$$

In addition,

$$\left|\int_a^t (q_\varepsilon - q_0)\,ds\right| \le \int_a^b |q_\varepsilon - q_0|\,ds \le \left(\int_a^b |q_\varepsilon - q_0|^2\,ds\right)^{1/2} (b-a)^{1/2} \to 0, \qquad \varepsilon \to +0,$$

$$\begin{aligned}\left|\int_a^t (q_\varepsilon^2 - q_0^2)\,ds\right| &\le \int_a^b |q_\varepsilon^2 - q_0^2|\,ds \le \int_a^b |q_\varepsilon - q_0|\,|q_\varepsilon + q_0|\,ds \\ &\le \left(\int_a^b |q_\varepsilon - q_0|^2\,ds\right)^{1/2}\left(\int_a^b |q_\varepsilon + q_0|^2\,ds\right)^{1/2} \to 0, \qquad \varepsilon \to +0.\end{aligned}$$

The example given below shows that Theorem 2 is stronger than Theorem 1.

Example 1. Suppose that $q_0(t) \equiv 0$, $q_\varepsilon(t) = e^{it/\varepsilon}$, $t \in [0,1]$.

The family of operators L_ε specified by these potentials does not satisfy the assumptions of Theorem 1, because

$$\|q_\varepsilon - q_0\|_2^2 = \|q_\varepsilon\|_2^2 = \int_0^1 |q_\varepsilon|^2\,ds \equiv 1.$$

It is readily verified that the functions $q_\varepsilon(\,\cdot\,)$ do not converge to zero even with respect to the Lebesgue measure. However, they satisfy conditions 1), 2), 3) of Theorem 2. Indeed, $\|q_\varepsilon\|_2 \le 1$. In addition,

$$\|q_\varepsilon^{\vee}\|_C = \left\|\int_0^t e^{is/\varepsilon}\,ds\right\|_C \le 2\varepsilon \to 0, \qquad \varepsilon \to +0,$$

$$\|(q_\varepsilon^2)^{\vee}\|_C = \left\|\int_0^t (e^{is/\varepsilon})^2\,ds\right\|_C \le \varepsilon \to 0, \qquad \varepsilon \to +0.$$

3. PRELIMINARY RESULT

Consider the boundary-value problem

$$y'(t;\varepsilon) = A(t;\varepsilon)y(t;\varepsilon) + f(t;\varepsilon), \qquad t \in [a,b], \quad \varepsilon \in [0,\varepsilon_0], \tag{3.1_ε}$$

$$U_\varepsilon y(\,\cdot\,;\varepsilon) = 0, \tag{3.2_ε}$$

where the matrix functions $A(\,\cdot\,,\varepsilon)$ belong to $L_1^{m\times m}$, the vector functions $f(\,\cdot\,,\varepsilon)$ belong to L_1^m and

$$U_\varepsilon\colon C([a,b];\mathbb{C}^m) \to \mathbb{C}^m$$

are linear continuous operators.

Following [3], [4], we introduce a descriptive definition.

Definition. Let $\mathscr{M}^m[a,b] =: \mathscr{M}^m$, $m \in \mathbb{N}$, be the class of all matrix functions $R(\,\cdot\,;\varepsilon)\colon [0,\varepsilon_0] \to L_1^{m\times m}$ parametrized by ε for which the solution of the Cauchy problem

$$Z'(t;\varepsilon) = R(t;\varepsilon)Z(t;\varepsilon), \qquad Z(a;\varepsilon) = I_m,$$

satisfies the limit relation

$$\lim_{\varepsilon\to+0} \|Z(\,\cdot\,;\varepsilon) - I_m\|_C = 0.$$

Constructive sufficient conditions for the inclusion $R(\,\cdot\,;\varepsilon) \in \mathscr{M}^m$ are the consequence of results from [5]. The simplest of them

$$\|R(\,\cdot\,;\varepsilon)\|_1 = O(1), \qquad \|R^\vee(\,\cdot\,;\varepsilon)\|_C \to 0,$$

where $\|\cdot\|_1$ is the norm on $L_1^{m\times m}$, is used in the proof of Theorem 2.

The following general theorem was established in [4].

Theorem 3. *Suppose that the following conditions hold*:

1) *the homogeneous limit boundary-value problem* (3.1_ε), (3.2_ε), $\varepsilon = 0$, *with* $f(\,\cdot\,;0) \equiv 0$, *has only the trivial solution*;

2) $A(\,\cdot\,;\varepsilon) - A(\,\cdot\,;0) \in \mathscr{M}^m$;

3) $\|U_\varepsilon - U_0\| \to 0$ *as* $\varepsilon \to +0$.

Then, for sufficiently small ε*, there exist Green matrices* $G(t,s;\varepsilon)$ *of problems* (3.1_ε), (3.2_ε) *and, on the square* $[a,b]\times[a,b]$,

$$\|G(\,\cdot\,,\,\cdot\,;\varepsilon) - G(\,\cdot\,,\,\cdot\,;0)\|_\infty \to 0, \qquad \varepsilon \to +0. \tag{3.3}$$

Condition 3) of Theorem 3 cannot be replaced by a weaker condition of the strong convergence of the operators $U_\varepsilon \xrightarrow{s} U_0$ [4]. However, as is readily verified, for the multipoint "boundary" operators

$$U_\varepsilon y := \sum_{k=1}^{n} B_k(\varepsilon)y(t_k), \qquad \{t_k\} \subset [a,b], \quad B_k(\varepsilon) \in \mathbb{C}^{m\times m}, \quad n \in \mathbb{N},$$

the conditions for both strong and norm convergence are equivalent to

$$\|B_k(\varepsilon) - B_k(0)\| \to 0, \qquad \varepsilon \to +0, \quad k \in \{1,\dots,n\}.$$

4. PROOF OF THEOREM 2

First, let us present two simple lemmas, which are used to reduce Theorem 2 to Theorem 3.

Lemma 1. *The function $y(t)$ is a solution of the boundary-value problem*

$$D_\varepsilon^{[2]} y(t) = f(t;\varepsilon) \in L_2, \qquad \varepsilon \in [0, \varepsilon_0], \tag{4.1}$$

$$\alpha(\varepsilon)\mathscr{Y}_a(\varepsilon) + \beta(\varepsilon)\mathscr{Y}_b(\varepsilon) = 0, \tag{4.2}$$

if and only if the vector function $w(t) = (y(t), D_\varepsilon^{[1]} y(t))$ is a solution of the boundary-value problem

$$w'(t) = A(t;\varepsilon)w(t) + \varphi(t;\varepsilon), \tag{4.3}$$

$$\alpha(\varepsilon)w(a) + \beta(\varepsilon)w(b) = 0, \tag{4.4}$$

where the square matrix is

$$A(\,\cdot\,;\varepsilon) := \begin{pmatrix} q_\varepsilon & 1 \\ -q_\varepsilon^2 & -q_\varepsilon \end{pmatrix} \in L_1^{2\times 2}, \tag{4.5}$$

and $\varphi(\,\cdot\,;\varepsilon) := (0, -f(\,\cdot\,;\varepsilon))$.

Proof. Consider the system of equations

$$\begin{cases} (D_\varepsilon^{[0]} y(t))' = q_\varepsilon(t) D_\varepsilon^{[0]} y(t) + D_\varepsilon^{[1]} y(t), \\ (D_\varepsilon^{[1]} y(t))' = -q_\varepsilon^2(t) D_\varepsilon^{[0]} y(t) - q_\varepsilon(t) D_\varepsilon^{[1]} y(t) - f(t;\varepsilon). \end{cases}$$

If $y(\,\cdot\,)$ is a solution of Eq. (4.1), then it follows from the definition of the quasiderivatives that $y(\,\cdot\,)$ is a solution of this system. On the other hand, setting

$$w(t) = (D_\varepsilon^{[0]} y(t), D_\varepsilon^{[1]} y(t)) \qquad \text{and} \qquad \varphi(t;\varepsilon) = (0, -f(t;\varepsilon)),$$

the system given above can be written in the form of Eq. (4.3).

Taking into account the equalities $\mathscr{Y}_a(\varepsilon) = w(a)$, $\mathscr{Y}_b(\varepsilon) = w(b)$, we can easily see that the boundary conditions (4.2) are equivalent to the boundary conditions (4.4). □

Lemma 2. *Suppose that the following assumption holds*:

the homogeneous boundary-value problem $D_0^{[2]} y(t) = 0$, $\alpha(0)\mathscr{Y}_a(0) + \beta(0)\mathscr{Y}_b(0) = 0$ has only the trivial solution. $(\mathscr{E})$

Then, for a sufficiently small ε, the Green function $\Gamma(t,s;\varepsilon)$ of the semihomogeneous boundary-value problems (4.1)*,* (4.2) *exists and*

$$\Gamma(t,s;\varepsilon) = -g_{12}(t,s;\varepsilon) \qquad \text{a.e.},$$

where $g_{12}(t,s;\varepsilon)$ is the corresponding element of the Green matrix

$$G(t,s;\varepsilon) = (g_{ij}(t,s;\varepsilon))_{i,j=1}^2$$

of the two-point vector boundary-value problem (4.3)*,* (4.4)*.*

Proof. In view of Theorem 3 and Lemma 1, assumption $(\mathscr{E})$ implies that the homogeneous boundary-value problem

$$w'(t) = A(t;\varepsilon)w(t), \qquad \alpha(\varepsilon)w(a) + \beta(\varepsilon)w(b) = 0,$$

has only the trivial solution for sufficiently small ε. Then, for problem (4.3), (4.4), there exists a Green matrix

$$G(t,s,\varepsilon) = (g_{ij}(t,s))_{i,j=1}^2 \in L_\infty^{2\times 2},$$

with the help of which the unique solution of problem (4.3), (4.4) is written in the form

$$w_\varepsilon(t) = \int_a^b G(t,s;\varepsilon)\varphi(s;\varepsilon)\,ds, \qquad t \in [a,b], \quad \varphi(\,\cdot\,;\varepsilon) \in L_2.$$

The last equality can be rewritten as

$$\begin{cases} D_\varepsilon^{[0]} y_\varepsilon(t) = \displaystyle\int_a^b g_{12}(t,s;\varepsilon)(-\varphi(s;\varepsilon))\,ds, \\ D_\varepsilon^{[1]} y_\varepsilon(t) = \displaystyle\int_a^b g_{22}(t,s;\varepsilon)(-\varphi(s;\varepsilon))\,ds, \end{cases}$$

where $y_\varepsilon(\,\cdot\,)$ is the unique solution of problem (4.1), (4.2). This implies the assertion of Lemma 2.

Now, passing to the proof of Theorem 2, we note that since

$$(q_\varepsilon + \mu)^2 - (q_0 + \mu)^2 = (q_\varepsilon^2 - q_0^2) + 2\mu(q_\varepsilon - q_0),$$

in view of conditions 2), 3), we can assume without loss of generality that $0 \in \rho(L_0)$. Let us show that

$$\sup_{\|f\|_2 = 1} \|L_\varepsilon^{-1} f - L_0^{-1} f\| \to 0, \qquad \varepsilon \to +0.$$

The equation $L_\varepsilon^{-1} f = y_\varepsilon$ is equivalent to the relation $L_\varepsilon y_\varepsilon = f$, i.e., y_ε is a solution of problem (4.1), (4.2). It follows from the inclusion $0 \in \rho(L_0)$ that the assumption ($\mathscr{E}$) of Lemma 2 holds. Conditions 1)–3) of Theorem 2 imply that $A(\,\cdot\,;\varepsilon) - A(\,\cdot\,;0) \in \mathscr{M}^2$, where $A(\,\cdot\,;\varepsilon)$ is given by (4.5). Therefore, it follows from the assumptions of Theorem 2 that the assumptions of Theorem 3 hold for problem (4.3), (4.4). This means that there exist Green matrices $G(t,s;\varepsilon)$ for problems (4.3), (4.4) and the limit relation (3.3) holds. Taking into account Lemma 2, this implies the limit equality

$$\|\Gamma(\,\cdot\,,\,\cdot\,;\varepsilon) - \Gamma(\,\cdot\,,\,\cdot\,;0)\|_\infty \to 0, \qquad \varepsilon \to +0.$$

Then

$$\begin{aligned} \|L_\varepsilon^{-1} - L_0^{-1}\| &= \sup_{\|f\|_2=1} \left\| \int_a^b [\Gamma(t,s;\varepsilon) - \Gamma(t,s;0)] f(s)\,ds \right\|_2 \\ &\le (b-a)^{1/2} \sup_{\|f\|_2=1} \left\| \int_a^b |\Gamma(t,s;\varepsilon) - \Gamma(t,s;0)|\,|f(s)|\,ds \right\|_C \\ &\le (b-a) \|\Gamma(\,\cdot\,,\,\cdot\,;\varepsilon) - \Gamma(\,\cdot\,,\,\cdot\,;0)\|_\infty \to 0, \qquad \varepsilon \to +0, \end{aligned}$$

which implies the assertion of Theorem 2. □

5. THREE EXTENSIONS OF THEOREM 2

As was already noted, the assumptions of Theorem 2 can be weakened. Let

$$R(\,\cdot\,;\varepsilon) := A(\,\cdot\,;\varepsilon) - A(\,\cdot\,;0),$$

where the matrix function $A(\,\cdot\,;\varepsilon)$ is defined by relation (4.5).

Theorem 4. *In the statement of Theorem* 2, *condition* 1) *can be replaced by any one of the following three more general* (*in view of* 2) *and* 3)) *asymptotic conditions as* $\varepsilon \to +0$:

(I) $\|R(\,\cdot\,;\varepsilon) R^\vee(\,\cdot\,;\varepsilon)\|_1 \to 0$;

(II) $\|R^\vee(\,\cdot\,;\varepsilon) R(\,\cdot\,;\varepsilon)\|_1 \to 0$;

(III) $\|R(\,\cdot\,;\varepsilon) R^\vee(\,\cdot\,;\varepsilon) - R^\vee(\,\cdot\,;\varepsilon) R(\,\cdot\,;\varepsilon)\|_1 \to 0$.

Proof. The proof of Theorem 4 is similar to that of Theorem 2 if the following remark is taken into account. For condition 2) of Theorem 3 to hold, it suffices (see [5]) that $\|R^{\vee}(\,\cdot\,;\varepsilon)\|_C \to 0$ and either the condition $\|R(\,\cdot\,;\varepsilon)\|_1 = O(1)$ (as in Theorem 2) or any one of three conditions (I), (II), (III) from Theorem 4 holds. □

The example given below shows that each part of Theorem 4 is stronger than Theorem 2.

Example 2. Suppose that $q_0(t) \equiv 0$, $q_\varepsilon(t) = \rho(\varepsilon)e^{it/\varepsilon}$, $t \in [0, 1]$.

Simple calculations show that if

$$\rho(\varepsilon) \uparrow \infty, \quad \varepsilon\rho^3(\varepsilon) \to 0, \qquad \varepsilon \to +0,$$

then assumptions 2), 3) of Theorem 4 hold as well as any one of conditions (I), (II), (III) of Theorem 4. However, condition 1) of Theorem 2 is violated, because $\|q_\varepsilon - q_0\|_2 \uparrow \infty$.

For Schrödinger operators of the form (1.1) on $\mathbb{R}$ with real-valued periodic potential q', where $q \in L_2^{\mathrm{loc}}$, self-adjointness and sufficient conditions for norm resolvent convergence were established in [6]. For other problems related to those studied in [1], see also [7], [8].

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