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Multi-interval Sturm–Liouville boundary-value problems with distributional potentials

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We study the multi-interval boundary-value Sturm–Liouville problems with distributional potentials. For the corresponding symmetric operators boundary triplets are found and the constructive descriptions of all self-adjoint, maximal dissipative and maximal accumulative extensions and generalized resolvents in terms of homogeneous boundary conditions are given. It is shown that all real maximal dissipative and maximal accumulative extensions are self-adjoint and all such extensions are described.

In recent years, the interest in multi-interval differential and quasi-differential operators has increased (see [1–4]). The main attention is paid to the case where a (quasi-)differential expression is formally self-adjoint. From the operator-theoretic point of view this corresponds to the situation where we investigate extensions of a symmetric (quasi-)differential operator with equal deficiency indices in the direct sum of Hilbert spaces on the basis of Glazman–Krein–Naimark theory [5–8]. In the present paper, we develop another approach to such problems based on the concept of boundary triplets [9, 10].

Let $m \in \mathbb{N}$, $a = a_0 < a_1 < \cdots < a_m = b$ be a partition of a finite interval $[a, b]$ into m parts and on every interval (a_{i-1}, a_i), $i \in \{1, \ldots, m\}$, let the formal Sturm–Liouville expression

$$l_i(y) = -(p_i(t)y')' + q_i(t)y \tag{1}$$

be given. Here the measurable finite functions p_i and Q_i are such that

$$\frac{1}{p_i},\ \frac{Q_i}{p_i},\ \frac{Q_i^2}{p_i} \in L_1([a_{i-1}, a_i], \mathbb{R}), \tag{2}$$

the potentials $q_i = Q_i'$, and the derivative is understood in the sense of distributions.

For $m = 1$ the boundary-value problems for the formal differential expression (1) under assumptions (2) were investigated in [11] on the basis of its regularization by Shin–Zettl quasi-derivatives. In this paper the most of the results of [11] is extended onto the case of an arbitrary $m \in \mathbb{N}$.

We introduce the quasi-derivatives

$$D_i^{[0]}y = y,$$

$$D_i^{[1]}y = p_i y' - Q_i y,$$

$$D_i^{[2]}y = (D_i^{[1]}y)' + \frac{Q_i}{p_i}D_i^{[1]}y + \frac{Q_i^2}{p_i}y$$

on every interval (a_{i-1}, a_i), as in [11].

Then the maximal and minimal operators

$$L_{i,1}\colon\quad y\to l_i[y],\quad \mathrm{Dom}(L_{i,1})\colon = \{y\in L_2 \mid y, D_i^{[1]}y\in AC([a_{i-1},a_i],\mathbb{C}), D_i^{[2]}y\in L_2\},$$

$$L_{i,0}\colon\quad y\to l_i[y],\quad \mathrm{Dom}(L_{i,0}) := \{y\in \mathrm{Dom}(L_{i,1}) \mid D_i^{[k]}y(a_{i-1}) = D_i^{[k]}y(a_i) = 0, k=0,1\}$$

are defined in the spaces $L_2((a_{i-1},a_i),\mathbb{C})$. According to [11] the operators $L_{i,1}$, $L_{i,0}$ are closed and densely defined in $L_2([a_{i-1},a_i],\mathbb{C})$. The operator $L_{i,0}$ is symmetric with the deficiency indices $(2,2)$ and

$$L_{i,0}^* = L_{i,1},\qquad L_{i,1}^* = L_{i,0}.$$

Recall that a *boundary triplet* of a closed densely defined symmetric operator T with equal (finite or infinite) deficiency indices is called a triplet (H,Γ_1,Γ_2) where H is an auxiliary Hilbert space and Γ_1, Γ_2 are the linear maps from $\mathrm{Dom}(T^*)$ to H such that

1) for any f, $g\in \mathrm{Dom}(T^*)$ there holds

$$(T^*f,g)_{\mathcal{H}} - (f,T^*g)_{\mathcal{H}} = (\Gamma_1 f,\Gamma_2 g)_H - (\Gamma_2 f,\Gamma_1 g)_H;$$

2) for any g_1, $g_2\in H$ there is a vector $f\in \mathrm{Dom}(T^*)$ such that $\Gamma_1 f = g_1$ and $\Gamma_2 f = g_2$.

It is proved in [11] that for every $i=1,\ldots,m$ the triplet $(\mathbb{C}^2,\Gamma_{1,i},\Gamma_{2,i})$, where $\Gamma_{1,i}$, $\Gamma_{2,i}$ are linear maps

$$\Gamma_{1,i}y := (D_i^{[1]}y(a_{i-1}+), -D_i^{[1]}y(a_i-)),\qquad \Gamma_{2,i}y := (y(a_{i-1}+), y(a_i-)),$$

from $\mathrm{Dom}(L_{i,1})$ to $\mathbb{C}^2$ is a boundary triplet for the operator $L_{i,0}$.

We consider the space $L_2([a,b],\mathbb{C})$ as a direct sum $\oplus_{i=1}^m L_2([a_{i-1},a_i],\mathbb{C})$ which consists of vector functions $f=\oplus_{i=1}^m f_i$ such that $f_i\in L_2([a_{i-1},a_i],\mathbb{C})$. In this space we consider operators $L_{\max} = \oplus_{i=1}^m L_{i,1}$ and $L_{\min} = \oplus_{i=1}^m L_{i,0}$.

Then the operators $L_{\max}$, $L_{\min}$ are closed and densely defined in $L_2([a,b],\mathbb{C})$. The operator $L_{\min}$ is symmetric with the deficiency indices $(2m,2m)$ and

$$L_{\min}^* = L_{\max},\qquad L_{\max}^* = L_{\min}.$$

Note that the minimal operator $L_{\min}$ may be not semi-bounded even in the case of a single-interval boundary-value problem since the function p may reverse sign.

Theorem 1. *The triplet* $(\mathbb{C}^{2m},\Gamma_1,\Gamma_2)$ *where* Γ_1, Γ_2 *are linear maps*

$$\Gamma_1 y := (\Gamma_{1,1}y,\Gamma_{1,2}y,\ldots,\Gamma_{1,m}y),\qquad \Gamma_2 y := (\Gamma_{2,1}y,\Gamma_{2,2}y,\ldots,\Gamma_{2,m}y)$$

from $\mathrm{Dom}(L_{\max})$ *onto* $\mathbb{C}^{2m}$ *is a boundary triplet for* $L_{\min}$.

Denote by L_K the restriction of $L_{\max}$ onto set of functions $y(t)\in \mathrm{Dom}(L_{\max})$ satisfying the homogeneous boundary condition

$$(K-I)\Gamma_1 y + i(K+I)\Gamma_2 y = 0.$$

Similarly, denote by L^K the restriction of $L_{\max}$ onto the set of functions $y(t)\in \mathrm{Dom}(L_{\max})$ satisfying the homogeneous boundary condition

$$(K-I)\Gamma_1 y - i(K+I)\Gamma_2 y = 0.$$

Here K is a bounded operator in $\mathbb{C}^{2m}$.

The constructive description of the various classes of extensions of the operator $L_{\min}$ is given by the following theorem.

Theorem 2. *Every L_K with K being a contracting operator in $\mathbb{C}^{2m}$ is a maximal dissipative extension of $L_{\min}$. Similarly, every L^K with K being a contracting operator in $\mathbb{C}^{2m}$ is a maximal accumulative extension of the operator $L_{\min}$.*

Conversely, for any maximal dissipative (respectively, maximal accumulative) extension $\widetilde{L}$ of the operator $L_{\min}$ there exists the unique contracting operator K such that $\widetilde{L} = L_K$ (respectively, $\widetilde{L} = L^K$).

The extensions L_K and L^K are self-adjoint if and only if K is a unitary operator on $\mathbb{C}^{2m}$.

Recall that a linear operator T acting in $L_2([a,b],\mathbb{C})$ is called *real* if:

1) for every function f from $\mathrm{Dom}(T)$ the complex conjugate function $\overline{f}$ also lies in $\mathrm{Dom}(T)$;

2) the operator T maps complex conjugate functions into complex conjugate functions, that is $T(\overline{f}) = \overline{T(f)}$.

One can see that the maximal and minimal operators are real.

Theorem 3. *All real maximal dissipative and maximal accumulative extensions of the minimal operator $L_{\min}$ are self-adjoint. The self-adjoint extension L_K or L^K is real if and only if the unitary matrix K is symmetric.*

Let us recall that a *generalized resolvent* of a closed symmetric operator T in a Hilbert space $\mathcal{H}$ is an operator-valued function $\lambda \mapsto R_\lambda$ defined on $\mathbb{C} \setminus \mathbb{R}$, which can be represented as

$$R_\lambda f = P^+(T^+ - \lambda I^+)^{-1} f, \qquad f \in \mathcal{H},$$

where T^+ is a self-adjoint extension of T which acts in a certain Hilbert space $\mathcal{H}^+ \supset \mathcal{H}$, I^+ is the identity operator on $\mathcal{H}^+$, and P^+ is the orthogonal projection operator from $\mathcal{H}^+$ onto $\mathcal{H}$. It is known that an operator-valued function R_λ ($\mathrm{Im}\,\lambda \neq 0$) is a generalized resolvent of a symmetric operator T if and only if it can be represented as

$$(R_\lambda f, g)_{\mathcal{H}} = \int\limits_{-\infty}^{+\infty} \frac{d(F_\mu f, g)}{\mu - \lambda}, \qquad f, g \in \mathcal{H},$$

where F_μ is a generalized spectral function of the operator T, i. e. $\mu \mapsto F_\mu$ is an operator-valued function F_μ defined on $\mathbb{R}$ and taking values in the space of continuous linear operators in $\mathcal{H}$ with the following properties:

1) for $\mu_2 > \mu_1$, the difference $F_{\mu_2} - F_{\mu_1}$ is a bounded non-negative operator;

2) $F_{\mu+} = F_\mu$ for any real μ;

3) for any $x \in \mathcal{H}$,

$$\lim_{\mu \to -\infty} \|F_\mu x\|_{\mathcal{H}} = 0, \qquad \lim_{\mu \to +\infty} \|F_\mu x - x\|_{\mathcal{H}} = 0.$$

The following theorem provides a description of all generalized resolvents of the operator $L_{\min}$.

Theorem 4. *1. Every generalized resolvent R_λ of the operator $L_{\min}$ in the half-plane* $\mathrm{Im}\,\lambda < 0$ *acts by the rule $R_\lambda h = y$, where y is a solution of the boundary-value problem*

$$l(y) = \lambda y + h,$$

$$(K(\lambda) - I)\Gamma_1 f + i(K(\lambda) + I)\Gamma_2 f = 0.$$

Here $h(x) \in L_2([a,b],\mathbb{C})$ and $K(\lambda)$ is a $2m \times 2m$ matrix-valued function which is holomorphic in the lower half-plane and satisfies $\|K(\lambda)\| \leqslant 1$.

2. In the half-plane $\operatorname{Im}\lambda > 0$, every generalized resolvent of $L_{\min}$ acts by the rule $R_\lambda h = y$, where y is a solution of the boundary-value problem

$$l(y) = \lambda y + h,$$

$$(K(\lambda) - I)\Gamma_1 f - i(K(\lambda) + I)\Gamma_2 f = 0.$$

Here $h(x) \in L_2([a,b],\mathbb{C})$ and $K(\lambda)$ is a $2m \times 2m$ matrix-valued function which is holomorphic in the lower half-plane and satisfies $\|K(\lambda)\| \leqslant 1$.

The parametrization of the generalized resolvents by the matrix-valued functions K is bijective.

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Багатоінтервальні крайові задачі Штурма–Ліувілля з потенціалами-розподілами

Вивчаються багатоінтервальні крайові задачі Штурма–Ліувілля з потенціалами-розподілами. Для відповідних симетричних операторів побудовано простори граничних значень і дано конструктивні описи всіх самоспряжених, максимальних дисипативних і максимальних акумулятивних розширень, а також узагальнених резольвент в термінах однорідних крайових умов. Показано, що всі дійсні максимальні дисипативні і максимальні акумулятивні розширення самоспряжені, і описано всі такі розширення.

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Многоинтервальные краевые задачи Штурма–Лиувилля с потенциалами-распределениями

Изучены многоинтервальные краевые задачи Штурма–Лиувилля с потенциалами-распределениями. Для соответствующих симметрических операторов построены пространства граничных значений и даны конструктивные описания всех самосопряженных, максимальных диссипативных и максимальных аккумулятивных расширений, а также обобщенных резольвент в терминах однородных краевых условий. Показано, что все вещественные максимальные диссипативные и максимальные аккумулятивные расширения самосопряжены, и описаны все такие расширения.