COMPATIBILITY OF PADÉ APPROXIMATIONS OF A COLLECTION OF DEGENERATE HYPERGEOMETRIC FUNCTIONS

A. P. Golub

UDC 517.53

1. Introduction

We give some information and definitions.

Definition 1 (cf., e.g., [1]). Let $F=\{f_k(z)\}_{k=1}^n$ be a collection of functions, analytic in a neighborhood of the point z = 0, and $\vec{r}=(r_1,\ldots,r_n)$ be a vector whose coordinates are nonnegative integers, whose sum is equal to a number $N=N(\vec{r})\in\mathbb{N}^1$. By compatible Padé approximations of the collection of functions $\{f_k(z)\}_{k=1}^n$ of order $([N/N];\vec{r})$ we mean rational polynomials $\pi_{N,N}^{(k)}\{F;\vec{r};z\}$, $k=\overline{1,n}$, of order [N/N] with common denominator, for which the following asymptotic equation holds

$$f_k(z)-\pi_{N,N}^{(k)}\{F;\vec{r};z\}=O(z^{N+r_k+1})\quad\text{as}\quad z\to 0,\quad k=\overline{1,n}. \tag{1}$$

Approximations of this type were first considered by Hermite for a system of exponentials [2] in connection with the question of the transcendence of the number e. Recently the theory of compatible Padé approximations has been extended by interesting new results. In the present paper we study the convergence of compatible Padé approximations for a collection of degenerate hypergeometric functions applying the generalized moment representations proposed by Dzyadyk [3].

Definition 2. For a collection $F=\{f_k(z)\}_{k=1}^n$ the vector-index $\vec{r}$ will be called normal, if the denominator of compatible Padé approximations $Q_N(z)$ of order $([N/N],\vec{r})$ exists and has degree exactly N.

Definition 3. The collection of functions F will be called perfect, if any vector-index is normal for it.

Definition 4 (cf. [3]). We shall say that there is constructed for the sequence of complex numbers $\{s_k\}_{k=0}^\infty$ a generalized moment representation, if on some set $X\subset\mathbb{R}$ there are defined a nondecreasing function $\mu(t)$ and two sequences of functions $\{a_i(t)\}_{i=0}^\infty$ and $\{b_j(t)\}_{j=0}^\infty$, for which, for arbitrary $i,j=\overline{0,\infty}$ one has

$$s_{i+j}=\int_X a_i(t)\,b_j(t)\,d\mu(t). \tag{2}$$

In [4, 5] generalized moment representations were used to study ordinary and two-point Padé approximations.

2. Connection of Generalized Moment Representations with Compatible Padé Approximations

THEOREM 1. Let $F=\{f_k\}_{k=1}^n$ be a collection of functions, analytic in a neighborhood of z = 0, having power series

$$f_k(z)=f_k(0)+\sum_{i=0}^{\infty}s_i^{(k)}z^{i+1},\quad |z|\leqslant R, \tag{3}$$

and for each of the sequences $\{s_i^{(k)}\}_{i=0}^\infty$, $k=\overline{1,n}$, let there be constructed generalized moment representations of the form

$$s_{i+j}^{(k)}=\int_0^1 a_i^{(k)}(t)\,b_j(t)\,d\mu(t),\qquad i,j=\overline{0,\infty};\ k=\overline{1,n}, \tag{4}$$

Institute of Problems of Design in Energetics, Academy of Sciences of the Ukrainian SSR, Kiev. Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 39, No. 6, pp. 701-706, November-December, 1987. Original article submitted September 27, 1985.

where $\mu(t)$ has an infinite number of points of growth on [0, 1], the system of functions $\{a_i^{(k)}(t): i=\overline{0, r_k-1},\ k=\overline{1, n}\}$ be Chebyshev on [0, 1] for any vector-index $\vec{r} = (r_1, \ldots, r_n)$, and the functions $b_j(t)$ have the form $b_j(t)=\beta_j t^j,\ \beta_j \neq 0,\ j=\overline{0, \infty}$.

Then the collection F will be perfect, and if we denote by $B_N(t)=\sum_{j=0}^{N} c_j^{(N)} b_j(t)$ a generalized polynomial, not identically zero, satisfying the biorthogonality conditions

$$\int_0^1 B_N(t)\, a_i^{(k)}(t)\, d\mu(t) = 0, \qquad i=\overline{0, r_k-1};\ k=\overline{1, n}, \tag{5}$$

then the rational polynomials

$$\pi_{N,N}^{(k)}\{F; \vec{r}; z\} = \frac{\sum_{j=0}^{N} c_j^{(N)} z^{N-j} T_j(f_k; z)}{\sum_{j=0}^{N} c_j^{(N)} z^{N-j}}, \qquad k=\overline{1, n}, \tag{6}$$

where $T_j(f_k;\ z)$ is the Taylor polynomial of the function $f_k(z)$ of order j, will realize compatible Padé approximations of the collection F of order ([N/N], $\vec{r}$).

If in addition we assume that the series $\sum_{i=0}^{\infty} a_i^{(k)}(z) z^i,\ k=\overline{1, n}$, converge uniformly to functions $A_k(z,\ t)$, which are analytic in z in some domain $D \subset \mathbb{C}$ and are term by term integrable, then the error of approximation can be written in the form

$$f_k(z) - \pi_{N,N}^{(k)}\{F; \vec{r}; z\} = \frac{z^{N+1}}{Q_N(z)} \int_0^1 A_k(z, t)\, B_N(t)\, d\mu(t) \quad \forall z \in D,\ k=\overline{1, n}, \tag{7}$$

where $Q_N(z) = \sum_{j=0}^{N} c_j^{(N)} z^{N-j}$.

Proof. First of all we note that under the hypotheses of the theorem, polynomials $B_N(t)$ satisfying (5) exist and have exactly N real simple roots on (0, 1) (cf., e.g., [1]). From this, it follows in particular that

We multiply (4) by z^i and we sum over i from −1 to ∞:

$$\frac{f_k(z) - T_j(f_k; z)}{z^j} = z \int_0^1 A_k(z, t)\, b_j(t)\, d\mu(t). \tag{8}$$

We multiply both sides of (8) by $c_j^{(N)}$ and we sum over j from 0 to N:

$$f_k(z)\, Q_N(z) - \sum_{j=0}^{N} c_j^{(N)} z^{N-j} T_j(f_k; z) = z^{N+1} \int_0^1 A_k(z, t)\, B_N(t)\, d\mu(t), \qquad k=\overline{1, n}. \tag{9}$$

Keeping in mind the biorthogonality properties of the polynomials $B_N(t)$, we see that (6) and (7) are valid. It is also easy to conclude that the collection of functions F is perfect, since the denominator $Q_N(z)$ found, according to the remark made at the beginning of the proof, has degree precisely N.

COROLLARY. The collection of degenerate hypergeometric functions

$$\{{}_1F_1(1; \nu_k+1; z)\}_{k=1}^{n},\quad \nu_k - \nu_m \notin \mathbb{Z} \ \text{ for } k \neq m,\quad \nu_k > -1, \qquad k=\overline{1, n}$$

is perfect.

Proof. The fact indicated follows from Theorem 2.1 of [5], in which generalized moment representations are constructed for the sequence of coefficients of the power series of the hypergeometric function:

$$ {}_1F_1(1; \nu+1; z) = 1 + \sum_{k=0}^{\infty} s_k z^{k+1}, \quad s_{i+j} = \int_0^1 \frac{(1-t)^{i+\nu}}{\Gamma(i+\nu+1)} \frac{t^j}{j!}\, dt, \qquad i, j = \overline{0, \infty},$$

and the fact that the system of functions $\{(1-t)^{\nu_k}\}_{k=1}^{n}$, $\nu_k-\nu_m \notin \mathbb{Z}$ for $k \neq m$, $\nu_k > -1$, $k = \overline{1, n}$ is an AT-system on [0, 1] (cf. [1]).

3. Location of Zeros of the Denominator

LEMMA 1. $\forall R>0\ \exists N_0 \in \mathbb{N}^1$ such that $\forall N \geqslant N_0$ and for an arbitrary algebraic polynomial of degree exactly N, all of whose zeros are located on (0, 1):

$$B_N(t) = \sum_{j=0}^{N} c_j^{(N)} \frac{t^j}{j!}, \quad c_N^{(N)} \neq 0,$$

the algebraic polynomial $Q_N(z) = \sum_{j=0}^{N} c_j^{(N)} z^{N-j}$ has no roots in the disc $K_R = \{z : |z| \leqslant R\}$.

Proof. Obviously one can represent the polynomial $B_N(t)$ in the form $B_N(t) = \beta_N \prod_{j=1}^{N} (t - t_j^{(N)})$, where $t_j^{(N)} \in (0, 1)$, $j = \overline{1, N}$.

We shall set $\beta_N = 1$, since a constant factor does not affect the location of the roots. Thus, $c_j^{(N)} = j!\,(-1)^{N-j} \sigma_{N-j}$, where $\sigma_j = \sum_{i_1=1}^{N} \sum_{i_2=1}^{i_1-1} \cdots \sum_{i_j=1}^{i_{j-1}-1} t_{i_1}^{(N)} t_{i_2}^{(N)} \cdots t_{i_j}^{(N)}$, $j = \overline{1, N}$, $\sigma_0 = 1$.

For each $N \in \mathbb{N}^1$ and each $B_N(t)$ we take the number

$$\alpha_N = \frac{t_1^{(N)} + t_2^{(N)} + \ldots + t_N^{(N)}}{N}$$

and we construct auxiliary functions

$$f_N(z) = \exp\{-\alpha_N z\}, \quad g_N(z) = \frac{1}{N!} Q_N(z) - \exp\{-\alpha_N z\}.$$

Obviously both functions are analytic in the whole complex plane. In order to use Rouchét's theorem we show that $|g_N(z)| \to 0$ as $N \to \infty$ uniformly on any compactum $K \subset \mathbb{C}$. We have

$$\frac{1}{N!} Q_N(z) = \frac{1}{N!} \sum_{j=0}^{N} c_j^{(N)} z^{N-j} = \frac{1}{N!} \sum_{j=0}^{N} j!\,(-1)^{N-j} \sigma_{N-j} z^{N-j} = \frac{1}{N!} \sum_{j=0}^{N} (N-j)!\,(-z)^j \sigma_j.$$

Consequently,

$$g_N(z) = \sum_{j=0}^{N} \frac{(N-j)!}{N!} (-z)^j \sigma_j - \sum_{j=0}^{N} \frac{(-z)^j}{j!} (\alpha_N)^j - \sum_{j=N+1}^{\infty} \frac{(-z)^j}{j!} (\alpha_N)^j.$$

We set $r_{N+1} \overset{\text{df}}{=} -\sum_{j=N+1}^{\infty} \frac{(-z)^j}{j!} (\alpha_N)^j$. Obviously r_{N+1} can be bounded above on any compactum by a quantity which tends to zero as $N \to \infty$ and independent of the location of the roots $t_1^{(N)}$, $t_2^{(N)}, \ldots, t_N^{(N)}$. Thus, $g_N(z) = \sum_{j=0}^{N} (-z)^j \left(\frac{(N-j)!}{N!} \sigma_j - \frac{(\alpha_N)^j}{j!} \right) + r_{N+1}$. We set $u_{N,j} = \frac{(N-j)!}{N!} \sigma_j - \frac{(\alpha_N)^j}{j!}$, $j = \overline{0, N}$.

It is easy to see that

$$u_{N,0} = u_{N,1} = 0,$$

$$u_{N,j} = \frac{(N-j)!\,N^{j-1} j!\,\sigma_j - (N-1)!\,(t_1^{(N)} + t_2^{(N)} + \ldots + t_N^{(N)})^j}{N!\,N^{j-1} j!} = (N-j)! \frac{j!\,\sigma_j - (t_1^{(N)} + t_2^{(N)} + \ldots + t_N^{(N)})^j}{N!\,j!}$$

$$\frac{[(N-1)! - N^{j-1}(N-j)!]\,(t_1^{(N)} + t_2^{(N)} + \ldots + t_N^{(N)})^j}{N!\,N^{j-1} j!} \overset{\text{df}}{=} v_{N,j} - w_{N,j}, \quad j = \overline{2, N}.$$

To estimate $v_{N,j}$ we note that among the N^j summands constituting $(t_1^{(N)} + t_2^{(N)} + \ldots + t_N^{(N)})^j$ there are $N(N-1)\ldots(N-j+1) = \frac{N!}{(N-j)!}$ whose sum is equal to $j!\sigma_j$. Thus, in the numerator after canceling there remain $N^j - \frac{N!}{(N-j)!}$ summands, each of which does not exceed one in modulus. Thus,

$$|v_{N,j}| \leqslant \frac{N^j - \frac{N!}{(N-j)!}}{N!\,j!}(N-j)! = \frac{1}{j!}\left[\frac{N^{j-1}}{(N-1)(N-2)\cdot\ldots\cdot(N-j+1)} - 1\right].$$

We estimate $|w_{N,j}|$:

$$|w_{N,j}| = \left|\frac{(N-j)!\,[(N-1)(N-2)\cdot\ldots\cdot(N-j+1) - N^{j-1}]}{N!\,N^{j-1}\,j!}\right.$$

$$\left.\times(t_1^{(N)} + \ldots + t_N^{(N)})^j\right| \leqslant \frac{1}{j!}\left[\frac{N^{j-1}}{(N-1)(N-2)\cdot\ldots\cdot(N-j+1)} - 1\right].$$

Thus, $|u_{N,j}| \leqslant \frac{2}{j!}\left[\frac{N^{j-1}}{(N-1)(N-2)\cdot\ldots\cdot(N-j+1)} - 1\right] \to 0$ as $N \to \infty$.

Using this inequality one can derive the applicability of Rouché's theorem [6, p. 425] to the functions $f_N(z)$ and $g_N(z)$, and consequently, $\forall R > 0$ the number of roots of the function $\frac{1}{N!}Q_N(z) = f_N(z) + g_N(z)$ in the disc $K_R = \{z : |z| \leqslant R\}$ must coincide with the number of roots of the function $f_N(z)$ in this disc, and therefore must be equal to zero.

COROLLARY. The set of zeros of the numerator of compatible Padé approximations of the system of degenerate hypergeometric functions $\{{}_1F_1(1;\ \nu_k + 1; z)\}_{k=1}^n$, $\nu_k - \nu_m \notin \mathbb{Z}$ for $k \neq m$, $\nu_k > -1$, $k = \overline{1, n}$ has the unique limit point $z = \infty$.

The proof follows from Theorem 1, the corollary to Theorem 1 (cf. also their proofs), and Lemma 1.

4. Convergence of Compatibile Padé Approximations for the Functions $\{{}_1F_1(1; \nu_k + 1; z)\}_{k=1}^n$, $\nu_k > -1$, $k = \overline{1, n}$, $\nu_k - \nu_m \notin \mathbb{Z}$, $k \neq m$.

THEOREM 2. Let $F = \{f_k(z)\}_{k=1}^n$ be a perfect system of functions. Then for the errors of compatible Padé approximations one has the analog of the Hermite iteration formula:

$$f_k(z) - \pi_{N,N}^{(k)}\{F; \vec{r}; z\} = \frac{1}{2\pi i}\int_{\Gamma_k}\left(\frac{z}{\xi}\right)^{N+r_k+1} f_k(\xi)\frac{Q_N(\xi)}{Q_N(z)}\frac{d\xi}{\xi - z}, \tag{10}$$

where $Q_N(z)$ is the denominator of the compatible Padé approximations of order $([N/N], \vec{r})$, $\vec{r} = (r_1, r_2, \ldots, r_n)$, $r_k \in \mathbb{N}^1 \cup \{0\}$, $\sum_{k=1}^n r_k = N$, Γ_k is a contour (for example, piecewise smooth), enveloping the origin and inside the domain of analyticity of the function $f_k(z)$.

The proof uses the definition of compatible Padé approximations and follows the scheme of the proof of the Hermite interpolation formula (cf., e.g., [7, p. 482]).

THEOREM 3. Compatible Padé approximations of the collection of degenerate hypergeometric functions $\{{}_1F_1(1; \nu_k + 1; z)\}_{k=1}^n$, $\nu_k - \nu_m \notin \mathbb{Z}$ for $k \neq m$, $\nu_k > -1$, $k = \overline{1, n}$, of order $([N/N], \vec{r})$ converge uniformly to the functions $f_k(z)$ on any compactum K of the complex plane as $N \to \infty$.

Proof. Let $K \subset K_R$, where K_R is a disc of sufficiently large radius R with center at the origin. We choose an $N_0 \in \mathbb{N}^1$, such that $\forall N \geqslant N_0$ the zeros of the denominator of compatible Padé approximations are outside the disc K_{8R}. For the estimate we use (10) with $\Gamma = \partial K_{2R} = \Gamma_{2R}$. For $z \in K_R$ we get

$$|f_k(z) - \pi_{N,N}^{(k)}\{F; \vec{r}; z\}| \leqslant \frac{1}{2\pi}\left(\frac{1}{2}\right)^{N+r_k+1}\sup_{\xi \in K_{2R}}|f_k(\xi)| \sup_{z_1, z_2, \ldots, z_N \in \mathbb{C}\setminus K_{8R}} \frac{\sup_{\xi \in K_{2R}}|\xi - z_1|\ldots|\xi - z_N|}{\inf_{z \in K_R}|z - z_1|\ldots|z - z_N|} \cdot \frac{1}{R}\cdot 2\pi R.$$

We note that if $|\xi - z_k| \leqslant 7R$, then $\frac{|\xi - z_k|}{\inf_{z \in K_R}|z - z_k|} \leqslant 1$, while if $|\xi - z_k| > 7R$, then $|z - z_k| \geqslant |\xi - z_k| - |\xi - z| \geqslant |\xi - z_k| - 3R$ and $\left|\frac{\xi - z_k}{z - z_k}\right| \leqslant \frac{|\xi - z_k|}{|\xi - z_k| - 3R} \leqslant 1 + 3R/4R = 7/4$. Thus, $|f_k(z) - \pi_{N,N}^{(k)}\{F; \vec{r}; z\}| \leqslant \left(\frac{7}{8}\right)^N \times \left(\frac{1}{2}\right)^{r_k+1}\sup_{\xi \in K_{2R}}|f_k(\xi)| \to 0$ as $N \to \infty$.

LITERATURE CITED

1. E. M. Nikitin, "Compatible Padé approximations," Mat. Sb., 113, No. 4, 499-519 (1980).
2. C. Hermite, "Sur la fonction exponentielle," in: Oeuvres, Vol. 3 (1973), pp. 151-181.
3. V. K. Dzyadyk, "Generalization of the problem of moments," Dokl. Akad. Nauk UkrSSR, Ser. A, No. 6, 8-12 (1981).
4. V. K. Dzyadyk and A. P. Golub, "Generalization of the problem of moments and Padé approximations," Preprint [in Russian], Kiev, 1981 (Akad. Nauk UkrSSR, Inst. Mat.; 81.58), pp. 3-15.
5. A. P. Golub, "Application of the generalized problem of moments to Padé approximations of some functions," Preprint [in Russian], Kiev, 1981 (Akad. Nauk UkrSSR, Inst. Mat.; 81.58), pp. 16-56.
6. A. I. Markushevich, Theory of Analytic Functions [in Russian], Vol. 1, Nauka, Moscow (1967).
7. V. K. Dzyadyk, Introduction to the Theory of Uniform Approximation of Functions by Polynomials [in Russian], Nauka, Moscow (1977).

LATTICE SEMICONTINUOUS POISSON PROCESSES ON MARKOV CHAINS

D. V. Gusak and A. I. Tureniyazova

UDC 519.21

We consider a two-dimensional Markov process $\{\xi(t), x(t)\}\,(t\geqslant 0)$, where x(t) is a finite regular Markov chain with values k = 1, 2,...,n and generating matrix $\mathbf{Q}=\mathbf{N}[\mathbf{P}-\mathrm{I}]$, $\xi(t)$ is a Poisson process, defined on the chain x(t), with values in $\mathbf{Z}=\{0,\pm 1,\pm 2,\ldots\}$, the matrix generating function of which has the form

$$\mathbf{\Phi}_t(u)=\|M[u^{\xi(t)}, x(t)=r\mid x(0)=k]\|=\|M_{kr}[u^{\xi(t)}]\|=\exp\{t\mathbf{\Psi}(u)\},$$

where $\mathbf{\Psi}(u)=\mathbf{\Lambda}(\tilde{\mathbf{P}}(u)-\mathrm{I})+\mathbf{N}(\tilde{\mathbf{F}}(u)-\mathrm{I})$, $\mathbf{\Psi}(1)=\mathbf{Q}$, $\mathbf{\Lambda}=\|\delta_{kr}\lambda_k\|$, $\mathbf{N}=\|\delta_{kr}n_k\|$, k = 1, 2,..., while $\tilde{\mathbf{F}}(u)$, $\tilde{\mathbf{P}}(u)$ are the generating matrices for the corresponding distributions $\mathbf{F}(m)=\|p_{kr}P\{\chi_{kr}=m\}\|$, $\mathbf{P}(m)=\|\delta_{kr}P\{\xi_k=m\}\|$.

For the brevity of the notation of the integral transformations we introduce the exponentially distributed random variable θ_s, s > 0. Then

$$\mathbf{P}_m(s)=s\int_0^\infty e^{-st}\|P_{kr}\{\xi(t)=m\}\|\,dt=\|P_{kr}\{\xi(\theta_s)=m\}\|,\quad \mathbf{P}(s,m)$$

$$=\|P_{kr}\{\xi(\theta_s)\leqslant m\}\|,\quad \mathbf{\Phi}(s,u)=\|M_{kr}[u^{\xi(\theta_s)}]\|=s(s\mathrm{I}-\mathbf{\Psi}(u))^{-1}.$$

We introduce also the notations $\xi^+(t)=\sup_{0\leqslant u\leqslant t}\xi(u)$, $\xi^-(t)=\inf_{0\leqslant u\leqslant t}\xi(u)$, $\tau_m^-=\inf\{t:\xi(t)=-m\}$, $\tau_m^+=\inf\{t:\xi(t)=m\}$, $m\geqslant 0$. In the same way as for processes with continuously distributed jumps [1], in the considered case, we have a factorization decomposition (the matrix analogue of the identity of the infinitely divisible factorization) for |u| = 1

$$\mathbf{\Phi}(s,u)=\begin{cases}\mathbf{\Phi}_+(s,u)\,\mathbf{P}_s^{-1}\mathbf{\Phi}^-(s,u),\\ \mathbf{\Phi}_-(s,u)\,\mathbf{P}_s^{-1}\mathbf{\Phi}^+(s,u),\end{cases}\tag{1}$$

where $\mathbf{\Phi}_\pm(s,u)=\mathbf{M}u^{\xi^\pm(\theta_s)}$, $\mathbf{\Phi}^\mp(s,u)=\mathbf{M}u^{\xi(\theta_s)-\xi^\pm(\theta_s)}$, $\mathbf{P}_s=s(s\mathrm{I}-\mathbf{Q})^{-1}$, $\mathbf{\Phi}_\pm$, $\mathbf{\Phi}_\mp$ are analytic for |u| < 1, |u| > 1, respectively.

Boundary value problems for semicontinuous processes on Markov chains have been investigated in [1-6] basically for the nonlattice case. The lattice case has been considered in [7] (when the matrix is a Jacobi matrix). We shall be interested in semicontinuous Poisson processes on chains with a lattice distribution of the jumps and also in certain boundary functionals of these processes. We shall assume that the distribution of the jumps $\{\chi_{kr}\}$ and

Institute of Mathematics, Academy of Sciences of the Ukrainian SSR, Kiev.

Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 39, No. 6, pp. 707-711, November-December, 1987. Original article submitted September 26, 1986; revision submitted November 21, 1986.

0041-5995/87/3906-0573$12.50