4. Proof of Theorems 1 and 2. Necessity. The Esseen and Janson theorem implies that (4) yields the upper bound in (5) and conditions (2) and (3). We shall now determine the lower bound in (5). To do this, we utilize Lemma 1 and show that (8) is fulfilled.

Set $Y_k = X_{2k} - X_{2k-1}$, $k = 1, 2, \ldots$ We shall now verify that for this sequence (4) is fulfilled. The upper bound follows from the corresponding bound on $\{X_k\}$ and the inequality $M|U + V|^p \le \max\{2^{p-1}, 1\}$ $(M|U|^p + M|V|^p)$ [2, p. 168].

If $0 < p < 1$, the lower bound follows from condition (6). For $p \ge 1$, this bound follows from (2) and the following well-known inequality [2, p. 277]. Let the r.v. U and V be independent, possess a finite absolute moment of the order $p \ge 1$ and $MU = 0$. Then $M|U + V|^p \ge M|V|^p$.

The r.v. possesses the c.f. $|f(t)|^2$. By Lemma 4 the bounds (8) are fulfilled for $|f(t)|^2$. We have

$$1-|f(t)|^2=(1-|f(t)|)(1+|f(t)|)\leqslant 2(1-\operatorname{Re} f(t)).$$

This yields the required lower bound for $1 - \operatorname{Re} f(t)$. The upper bound in (8) follows from the above-established upper bound in (5) and Lemma 1. Theorems 1 and 2 are thus proved.

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ONE TYPE OF GENERALIZED MOMENT REPRESENTATIONS

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In 1981, V. K. Dzyadyk [1] first investigated the problem of generalized moment representations of numerical sequences, a problem which has found useful applications in the study of rational approximations and integral representations of functions.

Definition 1. A generalized moment representation (GMR) of a sequence of complex numbers $\{s_k\}_{k=0}^{\infty}$ is defined to be the set of equations

$$s_{i+j}=\int_X a_i(t)\,b_j(t)\,d\mu(t),\quad i,j=\overline{0,\infty}, \tag{1}$$

in which $d\mu(t)$ is a measure on a set X (X is most often a segment of the real axis), and $a\{a_i(t)\}_{i=0}^{\infty}$ and $\{b_j(t)\}_{j=0}^{\infty}$ are sequences of measurable functions on X for which all integrals in (1) exist.

In the present article an analogous construction is investigated based on the concept of q-integral introduced by Jackson [2].

Definition 2. For some fixed generally speaking complex number q, $|q| < 1$, the q-integral of the function $\varphi(x)$ defined on [0, 1] is defined by the following formula:

$$\Phi(x)=\int_0^x \varphi(u)\,d_q u = x(1-q)\sum_{n=0}^{\infty}\varphi(xq^n)\,q^n,\quad x\in[0,1], \tag{2}$$

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provided the series on the right-hand side of (2) converges.

Remark 1. It is obvious that if we define the q-derivative by the formula (see, for example [3])

$$\frac{d_q}{d_q x}\Phi(x)=\frac{\Phi(qx)-\Phi(x)}{(q-1)x}, \tag{3}$$

then

$$\frac{d_q}{d_q x}\int_0^x \varphi(u)\,d_q u\equiv\varphi(u).$$

Thus, we introduce q-integral GMRs into consideration.

Definition 3. A q-integral generalized moment representation of a sequence of complex numbers $\{s_k\}_{k=0}^{\infty}$ is defined to be the set of equations

$$s_{i+j}=\int_0^1 a_i(t)\,b_j(t)\,d_q t,\quad i,j=\overline{0,\infty}, \tag{4}$$

in which $\{a_i(t)\}_{i=0}^{\infty}$ and $\{b_j(t)\}_{j=0}^{\infty}$ are sequences of functions such that all q-integrals in (4) exist.

It is known [4] that GMRs of type (1) under specific conditions can be represented in the form

$$s_k=\int_X (A^k a_0)(t)\,b_0(t)\,d\mu(t),\quad k=\overline{0,\infty},$$

where A is a linear bounded operator on a Banach space. A similarity transformation is possible also for q-integral GMRs. In fact, we introduce two infinite-dimensional spaces of functions $\mathfrak{M}$ and $\mathfrak{N}$ defined on [0, 1] such that $\forall\varphi\in\mathfrak{M}$ and $\forall\psi\in\mathfrak{N}$ the bilinear form

$$\langle\varphi,\psi\rangle:=\int_0^1\varphi(u)\,\psi(u)\,d_q u=(1-q)\sum_{n=0}^{\infty}\varphi(q^n)\,\psi(q^n)\,q^n \tag{5}$$

is defined. We consider a bounded linear operator $A:\mathfrak{M}\to\mathfrak{M}$ and assume that there exists a unique linear bounded operator $A^*:\mathfrak{N}\to\mathfrak{N}$ such that

$$\langle A\varphi,\psi\rangle=\langle\varphi,A^*\psi\rangle\quad\forall\varphi\in\mathfrak{M},\ \psi\in\mathfrak{N}.$$

We shall say that operator A^* is conjugate to A with respect to bilinear form (5). Now if we assume that $a_0(t)\in\mathfrak{M}$, $b_0(t)\in\mathfrak{N}$ and linear operator A has property $(Aa_i)(t)=a_{i+1}(t)$, $i=\overline{0,\infty}$, where $a_i(t)$, $i=\overline{0,\infty}$ are the functions appearing in Eqs. (4), then it is obvious as well that we will have

$$(A^*b_j)(t)=b_{j+1}(t),\quad j=\overline{0,\infty},$$

and, consequently, we obtain expressions

$$s_k=\int_0^1 (A^k a_0)(t)\,b_0(t)\,d_q t,\quad k=\overline{0,\infty}, \tag{6}$$

equivalent to (4).

It is particularly simple to carry Dzyadyk's theorem [5] on construction of diagonal Padé approximants over to the case of q-integral GMRs. We shall state a more general assertion here that is true for arbitrary bilinear forms.

THEOREM 1. Let an analytic function f(z) be representable in a neighborhood of z = 0 by the power series

$$f(z)=f(0)+\sum_{k=0}^{\infty}s_k z^{k+1}, \tag{7}$$

and for sequence $\{s_k\}_{k=0}^{\infty}$ let

$$s_{i+j}=\langle a_i, b_j\rangle, \quad i, j=\overline{0, \infty},$$

where $a_i \in \mathfrak{M}$, $i=\overline{0, \infty}$, $b_j \in \mathfrak{N}$, $j=\overline{0, \infty}$, $\mathfrak{M}$ and $\mathfrak{N}$ are infinite-dimensional linear spaces, and $\langle\cdot, \cdot\rangle$ is a bilinear form defined on the Cartesian product of these spaces. Further, let there exist nondegenerate biorthogonal sequences $\{A_M\}_{M=0}^{\infty}$, $\{B_N\}_{N=0}^{\infty}$:

$$A_M=\sum_{i=0}^{M} c_i^{(M)}a_i, \quad c_M^{(M)}\neq 0, \quad M=\overline{0, \infty}; \qquad B_N=\sum_{j=0}^{N} c_j^{(N)}b_j, \quad N=\overline{0, \infty},$$

for which

$$\langle A_M, B_N\rangle=\delta_{M,N}, \quad M, N=\overline{0, \infty}.$$

Then the diagonal Padé polynomials $[N/N]_f(z)$, $N=\overline{0, \infty}$ of function $f(z)$ can be represented in the form

$$[N/N]_f(z)=\frac{\sum_{j=0}^{N} c_j^{(N)}z^{N-j}T_j(f; z)}{\sum_{j=0}^{N} c_j^{(N)}z^{N-j}},$$

where $T_j(f; z)$ are partial sums of series (7) of order j.

Now we construct q-integral GMRs for certain functions.

Example. Consider the following function, sometimes called the exponent q-analog [6]

$$E_q(z)=1+\sum_{k=0}^{\infty}s_k z^{k+1}=\sum_{k=0}^{\infty}\frac{z^k}{k_q!},$$

where $k_q=\frac{1-q^k}{1-q}$, $k_q!=\prod_{i=1}^{k} i_q$, $0_q!:=1$.

In order to construct the q-integral GMR for $\{s_k\}_{k=0}^{\infty}$ we use the definitions and properties of q-gamma and q-beta functions (see, for example, [7]). The q-gamma function is defined by formula

$$\Gamma_q(x)=\frac{(q; q)_\infty}{(q^x; q)_\infty}(1-q)^{1-x},$$

where $(a; q)_\infty=\prod_{n=0}^{\infty}(1-aq^n)$. It is obvious that

$$\Gamma_q(n+1)=\frac{(1-q)(1-q^2)\dots(1-q^n)}{(1-q)^n}=n_q!.$$

The q-beta function is defined by means of q-integral

$$B_q(x, y)=\int_0^1 t^{x-1}\frac{(tq; q)_\infty}{(tq^y; q)_\infty}\,d_q t=(1-q)\sum_{n=0}^{\infty}q^{nx}\frac{(q^{n+1}; q)_\infty}{(q^{n+y}; q)_\infty}, \tag{8}$$

and can be expressed by means of the q-gamma function by the formula

$$B_q(x, y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}.$$

From (8) and (9) we get

$$\frac{1}{\Gamma_q(x+y)} = \int_0^1 \frac{t^{x-1}}{\Gamma_q(x)} \frac{(tq;q)_\infty}{\Gamma_q(y)(tq^y;q)_\infty} d_qt.$$

Setting x = i + 1, i = $\overline{0, \infty}$, y = j + 1, j = $\overline{0, \infty}$ we arrive at the q-integral GMR

$$s_{i+j} = \frac{1}{(i+j+1)_q!} = \int_0^1 \frac{t^i}{i_q!} \frac{\prod_{n=0}^{j-1}(1-tq^{n+1})}{j_q!} d_qt.$$

Thus, the upper part of the Padé table for $E_q(z)$ can be constructed in terms of polynomials $Q_n(t)$, n = $\overline{0, \infty}$ that are q-orthogonal on [0, 1], i.e., that satisfy the equations

$$\int_0^1 Q_n(t) Q_m(t) d_qt = (1-q)\sum_{i=0}^{\infty} Q_n(q^i) Q_m(q^i) q^i = 0$$

for m ≠ n. Such polynomials have been rather well studied up to the present time (see, for example, [3; 8, p. 92]).

Generalizing the arguments given above, we get the following result.

THEOREM 2. For a sequence $\{s_k\}_{k=0}^{\infty}$ of the coefficients of the power expansion of function

$$f(z) = \sum_{k=0}^{\infty} s_k z^k = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma_q(\mu k + \nu)}, \quad \mu, \nu > 0, \tag{10}$$

a q-integral GMR of the form

$$s_{i+j} = \frac{1}{\Gamma_q(\mu i + \mu j + \nu)} = \int_0^1 \frac{t^{\mu i + \nu_1 - 1}}{\Gamma_q(\mu_i + \nu_1)} \frac{(tq;q)_\infty}{\Gamma_q(\mu j + \nu_2)(tq^{\mu j + \nu_2};q)_\infty} d_qt, \tag{11}$$

occurs where $\nu_1, \nu_2 > 0$, $\nu_1 + \nu_2 = \nu$.

We note that GMR (11) admits an operator formulation of type (6) with operator

$$(Q^\mu \varphi)(x) = \frac{x^\mu}{\Gamma_q(\mu)} \int_0^1 \frac{(tq;q)_\infty}{(tq^\mu;q)_\infty} \varphi(xt) d_qt$$

and initial functions

$$a_0(t) = \frac{t^{\nu_1 - 1}}{\Gamma_q(\nu_1)}, \quad b_0(t) = \frac{(tq;q)_\infty}{\Gamma_q(\nu_2)(tq^{\nu_2};q)_\infty}. \tag{12}$$

More complicated q-integral GMRs can be constructed if instead of operator Q^μ we consider operator

$$(Q_\sigma^\mu \varphi)(x) = x^\mu \varphi(x) + \sigma(Q^\mu \varphi)(x)$$

with the same initial functions (12). In the particular case $\mu = 1$, $\nu_2 = 1$, $\nu_1 = \nu$ we get the following result.

THEOREM 3. For a sequence $\{s_k\}_{k=0}^{\infty}$ of the coefficients of the power expansion of function

$$f(z) = \sum_{k=0}^{\infty} s_k z^k = \sum_{k=0}^{\infty} \frac{\prod_{m=0}^{k-1}\left[\frac{1-q^{\nu+m}}{1-q}+\sigma\right]}{\Gamma_q(\nu+k+1)} z^k, \quad \nu > 0, \tag{13}$$

we get a q-integral generalized moment representation of form

$$s_{i+j} = \frac{\prod_{m=0}^{i+j-1}\left[\frac{1-q^{\nu+m}}{1-q}+\sigma\right]}{\Gamma_q(\nu+i+j+1)} = \int_0^1 a_i(t)\, b_j(t)\, d_q t, \quad i, j = \overline{0, \infty},$$

where polynomials $a_i(t)$, $i = \overline{0, \infty}$, $b_j(t)$, $j = \overline{0, \infty}$ can be expressed by formulas

$$a_i(t) = \frac{t^{\nu+i-1}}{\Gamma_q(\nu+i)} \prod_{m=0}^{i-1}\left[\frac{1-q^{\nu+m}}{1-q}+\sigma\right], \tag{14}$$

$$b_j(t) = t^j - \frac{1}{j!}\frac{d^j}{dz^j}\left\{\int_{qt}^{1} \Phi(zu)\, d_q u \,\frac{d_q}{d_q t}\left[\frac{1}{(1-zt)\,\Phi(zt)}\right]\right\}_{z=0}, \tag{15}$$

in which

$$\Phi(z) = \sum_{k=0}^{\infty} \frac{(\varkappa - q^k)(\varkappa - q^{k-1})\ldots(\varkappa - 1)}{(1-q^k)(1-q^{k-1})\ldots(1-q)} z^k, \quad \varkappa = 1 + \sigma - \sigma q. \tag{16}$$

Proof. Consider the operator

$$(Q_\sigma \varphi)(x) = x\varphi(x) + \sigma(Q\varphi)(x). \tag{17}$$

Construct its conjugate with respect to bilinear form (5)

$$(Q_\sigma^* \psi)(x) = x\psi(x) + \sigma \int_{qx}^{1} \psi(u)\, d_q u. \tag{18}$$

It is rather simple to establish the fact that successive application of operator (17) to initial function $a_0(t) = t/\Gamma_q(\nu)$ reduces to formulas (14); moreover, the corresponding generalized moments will equal the coefficients of the power expansion (13). In order to obtain formula (15), first we construct the resolvent of operator (17). For this it is necessary to solve the q-integral equation

$$\psi(x) - zx\psi(x) - z\sigma \int_0^x \psi(u)\, d_q u = \varphi(x),$$

which by an application of operator (3) reduces to the following q-differential equation:

$$(1 - zqx)\frac{d_q}{d_q x}\psi(x) - z(1+\sigma)\psi(x) = \frac{d_q}{d_q x}\varphi(x), \qquad \psi(0) = \varphi(0). \tag{19}$$

After applying the power series method, we shall establish that function $\lambda(x) = \Phi(zx)$ defined by formula (16) satisfies the homogeneous equation. For solution of the inhomogeneous equation we apply the method of variation of the constants

$$\psi(x)=C(x)\lambda(x). \tag{20}$$

Substituting expression (20) into Eq. (19), we get

$$\frac{d_q}{d_qx}C(x)=\frac{\frac{d_q}{d_qx}\varphi(x)}{(1-zqx)\lambda(qx)},$$

and, therefore,

$$C(x)=C_0+\int_0^x\frac{\frac{d_q}{d_qu}\varphi(u)}{(1-zqu)\lambda(qu)}d_qu.$$

As a result we get the formula for the resolvent

$$\psi(x)=(R_zQ_\sigma\varphi)(x)=\frac{\varphi(x)\Phi(zx)}{(1-zqx)\Phi(zqx)}-\Phi(zx)\int_0^{qx}\varphi(u)\frac{d_q}{d_qu}\left\{\frac{1}{(1-zu)\Phi(zu)}\right\}d_qu. \tag{21}$$

It is obvious that the resolvent of operator (18) will be conjugate to operator (21). Carrying out the necessary calculations, we get

$$(R_zQ_\sigma^*\psi)(x)=\frac{\psi(x)}{1-xz}-\int_{qx}^1\psi(u)\Phi(zu)d_qu\frac{d_q}{d_qx}\left\{\frac{1}{(1-xz)\Phi(zx)}\right\},$$

from which formula (15) follows immediately. Theorem 3 is proved.

Remark 2. Generalized moment representations of basis hypergeometric series of type (10) or (13) with continuous measures are constructed in [4]. In the same place, formulas are obtained for their Padé approximant in terms of special biorthogonal systems.

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