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TWO-DIMENSIONAL GENERALIZED MOMENT REPRESENTATIONS AND PADÉ APPROXIMATIONS FOR SOME HUMBERT SERIES

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By extending Dzyadyk's method of generalized moment representations to the case of two-dimensional number sequences, we construct and study Padé approximants for some confluent Humbert hypergeometric series.

The method of generalized moment representations proposed by Dzyadyk in [1] can be generalized to the case of two-dimensional number sequences [2].

Definition 1. *It is said that, for a two-dimensional number sequence* $\{s_{k,m}\}_{k,m=0}^{\infty}$, *a generalized moment representation on the product of linear spaces* $\mathscr{X}$ *and* $\mathscr{Y}$ *specified by a bilinear form* $\langle .\,,.\rangle$ *defined on this product exists if one can find a two-dimensional sequence of elements* $\{x_{k,m}\}_{k,m=0}^{\infty}$ *defined in the space* $\mathscr{X}$ *and a two-dimensional sequence of elements* $\{y_{j,n}\}_{j,n=0}^{\infty}$ *defined in the space* $\mathscr{Y}$. *such that*

$$s_{k+j,m+n} = \langle x_{k,m}, y_{j,n}\rangle, \quad k,j,m,n \in \mathbb{Z}_+, \tag{1}$$

These representations can be used for the construction of two-dimensional Padé approximations of formal power series of two variables.

Theorem 1 [2]. *Assume that a formal power series of two variables have the form*

$$f(z,w) = \sum_{k,m=0}^{\infty} s_{k,m} z^k w^m \tag{2}$$

and that a two-dimensional sequence $\{s_{k,m}\}_{k,m=0}^{\infty}$ *possesses a generalized moment representation of the form (1). If, for some* $N_1, N_2 \in \mathbb{N}$, *there exists a nontrivial generalized polynomial*

$$Y_{N_1,N_2} = \sum_{j=0}^{N_1}\sum_{n=0}^{N_2} c_{j,n}^{(N_1,N_2)} y_{j,n}$$

such that the conditions of biorthonormality

$$\langle x_{k,m}, Y_{N_1,N_2}\rangle = 0$$

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are satisfied for $(k,m) \in \mathscr{H}$, *where the domain* $\mathscr{H} \subset \mathbb{Z}_+^2$ *bounded by the graph of a certain function* $\rho = \rho(\varphi)$, $\varphi \in [0, \pi/2]$, *contains exactly* $(N_1+1)(N_2+1) - 1$ *points and, in addition,* $c_{N_1,N_2}^{(N_1,N_2)} \neq 0$, *then the rational function*

$$[\mathscr{N}/\mathscr{D}]_f(z,w) = \frac{1}{Q_{N_1,N_2}(z,w)} \Bigg\{ \sum_{k=0}^{N_1-1} \sum_{m=0}^{N_2-1} z^k w^m \sum_{j=0}^{k} \sum_{n=0}^{m} c_{N_1-j,N_2-n}^{(N_1,N_2)} s_{k-j,m-n}$$

$$+ z^{N_1} \sum_{m=0}^{N_2-1} \sum_{k=0}^{y(m)-N_1} z^k w^m \sum_{j=0}^{N_1} \sum_{n=0}^{m} c_{j,N_2-n}^{(N_1,N_2)} s_{k+j,m-n}$$

$$+ w^{N_2} \sum_{k=0}^{N_1-1} \sum_{m=0}^{x(k)-N_2} z^k w^m \sum_{j=0}^{k} \sum_{n=0}^{N_2} c_{N_1-j,n}^{(N_1,N_2)} s_{k-j,m+n} \Bigg\}, \tag{3}$$

where

$$Q_{N_1,N_2}(z,w) = \sum_{j=0}^{N_1} \sum_{n=0}^{N_2} c_{N_1-j,N_2-n}^{(N_1,N_2)} z^j w^n,$$

and $x(k)$ *and* $y(m)$ *are certain functions from* $\mathbb{Z}_+$ *in* $\mathbb{Z}_+$ *such that* $x(k) \geqslant N_2$ *and* $y(m) \geqslant N_1$ *for all values of* k *and* m, *admits the expansion in the power series whose coefficients coincide with the coefficients of series (2) for all*

$$(k,m) \in \mathscr{E} = ([0, N_1-1] \times [0, N_2-1]) \cup \{(k,m)\colon k \in [0, N_1-1], m \in [N_2, x(k)]\}$$

$$\cup \{(k,m)\colon m \in [0, N_2-1], k \in [N_1, y(m)]\}$$

$$\cup \{(k,m)\colon k \geqslant N_1, m \geqslant N_2, (k-N_1, m-N_2) \in \mathscr{H}\}.$$

Thus, the rational function (3) is the Padé approximant of series (2) with denominator whose coefficients have indices from the domain

$$\mathscr{D} = [0, N_1] \times [0, N_2]$$

and numerator whose coefficients have indices from the domain

$$\mathscr{N} = ([0, N_1-1] \times [0, N_2-1]) \cup \{(k,m)\colon k \in [0, N_1-1], m \in [N_2, x(k)]\}$$

$$\cup \{(k,m)\colon m \in [0, N_2-1], k \in [N_1, y(m)]\}.$$

For the construction of representations of the form (1), it is convenient to reformulate the problem of two-dimensional generalized moment representations, as well as the problem of one-dimensional generalized moment representations, in the operator form. Namely, we assume that the spaces $\mathscr{X}$ and $\mathscr{Y}$ are normed and, in the space $\mathscr{X}$, there exist commuting bounded linear operators $A,\ B\colon \mathscr{X} \to \mathscr{X}$ such that

$$Ax_{k,m} = x_{k+1,m},$$

$$Bx_{k,m} = x_{k,m+1}$$

for all $k, m \in \mathbb{Z}_+$. Moreover, we assume that the space $\mathscr{Y}$ contains bounded linear operators $A^\star$ and $B^\star\colon \mathscr{Y} \to \mathscr{Y}$ adjoint to the operators A and B with respect to the bilinear form $\langle .,. \rangle$ in a sense that, for any $x \in \mathscr{X}$ and $y \in \mathscr{Y}$,

$$\langle Ax, y\rangle = \langle x, A^\star y\rangle,$$

$$\langle Bx, y\rangle = \langle x, B^\star y\rangle.$$

Hence, representations (1) can be rewritten in the form

$$s_{k,m} = \left\langle A^k B^m x_{0,0}, y_{0,0} \right\rangle, \quad k, m \in \mathbb{Z}_+,$$

and series (2) converges in the neighborhood of the origin of coordinates to an analytic function admitting the representation

$$f(z,w) = \left\langle \widehat{R_z}(A)\widehat{R_w}(B)x_{0,0}, y_{0,0} \right\rangle, \tag{4}$$

where the resolvent of the function $\widehat{R_z}(A)$ is defined by the equality $\widehat{R_z}(A) = (I - zA)^{-1}$.

Under the conditions of Theorem 1, the following relation is true for the error of approximation:

$$f(z,w) - [\mathscr{N}/\mathscr{D}]_f(z,w) = \frac{1}{Q_{N_1,N_2}(z,w)} \Biggl\{ z^{N_1} w^{N_2} \left\langle \widehat{R_z}(A)\widehat{R_w}(B)x_{0,0}, Y_{N_1,N_2} \right\rangle$$

$$+ z^{N_1} \sum_{m=0}^{N_2-1} \sum_{k=y(m)-N_1+1}^{\infty} z^k w^m \sum_{j=0}^{N_1} \sum_{n=0}^{m} c_{j,N_2-n}^{(N_1,N_2)} s_{k+j,m-n}$$

$$+ w^{N_2} \sum_{k=0}^{N_1-1} \sum_{m=x(k)-N_2+1}^{\infty} z^k w^m \sum_{j=0}^{k} \sum_{n=0}^{N_2} c_{N_1-j,n}^{(N_1,N_2)} s_{k-j,m+n} \Biggr\}.$$

Let $\mathscr{X} = L\,[0,1/2] \cap C\,[1/2,1]$ be the space of functions on $[0,1]$ integrable on $[0,1/2]$ and continuous on $[1/2,1]$ with the norm

$$\|\cdot\|_{\mathscr{X}} = \|\cdot\|_{L[0,1/2]} + \|\cdot\|_{C[1/2,1]}.$$

In addition, let $\mathscr{Y} = C\,[0,1/2] \cap L\,[1/2,1]$ be the space of functions defined on $[0,1]$, continuous on $[0,1/2]$, and integrable on $[1/2,1]$ with the norm

$$\|\cdot\|_{\mathscr{Y}} = \|\cdot\|_{C[0,1/2]} + \|\cdot\|_{L[1/2,1]}.$$

It is clear that, in the product of these spaces, we can define the bilinear form

$$\langle x, y\rangle = \int_0^1 x(\tau)y(\tau)d\tau,$$

which is separately continuous (see [3, p. 63]).

In the space $\mathscr{X}$, we consider linear bounded operators

$$(A\varphi)(t) = (B\varphi)(t) = \int_0^t \varphi(\tau)d\tau. \tag{5}$$

It is easy to see (see [4, p. 36]) that the resolvent functions of these operators can be represented in the form

$$\left(\widehat{R_z}(A)\varphi\right)(t) = \varphi(t) + z\int_0^t \varphi(\tau)e^{z(t-\tau)}d\tau,$$

$$\left(\widehat{R_w}(B)\varphi\right)(t) = \varphi(t) + w\int_0^t \varphi(\tau)e^{w(t-\tau)}d\tau.$$

Setting $x_{0,0}(t) = y_{0,0}(t) \equiv 1$, we get

$$(\widehat{R_z}(A)x_{0,0})(t) = e^{zt},$$

$$(\widehat{R_w}(B)\widehat{R_z}(A)x_{0,0})(t) = \frac{ze^{zt} - we^{wt}}{z-w},$$

$$f(z,w) = \int_0^1 \frac{ze^{zt} - we^{wt}}{z-w}dt = \frac{e^z - e^w}{z-w}.$$

Moreover,

$$x_{k,m}(t) = (A^k B^m x_{0,0})(t) = \frac{t^{k+m}}{(k+m)!}$$

and, therefore,

$$s_{k,m} = \int_0^1 x_{k,m}(t)y_{0,0}(t)dt = \frac{1}{(k+m+1)!}.$$

If

$$x_{0,0}(t) = \frac{t^{\nu}}{\Gamma(\nu+1)}, \quad \nu > -1, \qquad y_{0,0}(t) = \frac{(1-t)^{\sigma}}{\Gamma(\sigma+1)}, \quad \sigma > -1, \tag{6}$$

then

$$x_{k,m}(t) = \Big(A^k B^m x_{0,0}\Big)(t) = \frac{t^{k+m+\nu}}{\Gamma(k+m+\nu+1)},$$

$$s_{k,m} = \int\limits_0^1 \frac{t^{k+m+\nu}}{\Gamma(k+m+\nu+1)} \frac{(1-t)^{\sigma}}{\Gamma(\sigma+1)} dt = \frac{1}{\Gamma(k+m+\nu+\sigma+2)}$$

and, hence, the function

$$f(z,w) = \sum_{k,m=0}^{\infty} \frac{z^k w^m}{\Gamma(k+m+\nu+\sigma+2)},$$

coincides, to within a constant factor, with a special case of the confluent Humbert hypergeometric series [5]

$$\Phi_2(\alpha,\beta,\gamma,z,w) = \sum_{k,m=0}^{\infty} \frac{(\alpha)_k(\beta)_m}{(\gamma)_{k+m} k! m!} z^k w^m$$

for $\alpha = \beta = 1$ and $\gamma = \nu + \sigma + 2$.

It is easy to see that whenever the operators A and B coincide, representation (4) can be rewritten in the form

$$f(z,w) = \Big\langle \widehat{R_z}(A)\widehat{R_w}(A)x_{0,0}, y_{0,0} \Big\rangle = \left\langle \left(\frac{z}{z-w}\widehat{R_z}(A) + \frac{w}{w-z}\widehat{R_w}(A) \right) x_{0,0}, y_{0,0} \right\rangle$$

$$= \frac{z}{z-w} \Big\langle \widehat{R_z}(A)x_{0,0}, y_{0,0} \Big\rangle + \frac{w}{w-z} \Big\langle \widehat{R_w}(A)x_{0,0}, y_{0,0} \Big\rangle .$$

In the case of operator (5) and initial functions (6), we get

$$f(z,w) = \frac{z\, {}_1F_1(1; \nu+\sigma+2; z) - w\, {}_1F_1(1; \nu+\sigma+2; w)}{z-w}, \tag{7}$$

where ${}_1F_1(a;b;z)$ is the degenerate Kummer hypergeometric function [6, p. 237].

We use these arguments for the construction of the rational approximations of function (7). Since function (7) is symmetric about its variables, it should be approximated only by symmetric rational functions. Thus, we immediately set $N_1 = N_2 = N$. To apply Theorem 1, it is necessary to construct a nontrivial generalized polynomial

$$Y_{N,N} = \sum_{j=0}^{N} \sum_{n=0}^{N} c_{j,n}^{(N,N)} y_{j,n}$$

satisfying the conditions of biorthormality

$$\langle x_{k,m}, Y_{N,N}\rangle = 0$$

for $(k,m) \in ([0,N]\times[0,N]) \setminus \{(N,N)\}$.

In the analyzed case, we have

$$x_{k,m}(t) = \frac{t^{k+m+\nu}}{\Gamma(k+m+\nu+1)}, \qquad y_{j,n}(t) = \frac{(1-t)^{j+n+\sigma}}{\Gamma(j+n+\sigma+1)}.$$

Therefore,

$$Y_{N,N}(t) = \sum_{j=0}^{N}\sum_{n=0}^{N} c_{j,n}^{(N,N)} \frac{(1-t)^{j+n+\sigma}}{\Gamma(j+n+\sigma+1)} = \gamma_N P_{2N}^{(\nu,\sigma)*}(t)(1-t)^{\sigma},$$

where $P_{2N}^{(\nu,\sigma)*}(t)$ is a shifted Jacobi polynomial of degree $2N$ orthonormal on $[0,1]$ with weight $t^{\nu}(1-t)^{\sigma}$ (see [7, p. 580]) and γ_N is a constant, which can be, without loss of generality, set equal to 1. Thus, we get

$$\sum_{j=0}^{N}\sum_{n=0}^{N} c_{j,n}^{(N,N)} \frac{t^{j+n}}{\Gamma(j+n+\sigma+1)} = P_{2N}^{(\nu,\sigma)*}(1-t).$$

By using the explicit form of the coefficients of the orthogonal Jacobi polynomials (see [7, p. 581]), we obtain (for the sake of convenience, the constant is again set equal to 1):

$$P_{2N}^{(\nu,\sigma)*}(1-t) = P_{2N}^{(\sigma,\nu)*}(t) = \sum_{m=0}^{2N} (-1)^m \binom{2N}{m} \frac{\Gamma(2N+\sigma+\nu+1+m)}{\Gamma(\sigma+1+m)} t^m.$$

Hence, it s necessary to choose the coefficients $\left\{c_{j,n}^{(N,N)}\right\}_{j,n=0}^{N}$ guaranteeing the validity of the equalities

$$\sum_{j=0}^{N}\sum_{n=0}^{N} c_{j,n}^{(N,N)} \frac{t^{j+n}}{\Gamma(j+n+\sigma+1)} = \sum_{m=0}^{2N} (-1)^m t^m \binom{2N}{m} \frac{\Gamma(2N+\sigma+\nu+1+m)}{\Gamma(\sigma+1+m)},$$

or

$$\sum_{\substack{0\leqslant j,n\leqslant N\\ j+n=m}} c_{j,n}^{(N,N)} = (-1)^m \binom{2N}{m} \Gamma(2N+\sigma+\nu+1+m).$$

We choose the indicated coefficients as follows:

$$c_{j,n}^{(N,N)} = \frac{(-1)^{j+n}\Gamma(2N+\sigma+\nu+1+j+n)}{2^{j+n}} \binom{j+n}{j}\binom{2N}{j+n} \quad \text{for} \quad 0\leqslant j+n\leqslant N,$$

$$c_{j,n}^{(N,N)} = \frac{(-1)^{j+n}\Gamma(2N+\sigma+\nu+1+j+n)}{2^{2N-j-n}} \binom{2N-j-n}{N-j}\binom{2N}{j+n} \quad \text{for} \quad N+1\leqslant j+n\leqslant 2N.$$

This enables us to use Theorem 1 for the construction of the Padé approximant of function (7) with denominator $Q_{N,N}(z,w)$ whose indices belong to the set $[0,N]^2 \subset \mathbb{Z}_+^2$. Since the polynomial $Y_{N,N}$ is orthogonal to $x_{k,m}$ not only for $(k,m) \in [0,N]^2 \setminus \{(N,N)\}$ but also for $(k,m) \in \{k, m \in \mathbb{Z}_+,\ k+m \leqslant 2N-1\}$, we set $x(k) = 4N-1-k$ and $y(m) = 4N-1-m$ in Theorem 1. This yields the following result:

Theorem 2. *For the confluent Humbert hypergeometric series*

$$f(z,w) = \Phi_2(1,1,\gamma,z,w) = \sum_{k,m=0}^{\infty} \frac{z^k w^m}{(\gamma)_{k+m}} = \frac{z\,{}_1F_1(1;\gamma;z) - w\,{}_1F_1(1;\gamma;w)}{z-w}$$

with $\gamma = \nu + \sigma + 2,\ \nu,\ \sigma > -1$, *for any* $N \in \mathbb{N}$, *the rational function*

$$[\mathscr{N}/\mathscr{D}]_f(z,w) = \frac{P_{\mathscr{N}}(z,w)}{Q_{N,N}(z,w)},$$

where

$$Q_{N,N}(z,w) = \sum_{m=0}^{N} (-1)^m \binom{2N}{m} \Gamma(2N+\sigma+\nu+1+m) z^{N-m} w^{N-m} \left(\frac{z+w}{2}\right)^m$$

$$+ \sum_{m=N+1}^{2N} (-1)^m \binom{2N}{m} \Gamma(2N+\sigma+\nu+1+m) \left(\frac{z+w}{2}\right)^{2N-m},$$

$$P_{\mathscr{N}}(z,w) = \sum_{m=0}^{N-1} \sum_{k=N-m}^{N-1} z^k w^m \sum_{j=N-k}^{N} \sum_{n=N-m}^{N-j} \frac{(-1)^{j+n}\Gamma(2N+\sigma+\nu+1+j+n)}{2^{j+n}}$$

$$\times \binom{j+n}{j} \binom{2N}{j+n} \frac{1}{\Gamma(k+m+j+n-2N+\nu+\sigma+2)}$$

$$+ \sum_{k=1}^{N-1} \sum_{m=0}^{N-1} z^k w^m \sum_{j=N-k}^{N-1} \sum_{n=N-j+1}^{N} \frac{(-1)^{j+n}\Gamma(2N+\sigma+\nu+1+j+n)}{2^{2N-j-n}}$$

$$\times \binom{2N-j-n}{N-j} \binom{2N}{j+n} \frac{1}{\Gamma(k+m+j+n-2N+\nu+\sigma+2)}$$

$$+ z^N \sum_{m=0}^{N-1} \sum_{k=0}^{3N-1-m} z^k w^m \sum_{j=0}^{N} \sum_{n=N-m}^{N-j} \frac{(-1)^{j+n}\Gamma(2N+\sigma+\nu+1+j+n)}{2^{j+n}}$$

$$\times \binom{j+n}{j} \binom{2N}{j+n} \frac{1}{\Gamma(k+m+j+n-N+\nu+\sigma+2)}$$

$$+ z^N \sum_{m=0}^{N-1} \sum_{k=0}^{3N-1-m} z^k w^m \sum_{j=0}^{N-1} \sum_{n=N-j+1}^{N} \frac{(-1)^{j+n}\Gamma(2N+\sigma+\nu+1+j+n)}{2^{2N-j-n}}$$

$$\times \binom{2N-j-n}{N-j}\binom{2N}{j+n} \frac{1}{\Gamma(k+m+j+n-N+\nu+\sigma+2)}$$

$$+ w^N \sum_{k=0}^{N-1} \sum_{m=0}^{3N-1-k} z^k w^m \sum_{n=0}^{k} \sum_{j=N-k}^{N-n} \frac{(-1)^{j+n}\Gamma(2N+\sigma+\nu+1+j+n)}{2^{j+n}}$$

$$\times \binom{j+n}{j}\binom{2N}{j+n} \frac{1}{\Gamma(k+m+j+n-N+\nu+\sigma+2)} +$$

$$+ w^N \sum_{k=0}^{N-1} \sum_{m=0}^{3N-1-k} z^k w^m \sum_{n=0}^{N-1} \sum_{j=N-n+1}^{N} \frac{(-1)^{j+n}\Gamma(2N+\sigma+\nu+1+j+n)}{2^{2N-j-n}}$$

$$\times \binom{2N-j-n}{N-j}\binom{2N}{j+n} \frac{1}{\Gamma(k+m+j+n-N+\nu+\sigma+2)},$$

admits an expansion in the power series whose coefficients coincide with the coefficients of series (2) for function (7) for all $(j,n) \in \mathscr{E} = \{(j,n) \in \mathbb{Z}_+^2 : j+n \leqslant 4N-1\}$.

On the basis of Theorem 2, we can establish the following result on the convergence of Padé approximants for confluent Humbert hypergeometric series:

Theorem 3. *The rational approximants of the function*

$$f(z,w) = \Phi_2(1,1,\gamma,z,w) = \sum_{k,m=0}^{\infty} \frac{z^k w^m}{(\gamma)_{k+m}} = \frac{z\,{}_1F_1(1;\gamma;z) - w\,{}_1F_1(1;\gamma;w)}{z-w}$$

with $\gamma = \nu+\sigma+2,\ \nu,\ \sigma > -1$, *constructed in Theorem 2 uniformly converge to the function* f *as* $N \to \infty$ *on each compact set from* $\mathbb{C}^2$. *Moreover, the denominators and numerators of the approximants satisfy the following asymptotic relations:*

$$Q_{N,N}(z,w) = \Gamma(4N+\sigma+\nu+1)\left(\exp\left(-\frac{z+w}{4}\right) + o(1)\right)$$

and

$$P_{\mathscr{N}}(z,w) = \Gamma(4N+\sigma+\nu+1)\left(\exp\left(\frac{z+w}{4}\right) f(z,w) + o(1)\right),$$

respectively, where the quantity $o(1) \to 0$ *as* $N \to \infty$ *on each compact set from* $\mathbb{C}^2$.

Proof. First, we consider the denominator

$$\frac{1}{\Gamma(4N+\sigma+\nu+1)}Q_{N,N}(z,w)=\sum_{m=0}^{N}(-1)^m\binom{2N}{m}\frac{\Gamma(2N+\sigma+\nu+1+m)}{\Gamma(4N+\sigma+\nu+1)}z^{N-m}w^{N-m}\left(\frac{z+w}{2}\right)^m$$

$$+\sum_{m=0}^{N-1}(-1)^m\binom{2N}{m}\frac{\Gamma(4N+\sigma+\nu+1-m)}{\Gamma(4N+\sigma+\nu+1)}\left(\frac{z+w}{2}\right)^m.$$

The following estimate is true for the first term:

$$\left|\sum_{m=0}^{N}(-1)^m\binom{2N}{m}\frac{\Gamma(2N+\sigma+\nu+1+m)}{\Gamma(4N+\sigma+\nu+1)}z^{N-m}w^{N-m}\left(\frac{z+w}{2}\right)^m\right|$$

$$\leqslant\sum_{m=0}^{N}\binom{2N}{m}\frac{|z|^{N-m}|w|^{N-m}\left|\frac{z+w}{2}\right|^m}{(3N+\sigma+\nu+1)(3N+\sigma+\nu+2)\ldots(4N+\sigma+\nu)}$$

$$\leqslant\frac{1}{(3N-1)^N}\sum_{m=0}^{2N}\binom{2N}{m}|z|^{N-m}|w|^{N-m}\left|\frac{z+w}{2}\right|^m\leqslant\frac{\left(|\,z\,||\,w\,|+\left|\frac{z+w}{2}\right|\right)^{2N}}{(3N-1)^N}.$$

For $|\,z\,|\leqslant R$ and $|\,w\,|\leqslant R,$ this estimate does not exceed the quantity

$$\frac{(R^2+R)^{2N}}{(3N-1)^N}$$

approaching zero as $N\to\infty$ for fixed $R>0.$

For some $0<M<N-1,$ we split the second term into two expressions:

$$\sum_{m=0}^{N-1}(-1)^m\binom{2N}{m}\frac{\Gamma(4N+\sigma+\nu+1-m)}{\Gamma(4N+\sigma+\nu+1)}\left(\frac{z+w}{2}\right)^m$$

$$=\sum_{m=0}^{M}(-1)^m\binom{2N}{m}\frac{\Gamma(4N+\sigma+\nu+1-m)}{\Gamma(4N+\sigma+\nu+1)}\left(\frac{z+w}{2}\right)^m$$

$$+\sum_{m=M+1}^{N-1}(-1)^m\binom{2N}{m}\frac{\Gamma(4N+\sigma+\nu+1-m)}{\Gamma(4N+\sigma+\nu+1)}\left(\frac{z+w}{2}\right)^m.$$

The second sum of this relation can be estimated as follows:

$$\left|\sum_{m=M+1}^{N-1}(-1)^m\binom{2N}{m}\frac{\Gamma(4N+\sigma+\nu+1-m)}{\Gamma(4N+\sigma+\nu+1)}\left(\frac{z+w}{2}\right)^m\right|$$

$$\leqslant \sum_{m=M+1}^{N-1} \frac{(2N-m+1)\ \dots\ (2N)}{m!} \frac{1}{(4N+\sigma+\nu+1-m)\ \dots\ (4N+\sigma+\nu)} \left|\frac{z+w}{2}\right|^m$$

$$= \sum_{m=M+1}^{N-1} \frac{\left|\frac{z+w}{2}\right|^m}{m!} \sum_{p=0}^{m-1} \frac{2N-p}{4N+\sigma+\nu-p} = \sum_{m=M+1}^{N-1} \frac{\left|\frac{z+w}{2}\right|^m}{m!} \sum_{p=0}^{m-1} \frac{1}{2+\dfrac{\sigma+\nu+p}{2N-p}}$$

$$= \sum_{m=M+1}^{N-1} \frac{\left|\frac{z+w}{2}\right|^m}{m!} \frac{1}{2^m} \sum_{p=0}^{m-1} \frac{1}{1+\dfrac{\sigma+\nu+p}{2(2N-p)}} \leqslant C \sum_{m=M+1}^{N-1} \frac{\left|\frac{z+w}{4}\right|^m}{m!}$$

$$\leqslant C \sum_{m=M+1}^{\infty} \frac{\left|\frac{z+w}{4}\right|^m}{m!} \leqslant C\frac{(R/2)^{M+1}}{(M+1)!},$$

where C is a constant.

We now subtract the expansion of the function $e^{-\frac{z+w}{4}}$ in the power series from the first sum and estimate the difference as follows:

$$\left|\sum_{m=0}^{M} (-1)^m \binom{2N}{m} \frac{\Gamma(4N+\sigma+\nu+1-m)}{\Gamma(4N+\sigma+\nu+1)} \left(\frac{z+w}{2}\right)^m - \exp\left(-\frac{z+w}{4}\right)\right|$$

$$\leqslant \left|\sum_{m=0}^{M} \frac{(-1)^m}{m!} \left(\frac{z+w}{2}\right)^m \left[\frac{(2N-m+1)\ \dots\ (2N)}{(4N+\sigma+\nu+1-m)\ \dots\ (4N+\sigma+\nu)} - \frac{1}{2^m}\right]\right.$$

$$\left. + \sum_{m=M+1}^{\infty} \frac{(-1)^m}{m!} \left(\frac{z+w}{4}\right)^m \right|$$

$$\leqslant \sum_{m=0}^{M} \frac{\left|\frac{z+w}{2}\right|^m}{m!} \left| \frac{1}{\left(2+\dfrac{\sigma+\nu+m-1}{2N-m+1}\right)\ \dots\ \left(2+\dfrac{\sigma+\nu}{2N}\right)} - \frac{1}{2^m}\right| + \sum_{m=M+1}^{\infty} \frac{1}{m!} \left|\frac{z+w}{4}\right|^m$$

$$\leqslant \varepsilon_N \sum_{m=0}^{M} \frac{1}{m!} \left|\frac{z+w}{4}\right|^m + \sum_{m=M+1}^{\infty} \frac{1}{m!} \left|\frac{z+w}{4}\right|^m \leqslant \varepsilon_N \exp\left(\frac{R}{2}\right) + \frac{1}{(M+1)!} \left(\frac{R}{2}\right)^{M+1},$$

where $\varepsilon_N \to 0$ as $N \to \infty$.

As a result, we get

$$\left|\frac{1}{\Gamma(4N+\sigma+\nu+1)} Q_{N,N}(z,w) - \exp\left(-\frac{z+w}{4}\right)\right|$$

$$\leqslant \frac{(R^2+R)^{2N}}{(3N-1)^N} + C\frac{(R/2)^{M+1}}{(M+1)!} + \varepsilon_N e^{R/2} + \frac{(R/2)^{M+1}}{(M+1)!}.$$

It is clear that this quantity tends to

$$(C+1)\frac{(R/2)^{M+1}}{(M+1)!},$$

as $N \to \infty$ and, hence, approaches zero as $M \to \infty$. Thus, on each polydisk $\{(z,w)\colon |z| \leqslant R, |w| \leqslant R\}$, we find

$$\frac{1}{\Gamma(4N+\sigma+\nu+1)} Q_{N,N}(z,w) \rightrightarrows \exp\left(-\frac{z+w}{4}\right).$$

By the Rouché theorem (see [8, p. 425]), the denominators $Q_{N,N}(z,w)$ do not have zeros on each compact set from $\mathbb{C}^2$ starting from a certain $N \in \mathbb{Z}_+$. In addition, this enables us to rewrite the denominators in the form

$$Q_{N,N}(z,w) = \Gamma(4N+\sigma+\nu+1)\left(\exp\left(-\frac{z+w}{4}\right) + o(1)\right),$$

where the quantity $o(1) \to 0$ as $N \to \infty$ on each compact set from $\mathbb{C}^2$.

We estimate the error of approximation as follows:

$$|f(z,w) - [\mathscr{N}/\mathscr{D}]_f(z,w)| = \frac{1}{|Q_{N,N}(z,w)|}\Bigg| z^N w^N \int_0^1 \left(\widehat{R_z}(A)\widehat{R_w}(B)x_{0,0}\right)(t)\, Y_{N,N}(t)dt$$

$$+ z^N \sum_{m=0}^{N-1} \sum_{k=3N-m}^{\infty} z^k w^m \sum_{j=0}^{N}\sum_{n=0}^{m} c_{j,N-n}^{(N,N)} s_{k+j,m-n}$$

$$+ w^N \sum_{k=0}^{N-1} \sum_{m=3N-k}^{\infty} z^k w^m \sum_{j=0}^{k}\sum_{n=0}^{N} c_{N-j,n}^{(N,N)} s_{k-j,m+n}\Bigg|.$$

Consider the first term

$$\int_0^1 \left(\widehat{R_z}(A)\widehat{R_w}(B)x_{0,0}\right)(t) Y_{N,N}(t)dt = \int_0^1 \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{t^{k+m+\nu} z^k w^m}{\Gamma(k+m+\nu+1)} Y_{N,N}(t)dt$$

$$= \int_0^1 \sum_{k+m \geqslant 2N} \frac{t^{k+m+\nu} z^k w^m}{\Gamma(k+m+\nu+1)} Y_{N,N}(t)dt$$

$$= \int_0^1 \sum_{p=2N}^{\infty} \frac{t^p}{\Gamma(p+\nu+1)} \sum_{k=0}^{p} z^k w^{p-k} Y_{N,N}(t) t^{\nu} dt$$

$$= \sum_{p=2N}^{\infty} \frac{1}{\Gamma(p+\nu+1)} \sum_{k=0}^{p} z^k w^{p-k} \int_0^1 t^p P_{2N}^{(\nu,\sigma)*}(t) t^{\nu} (1-t)^{\sigma} dt. \quad (8)$$

We can represent the function t^p as a linear combination of orthogonal shifted Jacobi polynomials $P_m^{(\nu,\sigma)*}(t)$:

$$t^p = \sum_{m=0}^{p} \alpha_m^{(p)} P_m^{(\nu,\sigma)*}(t). \quad (9)$$

Let $\overrightarrow{P_p}$ be a vector of the form

$$\overrightarrow{P_p} = \left(P_0^{(\nu,\sigma)*}(t), P_1^{(\nu,\sigma)*}(t), \ldots, P_p^{(\nu,\sigma)*}(t)\right)^T$$

and let $\overrightarrow{d_p}$ be a vector of the form

$$\overrightarrow{d_p} = (1, t, \ldots, t^p)^T.$$

Then we can rewrite representation (9) in the form

$$\overrightarrow{d_p} = A_p \overrightarrow{P_p}, \quad (10)$$

where the lower triangular matrix A_p has the form

$$A_p = \begin{pmatrix} \alpha_0^{(0)} & 0 & 0 & \ldots & 0 \\ \alpha_0^{(1)} & \alpha_1^{(1)} & 0 & \ldots & 0 \\ \alpha_0^{(2)} & \alpha_1^{(2)} & \alpha_2^{(2)} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \alpha_0^{(p)} & \alpha_1^{(p)} & \alpha_2^{(p)} & \ldots & \alpha_p^{(p)} \end{pmatrix}.$$

Earlier, it has been indicated that the following representation is true:

$$P_k^{(\nu,\sigma)*}(t) = \sum_{m=0}^{k} (-1)^m \binom{k}{m} \frac{\Gamma(k+\sigma+\nu+1+m)}{\Gamma(\nu+1+m)} t^m$$

or, in the vector form

$$\overrightarrow{P_p} = B_p \overrightarrow{d_p}, \quad (11)$$

Here, the lower triangular matrix B_p takes the form

$$B_p = \begin{pmatrix} \beta_0^{(0)} & 0 & 0 & \ldots & 0 \\ \beta_0^{(1)} & \beta_1^{(1)} & 0 & \ldots & 0 \\ \beta_0^{(2)} & \beta_1^{(2)} & \beta_2^{(2)} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \beta_0^{(p)} & \beta_1^{(p)} & \beta_2^{(p)} & \ldots & \beta_p^{(p)} \end{pmatrix}$$

and its elements have the form

$$\beta_m^{(k)} = (-1)^m \binom{k}{m} \frac{\Gamma(k+\sigma+\nu+1+m)}{\Gamma(\nu+1+m)}, \quad m = 0, 1, \ldots, k.$$

It follows from (10) and (11) that

$$A_p = B_p^{-1} \qquad \text{or} \qquad A_p B_p = I, \tag{12}$$

where I is the $(p+1)\times(p+1)$ identity matrix. By using (12), we obtain the following system of linear algebraic equations for the coefficients $\alpha_m^{(p)},\ m = 0, 1, \ldots, p$:

$$\begin{array}{cccc} \beta_0^{(0)}\alpha_0^{(p)} & +\beta_0^{(1)}\alpha_1^{(p)} & +\cdots+ & \beta_0^{(p)}\alpha_p^{(p)} = 0, \\ & \beta_1^{(1)}\alpha_1^{(p)} & +\cdots+ & \beta_1^{(p)}\alpha_p^{(p)} = 0, \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ & & & \beta_p^{(p)}\alpha_p^{(p)} = 1. \end{array}$$

The determinant of this system has the form

$$H_p = \beta_0^{(0)}\beta_1^{(1)}\ldots\beta_p^{(p)} \neq 0.$$

By using the Cramer formulas, the solution of this system is represented in the form

$$\alpha_j^{(p)} = \frac{1}{H_p} \begin{vmatrix} \beta_0^{(0)} & \beta_0^{(1)} & \ldots & \beta_0^{(j-1)} & 0 & \beta_0^{(j+1)} & \ldots & \beta_0^{(p)} \\ 0 & \beta_1^{(1)} & \ldots & \beta_1^{(j-1)} & 0 & \beta_1^{(j+1)} & \ldots & \beta_1^{(p)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \beta_{j-1}^{(j-1)} & 0 & \beta_{j-1}^{(j+1)} & \ldots & \beta_{j-1}^{(p)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & \beta_p^{(p)} \end{vmatrix}$$

$$
= \frac{(-1)^{p-j}}{\beta_j^{(j)} \dots \beta_p^{(p)}} \begin{vmatrix} \beta_j^{(j+1)} & \beta_j^{(j+2)} & \dots & \beta_j^{(p-1)} & \beta_j^{(p)} \\ \beta_{j+1}^{(j+1)} & \beta_{j+1}^{(j+2)} & \dots & \beta_{j+1}^{(p-1)} & \beta_{j+1}^{(p)} \\ 0 & \beta_{j+2}^{(j+2)} & \dots & \beta_{j+2}^{(p-1)} & \beta_{j+2}^{(p)} \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & \beta_{p-1}^{(p-1)} & \beta_{p-1}^{(p)} \end{vmatrix}
$$

This yields

$$
\alpha_{p-k}^{(p)} = (-1)^{p-k} \binom{p}{p-k} \frac{\Gamma(p+\nu+1)(2p+\sigma+\nu-2k+1)}{\Gamma(2p+\sigma+\nu+2-k)}. \tag{13}
$$

Remark. Relation (13) can also be derived from the corresponding relations for nonshifted orthogonal Jacobi polynomials (see [9, p. 471]).

In view of (9), we can rewrite relation (8) in the form

$$
\sum_{p=2N}^{\infty} \left(\sum_{k=0}^{p} z^k w^{p-k} \right) \frac{\alpha_{2N}^{(p)}}{\Gamma(p+\nu+1)} \int_0^1 \left[P_{2N}^{(\nu,\sigma)*}(t) \right]^2 t^{\nu} (1-t)^{\sigma} dt.
$$

Since the higher coefficient of the polynomial $P_{2N}^{(\nu,\sigma)*}$ is equal to $\frac{\Gamma(4N+\sigma+\nu+1)}{\Gamma(2N+\sigma+1)}$ and the square of the norm of the shifted Jacobi polynomial (with higher coefficient equal to 1) is given by the formula (see [7, p. 580])

$$
h_{2N} = \frac{(2N)!\, \Gamma(2N+\nu+1)\Gamma(2N+\sigma+\nu+1)\Gamma(2N+\sigma+1)}{(4N+\sigma+\nu+1)\Gamma^2(4N+\sigma+\nu+1)},
$$

we find

$$
\left\| P_{2N}^{(\nu,\sigma)*} \right\|^2_{L_2([0,1],t^{\nu}(1-t)^{\sigma})} = \int_0^1 \left[P_{2N}^{(\nu,\sigma)*}(t) \right]^2 t^{\nu}(1-t)^{\sigma} dt = \frac{(2N)!\Gamma(2N+\sigma+\nu+1)\Gamma(2N+\nu+1)}{(4N+\sigma+\nu+1)\Gamma(2N+\sigma+1)}. \tag{14}
$$

In view of (13) and (14), for the first term with $|z|,\ |w| \leqslant R,$ we obtain

$$
\left| \int_0^1 \left(\widehat{R_z}(A)\widehat{R_w}(B)x_{0,0} \right)(t) Y_{N,N}(t) dt \right| \leqslant \left\| P_{2N}^{(\nu,\sigma)*} \right\|^2 \sum_{p=2N}^{\infty} \left(\sum_{k=0}^{p} |z|^k |w|^{p-k} \right) \frac{\left| \alpha_{2N}^{(p)} \right|}{\Gamma(p+\nu+1)}
$$

$$
= \frac{(2N)!\Gamma(2N+\sigma+\nu+1)\Gamma(2N+\nu+1)}{\Gamma(2N+\sigma+1)(4N+\sigma+\nu+1)}
$$

$$
\times \sum_{p=2N}^{\infty} \left(\sum_{k=0}^{p} |z|^k |w|^{p-k} \right) \binom{p}{2N} \frac{\Gamma(p+\nu+1)(4N+\sigma+\nu+1)}{\Gamma(p+\nu+1)\Gamma(p+2N+\sigma+\nu+2)}
$$

$$= \frac{\Gamma(2N+\sigma+\nu+1)\Gamma(2N+\nu+1)}{\Gamma(2N+\sigma+1)} \sum_{p=2N}^{\infty} \left(\sum_{k=0}^{p} |z|^k |w|^{p-k} \right) \frac{p(p-1)\ldots(p-2N+1)}{\Gamma(p+2N+\sigma+\nu+2)}$$

$$\leqslant \frac{\Gamma(2N+\sigma+\nu+1)\Gamma(2N+\nu+1)}{\Gamma(2N+\sigma+1)} \sum_{p=2N}^{\infty} \frac{(p+1)p(p-1)\ldots(p-2N+1)R^p}{\Gamma(p+2N+\sigma+\nu+2)}$$

$$= \frac{\Gamma(2N+\sigma+\nu+1)\Gamma(2N+\nu+1)}{\Gamma(2N+\sigma+1)} R^{2N} \sum_{p=0}^{\infty} \frac{(p+2N+1)!R^p}{p!\Gamma(p+4N+\sigma+\nu+2)}.$$

Denote

$$I_N = \sum_{p=0}^{\infty} \frac{(p+2N+1)!R^p}{p!\Gamma(p+4N+\sigma+\nu+2)}$$

$$= \frac{(2N+1)!}{\Gamma(4N+\sigma+\nu+2)} \sum_{p=0}^{\infty} \frac{(2N+2)(2N+3)\ldots(2N+p+1)}{(4N+\sigma+\nu+2)(4N+\sigma+\nu+3)\ldots(4N+\sigma+\nu+p+1)} \frac{R^p}{p!}$$

$$\leqslant C \frac{(2N+1)!}{\Gamma(4N+\sigma+\nu+2)},$$

where C is a constant depending on R.

Thus, for the first term, we get

$$\left| \int_0^1 \left(\widehat{R_z}(A) \widehat{R_w}(B) x_{0,0} \right)(t) Y_{N,N}(t)\,dt \right| \leqslant C \frac{\Gamma(2N+\sigma+\nu+1)\Gamma(2N+\nu+1)R^{2N}(2N+1)!}{\Gamma(2N+\sigma+1)\Gamma(4N+\sigma+\nu+2)}.$$

By using the Stirling formula (see [7, p. 83]), we estimate this quantity by the expression

$$C \frac{(2N+2)^{1/2+\sigma-\nu}}{2^{4N}} R^{2N}.$$

The second term is estimated as follows:

$$\left| w^N \sum_{k=0}^{N-1} z^k \sum_{m=3N-k}^{\infty} w^m \sum_{j=0}^{k} \sum_{n=0}^{N} c_{N-j,n}^{(N,N)} s_{k-j,m+n} \right| \leqslant R^N \sum_{k=0}^{N-1} |z|^k \sum_{m=3N-k}^{\infty} |w|^m s_{0,m} \sum_{j=0}^{N} \sum_{n=0}^{N} \left| c_{j,n}^{(N,N)} \right|.$$

By using relations for the coefficients of shifted orthonormal Jacobi polynomials, we obtain

$$\sum_{n=0}^{N} \left| c_{j,n}^{(N,N)} \right| = \sum_{k=0}^{2N} \left| p_k^{(2N)} \right| = \sum_{k=0}^{2N} \binom{2N}{k} \frac{\Gamma(2N+\sigma+\nu+1+k)}{\Gamma(\sigma+1+k)} \leqslant C\Gamma(4N+\sigma+\nu+1)2^{4N}.$$

Hence, for the second term, we get

$$\left| w^N \sum_{k=0}^{N-1} z^k \sum_{m=3N-k}^{\infty} w^m \sum_{j=0}^{k} \sum_{n=0}^{N} c_{N-j,n}^{(N,N)} s_{k-j,m+n} \right|$$

$$\leqslant CR^N 2^{4N} \Gamma(4N+\sigma+\nu+1) \sum_{k=0}^{N-1} R^k \sum_{m=3N-k}^{\infty} \frac{R^m}{\Gamma(m+\nu+\sigma+2)}$$

$$\leqslant C \frac{R^{2N-1} 2^{4N} N \Gamma(4N+\sigma+\nu+1)}{\Gamma(2N+\nu+\sigma+2)}.$$

A similar estimate also holds for the third term.

By using the established estimates, we obtain the following relation for the error of approximation:

$$\left| f(z,w) - [\mathscr{N}/\mathscr{D}]_f(z,w) \right|$$

$$\leqslant \frac{C}{\Gamma(4N+\sigma+\nu+1)} \left(\frac{(2N+2)^{1/2+\sigma-\nu}}{2^{4N}} R^{2N} + \frac{R^{2N-1} 2^{4N} N \Gamma(4N+\sigma+\nu+1)}{\Gamma(2N+\nu+\sigma+2)} \right) \to 0$$

as $N \to \infty$. This yields the assertion of the theorem.

The theorem is proved.

To illustrate Theorem 2, we consider a special case $\nu = \sigma = 0$. As mentioned above, the function $f(z,w)$ has the form

$$f(z,w) = \frac{e^w - e^z}{w - z}. \tag{15}$$

First, we set $N = 1$. By Theorem 2, we obtain the following rational approximation:

$$\frac{P_{\mathscr{N}}(z,w)}{Q_{1,1}(z,w)} = \frac{24 + w^2 + z^2 + 6z + 6w}{2zw - 6z - 6w + 24}.$$

We now compare the values of the approximated function

$$f(z,w) = \frac{e^w - e^z}{w - z},$$

with the values of partial sum of the power series

$$P_3(z,w) = 1 + \frac{1}{2}(z+w) + \frac{1}{6}(z^2 + zw + w^2) + \frac{1}{24}(z^3 + z^2 w + z w^2 + w^3)$$

and the values of our approximation at points of the square $[1,0] \times [1,0]$ (see Table 1).

Table 1

w	z					
	0.0	0.2	0.4	0.6	0.8	1
0.0	1	1.107013790	1.229561745	1.370198000	1.531926160	1.718281828
	1	1.107000000	1.229333334	1.369000000	1.528000000	1.708333333
	1	1.107017544	1.229629630	1.370588235	1.533333333	1.722222222
0.2	1.107013790	1.221402758	1.352109700	1.501790105	1.673563617	1.871098837
	1.107000000	1.221333333	1.351666667	1.500000000	1.668333333	1.858666667
	1.107017544	1.221402214	1.352140078	1.502057613	1.674672489	1.874418605
0.4	1.229561745	1.352109700	1.491824698	1.651470510	1.834290575	2.044095217
	1.229333334	1.351666667	1.490666669	1.648333334	1.826666667	2.027666669
	1.229629630	1.352140078	1.491803279	1.651515152	1.834862385	2.046341463
0.6	1.370198000	1.501790105	1.651470510	1.822118800	2.017110640	2.240407570
	1.369000000	1.500000000	1.648333334	1.816000000	2.005000000	2.217333334
	1.370588235	1.502057613	1.651515152	1.821917808	2.016908213	2.241025641
0.8	1.531926160	1.673563617	1.834290575	2.017110640	2.225540928	2.46370450
	1.528000000	1.668333333	1.826666667	2.005000000	2.205333333	2.429666667
	1.533333333	1.674672489	1.834862385	2.016908213	2.224489796	2.462162162
1	1.718281828	1.871098837	2.044095217	2.240407570	2.46370450	2.718281828
	1.708333333	1.858666667	2.027666669	2.217333334	2.429666667	2.666666667
	1.722222222	1.874418605	2.046341463	2.241025641	2.462162162	2.714285714

Taking $N = 2,$ we get the following rational function:

$$\frac{P_{\mathcal{N}}(z,w)}{Q_{2,2}(z,w)} = \Big(40320 + 10080z + 10080w + 2760z^2 + 2760w^2 - 1200zw$$

$$+ 540z^3 + 540w^3 - 300z^2w - 300zw^2 + 96z^4 + 96w^4 - 84z^3w - 84zw^3$$

$$- 17zw^4 - 17zw^4 + 17w^5 + 17z^5 - 3zw^5 - 3z^5w + 3w^6 + 3z^6 + \frac{1}{2}z^7 + \frac{1}{2}w^7$$

Table 2

w	z					
	0.0	0.2	0.4	0.6	0.8	1
0.0	1	1.107013790	1.229561745	1.370198000	1.531926160	1.718281828
	1	1.107013790	1.229561743	1.370197951	1.531925658	1.718278770
	1	1.107013791	1.229561743	1.370197961	1.531925731	1.718279055
0.2	1.107013790	1.221402758	1.352109700	1.501790105	1.673563617	1.871098837
	1.107013790	1.221402753	1.352109694	1.501790031	1.673562944	1.871095015
	1.107013791	1.221402758	1.352109708	1.501790219	1.673564121	1.871099899
0.4	1.229561745	1.352109700	1.491824698	1.651470510	1.834290575	2.044095217
	1.229561743	1.352109694	1.491824685	1.651470370	1.834289575	2.044090124
	1.229561743	1.352109708	1.491824698	1.651470760	1.834292060	2.044100453
0.6	1.370198000	1.501790105	1.651470510	1.822118800	2.017110640	2.240407570
	1.370197951	1.501790031	1.651470370	1.822118351	2.017108779	2.240399999
	1.370197961	1.501790219	1.651470760	1.822118800	2.017108406	2.240416710
0.8	1.531926160	1.673563617	1.834290575	2.017110640	2.225540928	2.46370450
	1.531925658	1.673562944	1.834289575	2.017108779	2.225536363	2.463691216
	1.531925731	1.673564121	1.834292060	2.017108406	2.225540917	2.463714564
1	1.718281828	1.871098837	2.044095217	2.240407570	2.46370450	2.718281828
	1.718278770	1.871095015	2.044090124	2.240399999	2.463691216	2.718253968
	1.718279055	1.871099899	2.044100453	2.240416710	2.463714564	2.718281718

$$- \frac{1}{2}z^6 w - \frac{1}{2}zw^6\Big)\Big(24z^2w^2 - 240zw^2 - 240z^2w$$
$$+ 1080z^2 + 1080w^2 + 2160zw - 10080z - 10080w + 40320\Big)^{-1}.$$

The values of the approximated function, the values of a partial sum of the power series

$$P_7(z,w) = 1 + \frac{1}{2!}(z+w) + \frac{1}{3!}(z^2 + zw + w^2) + \frac{1}{4!}(z^3 + z^2w + zw^2 + w^3)$$

$$+\frac{1}{5!}(z^4+z^3w+z^2w^2+zw^3+w^4)$$

$$+\frac{1}{6!}(z^5+z^4w+z^3w^2+z^2w^3+zw^4+w^5)$$

$$+\frac{1}{7!}\big(z^6+z^5w+z^4w^2+z^3w^3+z^2w^4+zw^5+w^6\big)$$

$$+\frac{1}{8!}\big(z^7+z^6w+z^5w^2+z^4w^3+z^3w^4+z^2w^5+zw^6+w^7\big)$$

and the values of the constructed approximation are presented in Table 2.

The examples presented above demonstrate that the rational approximants constructed according to Theorem 2 approximate function (15) better than the partial sum of the power series with the same number of free coefficients.

Note that the numerical examples presented in [10] are connected with the computation of rational approximations obtained as certain two-dimensional generalizations of Padé approximations for the function

$$f(z,w)=\frac{we^w-ze^z}{w-z}.$$

This function is also a special case of function (7) for $\nu+\sigma=-1$.

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