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GENERALIZED MOMENT REPRESENTATIONS, BIORTHOGONAL POLYNOMIALS, AND PADÉ APPROXIMANTS

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By using the method of generalized moment representations and certain properties of biorthogonal polynomials, we establish new invariance properties of the Padé approximants.

1. Padé Approximants and Generalized Moment Representations

The method of Padé approximants is one of the most efficient and widely used methods for rational approximation of analytic functions. Padé approximants and their applications are studied in numerous papers (see, e.g., the bibliography in [1]).

Definition 1 [1, p. 311]. *Assume that a function $f(z)$ can be expanded in a power series of the form*

$$f(z) = \sum_{k=0}^{\infty} s_k z^k \tag{1}$$

in a neighborhood of the point $z = 0$. The rational function

$$[M/N]_f(z) = \frac{P_M(z)}{Q_N(z)}, \tag{2}$$

where $P_M(z)$ and $Q_N(z)$ are algebraic polynomials of degree not higher than M and N, respectively, is called the Padé approximant of degree $[M/N]$ for the function $f(z)$ if $f(z) - [M/N]_f(z) = O(z^{M+N+1})$ for $z \to 0$.

It was shown by Jacobi that the construction of the Padé approximants for functions defined by their power expansions (1) can be reduced to the solution of linear algebraic equations and, hence, approximants (2) can be represented as a ratio of determinants [1, p. 18]. In many cases, however, this approach is not efficient. If $f(z)$ is a Markov function, i.e., if it can be represented in the form of an integral

$$f(z) = \int_{\Delta} \frac{d\mu(t)}{1 - zt}, \tag{3}$$

where $\mu(t)$ is a nondecreasing function with infinitely many points of growth on a real interval Δ, then the construction of the Padé approximants of degree $[N-1/N]$, $N \in \mathbb{N}$, for $f(z)$ can be reduced to the construction of polynomials orthogonal on Δ with weight $d\mu(t)$ [2, p. 34].

In 1981, Dzyadyk [3] suggested the method of generalized moment representations, which enables one to construct and investigate the Padé approximants of functions more general than (3).

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Definition 2 [3]. *The generalized moment representation of a numerical sequence* $\{s_k\}_{k=0}^{\infty}$ *in a Banach space* X *is the following collection of equalities:*

$$s_{k+j} = \langle x_k, y_j \rangle, \quad k, j = \overline{0, \infty}, \tag{4}$$

where $x_k \in X$, $k = \overline{0, \infty}$, $y_j \in X^*$, $j = \overline{0, \infty}$, *and* $\langle x, y \rangle$ *denotes the action of a functional* $y \in X^*$ *on an element* $x \in X^*$.

However, it turns out that the construction of the Padé approximants for functions expandable in power series and representable in the form (4) requires not orthogonal polynomials as in the case of functions of the form (3) but biorthogonal ones — much more complicated and inadequately studied objects. Namely, if $f(z)$ is expandable in series (1) and the sequence $\{s_k\}_{k=0}^{\infty}$ is representable in the form (4), the Padé approximant of degree $[N-1/N]$ for the function $f(z)$ can be written as follows:

$$[N-1/N]_f(z) = \frac{P_{N-1}(z)}{Q_N(z)},$$

where

$$P_{N-1}(z) = \sum_{j=1}^{N} c_j^{(N)} z^{N-j} \sum_{k=0}^{j-1} s_k z^k,$$

$$Q_N(z) = \sum_{j=0}^{N} c_j^{(N)} z^{N-j},$$

and the coefficients $c_j^{(N)}$, $j = \overline{0, N}$, are determined by the biorthogonal relations for the generalized polynomial $Y_N = \sum_{j=0}^{N} c_j^{(N)} y_j$: $\langle x_k, Y_N \rangle = 0$, $k = \overline{0, N-1}$.

In this connection, it became quite important to study the properties of biorthogonal polynomials (see [4–7]). In this paper, we establish some new properties of biorthogonal polynomials and apply them to the investigation of Padé approximants.

2. Generalized Moment Representations and Biorthogonal Polynomials

Let $\{s_k\}_{k=0}^{\infty}$ be the sequence of coefficients of the expansion of an analytic function $f(z)$ in a power series

$$f(z) = \sum_{k=0}^{\infty} s_k z^k. \tag{5}$$

Below, we assume that this sequence admits the following generalized moment representation in a Banach space X:

$$s_{k+j} = \langle x_k, y_j \rangle, \quad k, j = \overline{0, \infty}. \tag{6}$$

Furthermore, we assume that there exists a linear continuous operator $A : X \to X$ such that $A x_k = x_{k+1}$, $k = \overline{0, \infty}$. In this case, clearly, we have [8] $A^* y_j = y_{j+1}$, $j = \overline{0, \infty}$, the generalized moment representation (6) is equivalent to the representation $s_k = \langle A^k x_0, y_0 \rangle$, $k = \overline{0, \infty}$, whereas representation (5) can be rewritten in the form $f(z) = \langle R_z(A) x_0, y_0 \rangle$, where $R_z(A) = (I - zA)^{-1}$ is the resolvent of the operator A.

We also assume that the systems $x_k = A^k x_0$, $k = \overline{0,\infty}$, and $y_j = A^{*j} y_0$, $j = \overline{0,\infty}$, admit nondegenerate biorthogonalization, i.e., for any $M, N \in \mathbb{N}$, there exist polynomials $X_M = \sum_{k=0}^{M} c_k^{(M)} x_k$ and $Y_N = \sum_{j=0}^{N} c_j^{(N)} y_j$ with nonzero leading coefficients such that

$$\langle X_M, y_j \rangle = 0, \quad j = \overline{0, M-1}, \quad \langle X_M, y_M \rangle \neq 0,$$

$$\langle x_k, Y_N \rangle = 0, \quad k = \overline{0, N-1}, \quad \langle x_N, Y_N \rangle \neq 0.$$

Clearly, this is equivalent to the assumption that the Hankel determinants of the sequence $\{s_k\}_{k=0}^{\infty}$ are nonzero [1, p. 18]. We consider the problem of biorthogonalization of the systems $\tilde{x}_k = A^k \tilde{x}_0$, $k = \overline{0,\infty}$, and $y_j = A^{*j} y_0$, $j = \overline{0,\infty}$, where $\tilde{x}_0 \in X$ satisfies the operator equation

$$\prod_{m=1}^{M} (1 - \beta_m A) \tilde{x}_0 = \sum_{m=0}^{M} \alpha_m A^m \tilde{x}_0 = x_0.$$

Before formulating the corresponding result, we introduce some additional notation:

$$\tilde{f}(z) := \sum_{k=0}^{\infty} \tilde{s}_k z^k = \langle R_z(A) x_0, y_0 \rangle,$$

$$[N-1/N]_f(z) =: \frac{P_{N-1}(z)}{Q_N(z)}, \quad N = \overline{1,\infty},$$

$$[N-1/N]_{\tilde{f}}(z) =: \frac{\tilde{P}_{N-1}(z)}{\tilde{Q}_N(z)}, \quad N = \overline{1,\infty},$$

$$\varepsilon_N(z) := \frac{f(z) Q_N(z) - P_{N-1}(z)}{z^N}.$$

Theorem 1. *Let* $\tilde{x}_0$ *be a solution of the operator equation*

$$\prod_{m=1}^{p} (1 - \beta_m A)^{r_m} \tilde{x}_0 = x_0, \tag{7}$$

where all numbers β_m, $m = \overline{1,p}$, *are different. Then, for all* $N \geq M = r_1 + r_2 + \ldots + r_p$ *nontrivial generalized polynomials* $\tilde{Y}_N$ *in the system of functionals* $y_j = A^{*j} y_0$, $j = \overline{0,N}$, *possessing the biorthogonal properties*

$$\langle \tilde{x}_k, \tilde{Y}_N \rangle = 0, \quad k = \overline{0, N-1},$$

where $\tilde{x}_k = A^k \tilde{x}_0$, $k = \overline{0,\infty}$, *can be represented in terms of biorthogonal polynomials* Y_m, $m = \overline{N-M, N}$, *possessing the biorthogonal properties*

$$\langle x_k, Y_m \rangle = 0, \quad k = \overline{0, m-1},$$

by the relations

$$\tilde{Y}_N = \det \begin{Vmatrix} \varepsilon_{N-M}(\beta_1) & \varepsilon_{N-M+1}(\beta_1) & \dots & \varepsilon_N(\beta_1) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \varepsilon_{N-M}^{(r_1-1)}(\beta_1) & \varepsilon_{N-M+1}^{(r_1-1)}(\beta_1) & \dots & \varepsilon_N^{(r_1-1)}(\beta_1) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \varepsilon_{N-M}^{(r_p-1)}(\beta_p) & \varepsilon_{N-M+1}^{(r_p-1)}(\beta_p) & \dots & \varepsilon_N^{(r_p-1)}(\beta_p) \\ Y_{N-M} & Y_{N-M+1} & \dots & Y_N \end{Vmatrix}.$$

Proof. First, we consider the case where $r_1 = r_2 = \ldots = r_p = 1$. Clearly, the following expansion is possible for any $N \geq M$:

$$\tilde{Y}_N = \sum_{i=0}^{N} \gamma_i^{(N)} Y_i. \tag{8}$$

By applying functional (8) to x_k, $k = \overline{0, N-M-1}$, we get

$$0 = \sum_{i=0}^{k} \gamma_i^{(N)} \langle x_k, Y_i \rangle, \qquad k = \overline{0, N-M-1}. \tag{9}$$

Since the biorthogonalization is nondegenerate, we have $\langle x_i, Y_i \rangle \neq 0$ for any $i \in \mathbb{N} \cup \{0\}$. Therefore, relations (9) imply that $\gamma_i^{(N)} = 0$, $i = \overline{0, N-M-1}$. Thus, we can rewrite (8) in the form

$$\tilde{Y}_N = \sum_{i=N-M}^{N} \gamma_i^{(N)} Y_i. \tag{10}$$

The coefficients $\gamma_i^{(N)}$, $i = \overline{N-M, N}$, can be determined (to within a constant factor) from the conditions of orthogonality of $\tilde{Y}_N$ to $\tilde{x}_k$, $k = \overline{0, M-1}$,

$$0 = \langle \tilde{x}_k, \tilde{Y}_N \rangle = \sum_{i=N-M}^{N} \gamma_i^{(N)} \langle \tilde{x}_k, Y_i \rangle, \qquad k = \overline{0, M-1}. \tag{11}$$

It follows from (11) that

$$\sum_{i=N-M}^{N} \gamma_i^{(N)} \left\langle \prod_{\substack{m=1 \\ m \neq k}}^{M} (1 - \beta_m A)\tilde{x}_0, Y_i \right\rangle = 0, \quad k = \overline{1, M}. \tag{12}$$

Consider the algebraic polynomials

$$b_k(x) = \prod_{\substack{m=1 \\ m \neq k}}^{M} (1 - \beta_m x).$$

These polynomials are linearly independent. Indeed, assume the contrary, i.e., let

$$0 \equiv \sum_{k=1}^{M} \xi_k b_k(x) = \sum_{k=1}^{M} \xi_k \prod_{\substack{m=1 \\ m \neq k}}^{M} (1-\beta_m x). \tag{13}$$

By setting $x = 1/\beta_j$, $j = \overline{1, M}$, in (13), we get $\xi_j = 0$, $j = \overline{1, M}$, which contradicts the assumption. This implies that conditions (11) and (12) are equivalent. In view of (7), equalities (12) can be rewritten in the form

$$\sum_{i=N-M}^{N} \gamma_i^{(N)} \langle (I-\beta_k A)^{-1} x_0, Y_i \rangle = 0, \quad k = \overline{1, M}. \tag{14}$$

It can be shown that [8]

$$\langle (I-\beta_k A)^{-1} x_0, Y_i \rangle = \frac{f(\beta_k) Q_i(\beta_k) - P_{i-1}(\beta_k)}{\beta_k^i} = \varepsilon_i(\beta_k), \tag{15}$$

where $i = \overline{N-M, N}$, $k = \overline{1, M}$. Taking (10), (14), and (15) into account, we obtain

$$\tilde{Y}_N = \det \begin{Vmatrix} \varepsilon_{N-M}(\beta_1) & \varepsilon_{N-M+1}(\beta_1) & \dots & \varepsilon_N(\beta_1) \\ \varepsilon_{N-M}(\beta_2) & \varepsilon_{N-M+1}(\beta_2) & \dots & \varepsilon_N(\beta_2) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \varepsilon_{N-M}(\beta_M) & \varepsilon_{N-M+1}(\beta_M) & \dots & \varepsilon_N(\beta_M) \\ Y_{N-M} & Y_{N-M+1} & \dots & Y_N \end{Vmatrix}.$$

To prove Theorem 1 in the general case, i.e., for multiple β_m, it suffices to replace each number β_m of multiplicity r_m by r_m distinct numbers β_m, $\beta_m + h, \dots, \beta_m + (r_m - 1)h$ and then pass to the limit as $h \to 0$.

3. Biorthogonal Polynomials and Padé Approximants

The properties of biorthogonal polynomials established in Theorem 1 involve certain invariance properties of the Padé approximants.

Theorem 2. *Suppose that there exist nondegenerate Padé approximants of degree* $[N-1/N]$ *and* $[N-2/N-1]$, $N \geq 2$, *for a function* $f(z) = \sum_{k=0}^{\infty} s_k z^k$. *Assume also that, for some point* β,

$$\varepsilon_{N-1}(\beta) = \frac{f(\beta) Q_{N-1}(\beta) - P_{N-2}(\beta)}{\beta^{N-1}} \neq 0.$$

Then there exists a nondegenerate Padé approximant of degree $[N-1/N]$ *for the function*

$$\tilde{f}(z) = \frac{z f(z) - \beta f(\beta)}{z - \beta}.$$

In this case,

$$[N-1/N]_{\tilde{f}}(z) = \frac{\tilde{P}_{N-1}(z)}{\tilde{Q}_N(z)},$$

where

$$\tilde{P}_{N-1}(z) = \frac{\varepsilon_{N-1}(\beta)}{\beta - z}[\beta f(\beta)Q_N(z) - zP_{N-1}(z)] - \frac{z\varepsilon_N(\beta)}{\beta - z}[\beta f(\beta)Q_{N-1}(z) - zP_{N-2}(z)],$$

$$\tilde{Q}_N(z) = \varepsilon_{N-1}(\beta)Q_N(z) - z\varepsilon_N(\beta)Q_{N-1}(z).$$

Proof. Without loss of generality, we can assume [9] that the sequence $\{s_k\}_{k=0}^{\infty}$ admits the generalized moment representation $s_k = \langle A^k x_0, y_0 \rangle$, $k = \overline{0, \infty}$. We set $\tilde{x}_0 = (I - \beta A)^{-1} x_0$. Then

$$\tilde{s}_k = \langle A^k \tilde{x}_0, y_0 \rangle = \langle (I - \beta A)^{-1} x_0, A^{*k} y_0 \rangle = \langle R_\beta(A) x_0, y_k \rangle = \frac{f(\beta) - T_{k-1}(f;\beta)}{\beta^k},$$

where $T_k(f, \beta)$ are Taylor polynomials of degree k for the function $f(z)$ [8]. Further, we have

$$f(z) = \sum_{k=0}^{\infty} \tilde{s}_k z^k = \sum_{k=0}^{\infty} \frac{f(\beta) - T_{k-1}(f;\beta)}{\beta^k} z^k = \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} s_j \beta^{j-k} z^k$$

$$= \sum_{j=0}^{\infty} s_j \sum_{k=0}^{j} \beta^{j-k} z^k = \sum_{j=0}^{\infty} s_j \beta_j \sum_{k=0}^{j} (z/\beta)^k = \sum_{j=0}^{\infty} s_j \beta^j \frac{1-(z/\beta)^{j+1}}{1-z/\beta}$$

$$= \sum_{j=0}^{\infty} s_j \frac{\beta^{j+1} - z^{j+1}}{\beta - z} = \frac{zf(z) - \beta f(\beta)}{z - \beta}.$$

By virtue of Theorem 1, we get

$$\tilde{Y}_N = \varepsilon_{N-1}(\beta) Y_N - \varepsilon_N(\beta) Y_{N-1}$$

or

$$\sum_{j=0}^{N} c_j^{(N)} y_j = \sum_{j=0}^{N-1} [\varepsilon_{N-1}(\beta) c_j^{(N)} - \varepsilon_N(\beta) c_j^{(N-1)}] y_j + \varepsilon_{N-1}(\beta) c_N^{(N)} y_N.$$

Hence,

$$\tilde{c}_j^{(N)} = \varepsilon_{N-1}(\beta) c_j^{(N)} - \varepsilon_N(\beta) c_j^{(N-1)}, \quad j = \overline{0, N-1}, \quad \tilde{c}_N^{(N)} = \varepsilon_{N-1}(\beta) c_N^{(N)}.$$

Therefore, the denominator of the Padé approximant of degree $[N-1/N]$ for the function $\tilde{f}(z)$ can be written as follows:

$$\tilde{Q}_N(z) = \sum_{j=0}^{N} \tilde{c}_j^{(N)} z^{N-j} = \sum_{j=0}^{N-1} [\varepsilon_{N-1}(\beta) c_j^{(N)} - \varepsilon_N(\beta) c_j^{(N-1)}] z^{N-j} + \varepsilon_{N-1}(\beta) c_N^{(N)}$$

$$= \varepsilon_{N-1}(\beta) \sum_{j=0}^{N} c_j^{(N)} z^{N-j} - \varepsilon_N(\beta) \sum_{j=0}^{N-1} c_j^{(N-1)} z^{N-j} = \varepsilon_{N-1}(\beta) Q_N(z) - \varepsilon_N(\beta) z Q_{N-1}(z).$$

The numerator of the Padé approximant has the form

$$\tilde{P}_{N-1}(z) = \sum_{j=0}^{N} \tilde{c}_j^{(N)} z^{N-j} T_{j-1}[\tilde{f}; z]. \tag{16}$$

We have

$$T_{j-1}[\tilde{f}; z] = \sum_{p=0}^{j-1} \tilde{s}_p z^p = \sum_{p=0}^{j-1} f_p(\beta) z^p = \sum_{p=0}^{j-1} \frac{f(\beta) - T_{p-1}(f;\beta)}{\beta^p} z^p$$

$$= \sum_{k=0}^{j-1} \sum_{k=p}^{\infty} s_k \beta^{k-p} z^p = \sum_{p=0}^{j-1} \left[\sum_{k=p}^{j-1} s_p \beta^{k-p} + \sum_{k=j}^{\infty} s_p \beta^{k-p} \right] z^p$$

$$= \sum_{k=0}^{j-1} s_k \beta^k \sum_{p=0}^{k} \left(\frac{z}{\beta}\right)^p + \sum_{k=j}^{\infty} s_k \beta^k \sum_{p=0}^{j-1} \left(\frac{z}{\beta}\right)^p$$

$$= \sum_{k=0}^{j-1} s_k \beta^k \frac{1 - \left(\frac{z}{\beta}\right)^{k+1}}{1 - \frac{z}{\beta}} + \sum_{k=j}^{\infty} s_k \beta^k \frac{1 - \left(\frac{z}{\beta}\right)^{j}}{1 - \frac{z}{\beta}}$$

$$= \frac{1}{\beta - z} \sum_{k=0}^{j-1} s_k (\beta^{k+1} - z^{k+1}) + \frac{1}{\beta - z} \sum_{k=j}^{\infty} s_k \beta^{k-j+1} (\beta^j - z^j)$$

$$= \frac{1}{\beta - z} \{ \beta T_{j-1}[f;\beta] - z T_{j-1}[f;z] \} + \frac{1}{\beta - z} \sum_{k=j}^{\infty} s_k (\beta^{k+1} - \beta^{k-j+1} z^j)$$

$$= \frac{1}{\beta - z} \{ \beta T_{j-1}[f;\beta] - z T_{j-1}[f;z] \} + \frac{\beta}{\beta - z} \left[1 - \left(\frac{z}{\beta}\right)^j \right] \{ f(\beta) - z T_{j-1}[f;\beta] \}$$

$$= \frac{1}{\beta - z} \left\{ \beta f(\beta) - \beta \left(\frac{z}{\beta}\right)^j f(\beta) + \beta \left(\frac{z}{\beta}\right)^j T_{j-1}[f;\beta] - z T_{j-1}[f;z] \right\}. \tag{17}$$

By inserting (17) in (16), we obtain

$$\tilde{P}_{N-1}(z) = \varepsilon_{N-1}(\beta) \sum_{j=0}^{N} c_j^{(N)} z^{N-j} \frac{1}{\beta - z} \left\{ \beta f(\beta) - \beta \left(\frac{z}{\beta}\right)^j f(\beta) + \beta \left(\frac{z}{\beta}\right)^j T_{j-1}[f;\beta] - zT_{j-1}[f;z] \right\}$$

$$- \varepsilon_N(\beta) \sum_{j=0}^{N} c_j^{(N-1)} z^{N-j} \frac{1}{\beta - z} \left[\beta f(\beta) - \beta \left(\frac{z}{\beta}\right)^j f(\beta) + \beta \left(\frac{z}{\beta}\right)^j T_{j-1}(\beta) - zT_{j-1}(z) \right]$$

$$= \frac{\varepsilon_{N-1}(\beta)}{\beta - z} \left\{ \beta f(\beta) Q_N(z) - zP_{N-1}(z) - \beta \left(\frac{z}{\beta}\right)^N \varepsilon_N(\beta)\beta^N \right\}$$

$$- \frac{\varepsilon_N(\beta)}{\beta - z} \left\{ z\beta f(\beta) Q_{N-1}(z) - z^2 P_{N-2}(z) - \beta^2 \left(\frac{z}{\beta}\right)^N \varepsilon_{N-1}(\beta)\beta^{N-1} \right\}$$

$$= \frac{\varepsilon_{N-1}(\beta)}{\beta - z} \{ \beta f(\beta) Q_N(z) - zP_{N-1}(z) \} - \frac{\varepsilon_N(\beta)}{\beta - z} \{ z\beta f(\beta) Q_{N-1}(z) - z^2 P_{N-2}(z) \}.$$

Thus, Theorem 2 is proved.

Example. Consider the function

$$f(z) = (\exp z - 1)/z = \sum_{k=0}^{\infty} z^k/(k+1)!.$$

The sequence $s_k = 1/(k+1)!$, $k = \overline{0, \infty}$, admits the generalized moment presentation

$$s_{k+j} = \frac{1}{(k+j+1)!} = \int_0^1 \frac{t^k}{k!} \frac{(1-t)^j}{j!} dt, \qquad k, j = \overline{0, \infty},$$

or

$$s_k = \langle A^k x_0, y_0 \rangle, \qquad k, j = \overline{0, \infty},$$

where $X = X^* = L_2[0, 1]$, the operator A is determined by the relation

$$(A\varphi)(t) = \int_0^t \varphi(\tau) d\tau,$$

and the initial elements (functions) x_0 and y_0 are identically equal to one. We now take $\tilde{x}_0(t) = \cos \alpha t$, $\alpha \in (0, \pi/2]$. It is easy to see that

$$(A\tilde{x}_0)(t) = \int_0^t \cos \alpha\tau \, d\tau = \frac{\sin \alpha t}{\alpha},$$

$$(A^2 \tilde{x}_0)(t) = \int_0^t \frac{\sin \alpha\tau}{\alpha} d\tau = \frac{1 - \cos \alpha t}{\alpha^2}.$$

Hence, $[(I+\alpha^2A^2)\tilde{x}_0](t) = 1 = x_0(t)$ and, thus, we can use the argument presented above. Moreover, for any $N \in \mathbb{N} \cup \{0\}$ and $\alpha \in (0, \pi/2]$, the system of functions $\{\tilde{x}_k(t)\}_{k=0}^N$ is a Chebyshev system on [0, 1] [10, p. 13]. Indeed, for $N \le 2$, this is obvious and, for $N > 2$, this statement follows from the fact that $\{\tilde{x}_k(t)\}_{k=0}^{N-2} \cup \{\tilde{x}_k(t)\}_{k=0}^{1}$ is a Chebyshev system of functions. For the Wronskian, we can write

$$W_N = \det \begin{Vmatrix} 1 & t & \ldots & t^{N-2}/(N-2)! & \cos\alpha t & \alpha^{-1}\sin\alpha t \\ 0 & 1 & \ldots & t^{N-3}/(N-3)! & -\alpha\sin\alpha t & \cos\alpha t \\ \cdots & & & & & \\ 0 & 0 & \ldots & 1 & (\cos\alpha t)^{(N-2)} & (\cos\alpha t)^{(N-3)} \\ 0 & 0 & \ldots & 0 & (\cos\alpha t)^{(N-1)} & (\cos\alpha t)^{(N-2)} \\ 0 & 0 & \ldots & 0 & (\cos\alpha t)^{(N)} & (\cos\alpha t)^{(N-1)} \end{Vmatrix}$$

$$= \det \begin{Vmatrix} \alpha^{N-1}\cos\left[\alpha t + \frac{(N-1)\pi}{2}\right] & \alpha^{N-2}\sin\left[\alpha t + \frac{(N-1)\pi}{2}\right] \\ \alpha^{N}\cos\left[\alpha t + \frac{N\pi}{2}\right] & \alpha^{N-1}\sin\left[\alpha t + \frac{N\pi}{2}\right] \end{Vmatrix} = \alpha^{4N-4} \neq 0$$

for any $t \in [0, 1]$.

Note that $\{y_k(t)\}_{k=0}^N$ is also a Chebyshev system of functions for all $N \in \mathbb{N} \cup \{0\}$. Therefore, we can use the results obtained in [8] and conclude that the Padé approximants of degree $[N-1/N]$, $N \in \mathbb{N}$, for the function

$$\tilde{f}(z) = \langle R_z(A)\tilde{x}_0, y_0 \rangle = \int_0^1 \cos\alpha t\, e^{z(1-t)}dt = \frac{ze^z + \alpha\sin\alpha - z\cos\alpha}{z^2 + \alpha^2}$$

exist, are nondegenerate, and converge uniformly on compact sets of the complex plane.

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